

Antti Hannukainen; Sergey Korotov; Michal Křížek  
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ON SYNGE-TYPE ANGLE CONDITION FOR  $d$ -SIMPLICES

ANTTI HANNUKAINEN, Espoo, SERGEY KOROTOV, Bergen,  
MICHAL KRÍŽEK, Praha

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*Abstract.* The maximum angle condition of J. L. Synge was originally introduced in interpolation theory and further used in finite element analysis and applications for triangular and later also for tetrahedral finite element meshes. In this paper we present some of its generalizations to higher-dimensional simplicial elements. In particular, we prove optimal interpolation properties of linear simplicial elements in  $\mathbb{R}^d$  that degenerate in some way.

*Keywords:* simplicial element; maximum angle condition; interpolation error; higher-dimensional problem;  $d$ -dimensional sine; semiregular family of simplicial partitions

*MSC 2010:* 65N30, 65N50, 65N12, 65N15

## 1. INTRODUCTION

The regularity of finite element meshes is important both in analysis and practical applications of the finite element method, as it greatly influences interpolation properties of finite element spaces and through Cea's lemma also the convergence rate of the finite element method. One of the most famous uses of the mesh regularity concept is the so-called Zlámal minimum angle condition (see Remark 4 and also [29], [8], [10], [28]), which can be roughly described as a general requirement on triangular elements to guarantee that their shape does not degenerate in the course of refinement.

However, in practical calculations we sometimes produce simplicial elements degenerating in some way [9], see Figure 1. Flat and narrow elements are also commonly used in covering thin slots, gaps or strips of different materials or to approximate functions that change more rapidly in one direction than in another direction [1].

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Therefore, much effort has been devoted to finding suitable (and practical) concepts, which are weaker than the Zlámal-type conditions. The first attempt was done by Syngé [27] without any application to the convergence of the finite element method. Already in 1957, he proved that linear triangular elements yield the optimal interpolation order in the maximum norm under the so-called maximum angle condition (see Remark 5). This condition is weaker than the minimum angle condition. Later, Jamet [14] proved optimal interpolation properties of triangular finite elements of  $k$ -th degree in the  $W^{m,p}$ -Sobolev norm provided

$$1 < p \leq \infty \quad \text{and} \quad k + 1 - m > \frac{2}{p}$$

and edges do not degenerate to one line. However, we immediately observe that the important case  $p = 2$  and  $k = m$ , used e.g. in convergence analysis of the finite element method, is not covered by the above inequalities (see also [15], p. 493). This gap was resolved in [15]. The maximum angle condition for triangular elements was also investigated in terms of various norms in [2], [3], [4], [16], [15], [17], [18], [20], [22], [23], [24]. To obtain optimal interpolation properties of linear tetrahedral elements one has to impose (see [19]) the maximum angle condition for all triangular faces as well as a similar condition for all dihedral angles between faces. In [19] it is shown that these two conditions are independent. For the cap (and also sliver) tetrahedron from Figure 1, the Syngé condition holds for any triangular face, but some of the dihedral angles between faces may converge to  $\pi$ . On the other hand, for the spike tetrahedron the Syngé condition is violated for some faces, but all dihedral angles are less than some positive constant  $C < \pi$  for a certain manner of degeneracy.

Finally note that the maximum angle condition is, in fact, not necessary for convergence of finite element approximations as shown in [13], [20], [21], and [24].

## 2. NOTATION AND DEFINITIONS

A  $d$ -simplex (or just simplex)  $S$  in  $\mathbb{R}^d$ ,  $d \in \{1, 2, 3, \dots\}$ , is the convex hull of  $d + 1$  vertices  $A_0, A_1, \dots, A_d$  that do not belong to the same  $(d - 1)$ -dimensional hyperplane, i.e.,

$$S = \text{conv}\{A_0, A_1, \dots, A_d\}.$$

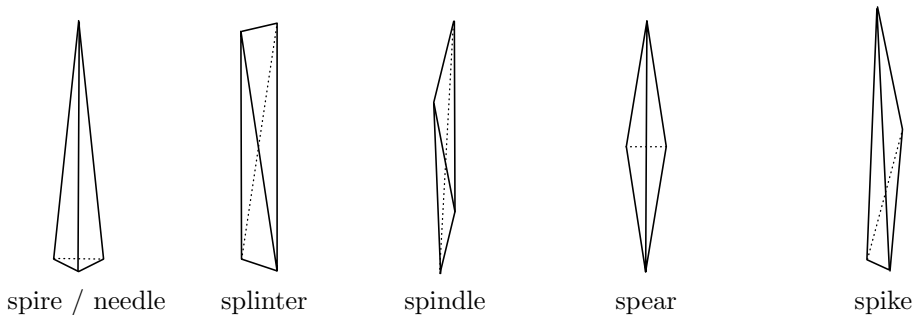
We denote by  $h_S$  the length of the longest edge of  $S$ .

Let

$$F_i = \text{conv}\{A_0, \dots, A_{i-1}, A_{i+1}, \dots, A_d\}$$

be the facet of  $S$  opposite to vertex  $A_i$  for  $i \in \{0, \dots, d\}$ . The dihedral angle  $\alpha$  between two such facets is defined by means of the inner product of their outward

### SKINNY TETRAHEDRA



### FLAT TETRAHEDRA

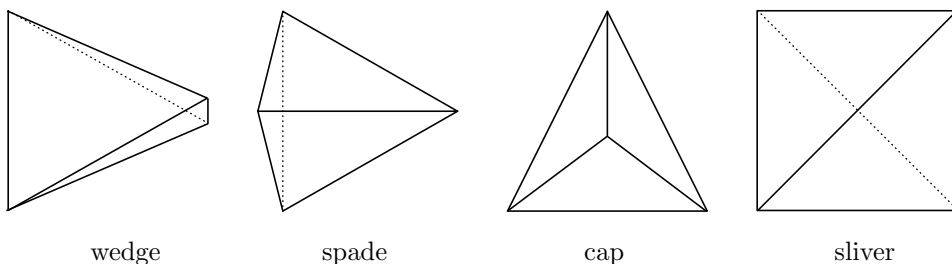


Figure 1. Classification of degenerated tetrahedra according to [9] and [11].

unit normals  $n_1$  and  $n_2$  (see also [12], p. 74),

$$\cos \alpha = -n_1 \cdot n_2.$$

Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain. Assume that  $\overline{\Omega}$  is *polytopic* (or just a *polytope*). By this we mean that  $\overline{\Omega}$  is the closure of a domain whose boundary  $\partial\overline{\Omega}$  is contained in a finite number of  $(d - 1)$ -dimensional hyperplanes.

Next we define a simplicial partition of a bounded closed polytopic domain  $\overline{\Omega} \subset \mathbb{R}^d$  as follows. We subdivide  $\overline{\Omega}$  into a finite number of simplices (called *elements* and denoted by  $S$ ), so that their union is  $\overline{\Omega}$ , any two distinct simplices have disjoint interiors, and any facet of any simplex is either a facet of another simplex from the partition or belongs to the boundary  $\partial\overline{\Omega}$ . The set of such simplices will be called a *simplicial face-to-face partition* (shortly only *partition*) and denoted by  $\mathcal{T}_h$ , where

$$h = \max_{S \in \mathcal{T}_h} h_S.$$

The sequence of partitions  $\mathcal{F} = \{\mathcal{T}_h\}_{h \rightarrow 0}$  of  $\overline{\Omega}$  is called a *family of partitions* if for every  $\varepsilon > 0$  there exists

$$\mathcal{T}_h \in \mathcal{F} \quad \text{with} \quad h < \varepsilon.$$

In what follows, all constants  $C_i$  are independent of  $S$  and  $h$ , but can depend on the dimension  $d$ . By  $\text{meas}_p$  we denote the  $p$ -dimensional volume ( $p \leq d$ ).

Let us recall the definition of the sine of a  $d$ -dimensional angle in  $\mathbb{R}^d$  from [12], p. 72 (see also [5]). Denote by  $\widehat{A}_i$  the angle at the vertex  $A_i$  of the simplex  $S$ . The  $d$ -sine of  $\widehat{A}_i$  for  $d > 1$  is defined as

$$(1) \quad \sin_d(\widehat{A}_i | A_0 A_1 \dots A_d) = \frac{d^{d-1} (\text{meas}_d S)^{d-1}}{(d-1)! \prod_{j=0, j \neq i}^d \text{meas}_{d-1} F_j}.$$

**Remark 1.** For  $d = 2$ ,  $\sin_2(\widehat{A}_i | A_0 A_1 A_2)$  is the standard sine of the angle  $\widehat{A}_i$  in the triangle  $A_0 A_1 A_2$ , due to the following well-known formula, e.g. for  $i = 0$ ,

$$(2) \quad \text{meas}_2(A_0 A_1 A_2) = \frac{1}{2} |A_0 A_1| |A_0 A_2| \sin \widehat{A}_0.$$

**Remark 2.** Consider the standard cube-corner tetrahedron with vertices

$$A_0 = (0, 0, 0), \quad A_1 = (1, 0, 0), \quad A_2 = (0, 1, 0), \quad \text{and} \quad A_3 = (0, 0, 1).$$

Its solid (spatial) angle at  $A_0$ , i.e., the solid angle of the first octant, is  $\frac{4}{8}\pi = \frac{1}{2}\pi$ . Its 3-sine is by (1) equal to (see [12], p. 76)

$$\sin_3(\widehat{A}_0 | A_0 A_1 A_2 A_3) = \frac{3^2 (1/6)^2}{2! (1/2)^3} = 1.$$

For any  $d > 1$  and a general hypercube-corner  $d$ -simplex  $S$  with vertices

$$A_0 = (0, \dots, 0), \quad A_1 = (1, 0, \dots, 0), \dots, \quad A_d = (0, \dots, 0, 1),$$

we have  $\text{meas}_d S = 1/d!$  and  $\text{meas}_{d-1} F_j = 1/(d-1)!$  for  $j = 1, \dots, d$ . Substituting these values into (1), a simple calculation also leads to

$$\sin_d(\widehat{A}_0 | A_0 A_1 \dots A_d) = 1.$$

**Remark 3.** Denoting by  $A_0, A_1, A_2, A_3$  the vertices of the cap or sliver tetrahedron from Figure 1 in an arbitrary way, we find by (1) that

$$\sin_3(\widehat{A}_0 | A_0 A_1 A_2 A_3) \rightarrow 0.$$

**Definition 1.** A family  $\mathcal{F}$  is called a regular family of partitions of a polytope into simplices if there exists  $C > 0$  such that for any  $\mathcal{T}_h \in \mathcal{F}$  and any  $S = \text{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$  we have

$$(3) \quad \forall i \in \{0, 1, \dots, d\} \quad \sin_d(\widehat{A}_i | A_0 A_1 \dots A_d) \geq C > 0,$$

where  $\sin_d$  is defined in (1).

**Remark 4.** The above definition is a natural  $d$ -dimensional generalization (see [8]) of the Zlámál minimum angle condition, which is formulated as follows: There exists a constant  $\alpha_0$  such that for any triangulation  $\mathcal{T}_h \in \mathcal{F}$  and any triangle  $T \in \mathcal{T}_h$  we have

$$\alpha_T \geq \alpha_0 > 0, \quad (\text{Zlámál condition})$$

where  $\alpha_T$  is the minimum angle of the triangle  $T$ . Further (equivalent) definitions of the regularity of a family of partitions of a polytope into simplices are presented in [6], [7].

**Definition 2.** A family  $\mathcal{F}$  is called a semiregular family of partitions of a polytope into simplices if there exists  $C > 0$  such that for any  $\mathcal{T}_h \in \mathcal{F}$  and any  $S = \text{conv}\{A_0, \dots, A_d\} \in \mathcal{T}_h$  we can always find  $d$  edges of  $S$  which, when considered as vectors, constitute a (higher-dimensional) angle whose  $d$ -sine is bounded from below by the constant  $C$ .

**Remark 5.** We observe that for the case  $d = 2$ , Definition 2 is equivalent to the maximum angle condition of Synge (see e.g. [27] and [3]): There exists a constant  $\gamma_0$  such that for any triangulation  $\mathcal{T}_h \in \mathcal{F}$  and any triangle  $T \in \mathcal{T}_h$  we have

$$\gamma_T \leq \gamma_0 < \pi, \quad (\text{Synge condition})$$

where  $\gamma_T$  is the maximum angle of the triangle  $T$ . To see this, we denote by  $\widehat{A}_0$  the largest angle of  $T$  and then apply formula (2). We observe that in the family of triangulations which satisfies the maximum angle condition of Synge, we can have, in principle, an infinite sequence of triangles with decaying sizes whose minimum angles approach zero, which is not possible for the Zlámál condition, see e.g. [18] for some examples on this.

**Remark 6.** Let us point out that the Synge condition guarantees the optimal interpolation order for triangles, which is not true if the maximum angle tends to  $\pi$ , see [13], [26].

**Remark 7.** Each  $d$ -simplex has  $\binom{d+1}{2} = \frac{1}{2}(d+1)d$  edges. The  $d$  edges mentioned in the above definition do not necessarily emanate from the same vertex. An example is a path simplex with its  $d$  orthogonal edges forming a path (in the sense of graph theory). The path element can degenerate (e.g. the  $d$ -dimensional sine of some of its angles can be close to zero) but the  $d$ -sine made by these orthogonal  $d$  edges stays the same.

### 3. MAIN RESULTS

Below we will use the standard definitions of norms and seminorms of various function spaces, see e.g. [10] for details.

**Lemma 1.** *Any regular family of partitions of a polytope into simplices is semiregular.*

**Proof.** For any simplex from the regular family of partitions the required  $d$  edges in the definition of semiregularity are just  $d$  edges emanating from some of its vertices.  $\square$

**Lemma 2.** *Consider a simplex  $S$  with vertices  $A_0, A_1, \dots, A_d$ . Let  $A_{i_l}$  denote any of its vertices. We have the following relations between sines of angles of various dimensions based on  $A_{i_l}$ :*

$$(4) \quad \sin_2(\widehat{A}_{i_l} | A_{i_0} A_{i_1} A_{i_2}) \geq \sin_3(\widehat{A}_{i_l} | A_{i_0} A_{i_1} A_{i_2} A_{i_3}) \geq \dots \geq \sin_d(\widehat{A}_{i_l} | A_0 \dots A_d),$$

where  $i_0, i_1, i_2, i_3, \dots$  are distinct indices from the set  $\{0, 1, \dots, d\}$  and  $i_l \in \{i_0, i_1, i_2\}$ .

For the proof see [12], p. 76. By Remark 2 we see that the inequalities in (4) may become equalities for the hypercube-corner simplex.

**Lemma 3.** *Let  $\mathcal{F}$  be a semiregular family of partitions of a polytope into simplices. Then there exists a constant  $C > 0$  (depending on  $d$  only) such that for any  $\mathcal{T}_h \in \mathcal{F}$  and any  $S \in \mathcal{T}_h$  we have*

$$(5) \quad |\det \mathcal{M}_S| > C,$$

where the entries of the matrix  $\mathcal{M}_S$  are the scalar products  $t_i^T t_j$ ,  $i, j = 1, \dots, d$ , of the unit vectors  $t_1, t_2, \dots, t_d$  parallel to those  $d$  edges of  $S$  which are required in Definition 2.

**Proof.** It is well-known that the volume of the parallelotope  $\mathcal{P}$  generated by the vectors  $t_1, t_2, \dots, t_d$  is equal to  $|\det \mathcal{M}|$ , where

$$\mathcal{M} = (t_i^T t_j)_{i,j=1}^d.$$

For convenience, we can suppose that these  $d$  vectors originate from the same vertex of the parallelotope. Let us denote it as  $A_i$ , where  $i \in \{0, 1, \dots, d\}$ , and let the  $d$ -simplex formed by  $A_i$  and the endpoints of these  $d$  vectors be denoted by  $\widetilde{S}$ . Then

$$(6) \quad |\det \mathcal{M}_{\widetilde{S}}| = \text{meas}_d \mathcal{P} = d! \text{meas}_d \widetilde{S}.$$

Let us use the notations  $A_0, A_1, \dots, A_d$  for the vertices of the simplex  $\tilde{S}$ . From (1), we have

$$(7) \quad (\text{meas}_d \tilde{S})^{d-1} = \frac{(d-1)!}{d^{d-1}} \prod_{j=0, j \neq i}^d \text{meas}_{d-1} \tilde{F}_j \cdot \sin_d(\hat{A}_i | A_0 A_1 \dots A_d),$$

where  $\tilde{F}_j$  are the facets of  $\tilde{S}$  forming the  $d$ -dimensional angle at the vertex  $A_i$ . Further, due to the property of semiregularity,  $\sin_d(\hat{A}_i | A_0 A_1 \dots A_d)$  is bounded from below by a positive constant, i.e.

$$(8) \quad (\text{meas}_d \tilde{S})^{d-1} \geq C \frac{(d-1)!}{d^{d-1}} \prod_{j=0, j \neq i}^d \text{meas}_{d-1} \tilde{F}_j,$$

where  $C$  is a constant from the definition of semiregularity. Now, we notice that in (8) each term  $\text{meas}_{d-1} \tilde{F}_j$  can be computed (and further estimated from below via a product of areas of the relevant  $(d-2)$ -dimensional facets) in a similar way to (7) and (8) using the semiregularity property and relations between sines of various dimensions (4). Finally, in order to get (5) proved, we apply the above arguments recursively to all involved areas of facets of dimensions  $d-2, d-3, \dots, 2$  and notice at the end that after finitely many steps the areas of one-dimensional facets are just  $\|t_j\| = 1$ . Lemma 3 thus follows from (6) and (8) by induction.  $\square$

The next theorem gives the optimal interpolation order in the following maximum norm over  $S$

$$(9) \quad \|w\|_{1,\infty} = \max \left\{ \max_{x \in S} |w(x)|, \max_{x \in S} \left| \frac{\partial w(x)}{\partial x_1} \right|, \dots, \max_{x \in S} \left| \frac{\partial w(x)}{\partial x_d} \right| \right\}$$

for any Lipschitz function  $w$ . In what follows, we will also use the matrix norm

$$\|A\|_1 = \max_j \sum_i |a_{ij}|.$$

**Theorem 1.** *Let  $\mathcal{F}$  be a semiregular family of partitions of a polytope into simplices. Then there exists a constant  $C > 0$  such that for any  $\mathcal{T}_h \in \mathcal{F}$  and any  $S \in \mathcal{T}_h$  we have*

$$(10) \quad \|v - \pi_S v\|_{1,\infty} \leq C h_S \|v\|_{2,\infty} \quad \forall v \in \mathcal{C}^2(S),$$

where  $\pi_S$  is the standard Lagrange linear interpolant and  $h_S = \text{diam } S$ .



*Proof.* Consider a simplex  $S \in \mathcal{T}_h \in \mathcal{F}$ . Let  $v \in \mathcal{C}^2(S)$  be arbitrary and let  $e$  be an arbitrary edge of  $S$ . Set

$$(11) \quad w = v - \pi_S v \quad \text{on } S.$$

Since  $w = 0$  at all vertices of  $S$ , there exists by the Rolle's theorem [25], p. 387 a point  $Q \in e$  such that

$$(12) \quad t^T \text{grad } w(Q) = 0,$$

where  $t$  is the unit vector parallel to  $e$ . Set

$$(13) \quad z = t^T \text{grad } w$$

in  $S$ . Let  $P$  be an arbitrary fixed point in the interior of  $S$  and let  $u$  be the unit vector parallel to  $QP$ . Then by (12) and (13) we have  $z(Q) = 0$  and

$$(14) \quad z(P) = \int_Q^P u^T \text{grad } z \, ds = \int_Q^P u^T (\text{hes } w) t \, ds,$$

where  $\text{hes } w$  is the matrix of the second derivatives (Hessian) of  $w$ . Thus, by (13), (14), and (11),

$$(15) \quad |t^T \text{grad } w(P)| \leq dh_S |w|_{2,\infty} = dh_S |v|_{2,\infty},$$

where the upper estimate does not depend on  $P$ .

Consider now those  $d$  edges of  $S$  required in Definition 2 and  $d$  corresponding unit vectors  $t_1, t_2, \dots, t_d$  parallel with them. Writing

$$(16) \quad \text{grad } w(P) = \sum_{j=1}^d c_j(P) t_j, \quad c_j \in \mathbb{R}^1,$$

we see that the coefficients  $c_j = c_j(P)$  fulfil the linear system of algebraic equations

$$(17) \quad \sum_{j=1}^d t_i^T t_j c_j(P) = t_i^T \text{grad } w(P), \quad i = 1, 2, \dots, d,$$

with the Gram matrix

$$\mathcal{M}_S = (t_i^T t_j)_{i,j=1}^d.$$

We can concisely write the above system as  $\mathcal{M}_S \mathbf{c}(P) = \mathbf{b}(P)$ .

Using standard definitions of vector and matrix norms and applying the constants from the relevant equivalence relations, we observe from the equalities (16) and (17) that

$$(18) \quad \|\text{grad } w(P)\|_\infty \leq \|\mathbf{c}(P)\|_1 \leq \|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_1 \leq d \|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_\infty$$

is valid for any point  $P$ . (In the above we also used the fact that  $\|t_j\|_2 = 1$  for  $j = 1, 2, \dots, d$ .) Let  $\mathcal{M}_S^*$  be the algebraic adjoint of the entries of  $\mathcal{M}_S$ . Then

$$\mathcal{M}_S^{-1} = \frac{1}{\det \mathcal{M}_S} \mathcal{M}_S^*,$$

and a straightforward calculation leads to the following estimation of the  $\|\cdot\|_1$ -norm of  $\mathcal{M}_S^*$ :

$$(19) \quad \|\mathcal{M}_S^*\|_1 \leq d \cdot (d-1)! = d!.$$

Hence, from (18), (19), and (15) we get

$$(20) \quad \|\text{grad } w(P)\|_\infty \leq d \|\mathcal{M}_S^{-1}\|_1 \|\mathbf{b}(P)\|_\infty \leq \frac{d^2 d!}{|\det \mathcal{M}_S|} h_S |v|_{2,\infty},$$

valid for any point  $P \in S$ . Now, using (20) and Lemma 3, we get that for any  $i = 1, \dots, d$

$$(21) \quad \max_{x \in S} \left| \frac{\partial w(x)}{\partial x_i} \right| \leq C h_S |v|_{2,\infty}.$$

By [10], pp. 118–120, we have the estimate

$$(22) \quad \max_{x \in S} |w(x)| \leq C h_S^2 |v|_{2,\infty} \quad \forall v \in W_\infty^2(S)$$

without any regularity assumption on the family  $\mathcal{F}$ . The relations (9), (21), and (22) now imply (10).  $\square$

#### 4. EXAMPLES

**Example 1.** The needle, splinter, and wedge elements from Figure 1 satisfy Definition 2. They yield the optimal interpolation order of linear elements provided the lengths of their edges are as indicated in Figure 2. For the splinter tetrahedron from Figure 1, we have to take two short opposite edges and one long edge to satisfy Definition 2.

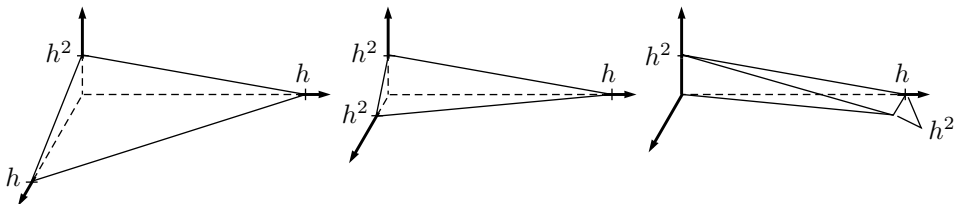


Figure 2. Three types of degenerating tetrahedra which do not deteriorate the optimal interpolation order [19]. The length  $h^2$  can be replaced by  $h^{1+\varepsilon}$  for any  $\varepsilon > 0$ .

**Example 2.** A higher-dimensional example can be constructed as follows. Consider positive numbers  $r_1, r_2, \dots, r_d$  and a simplex with vertices  $A_0, A_1, \dots, A_d$ . We fix some number  $k$  such that  $0 \leq k \leq d$ . The first  $k + 1$  vertices of the simplex are defined as follows. Let  $A_0 = (0, 0, \dots, 0, \dots, 0)$ . Further, let

$$\begin{aligned} A_1 &= (r_1, 0, \dots, 0, \dots, 0), \\ A_2 &= (0, r_2, \dots, 0, \dots, 0), \\ &\vdots \\ A_k &= (0, \dots, 0, r_k, 0, \dots, 0). \end{aligned}$$

The remaining vertices are:

$$\begin{aligned} A_{k+1} &= (0, \dots, 0, r_{k+1}, 0, \dots, 0), \\ A_{k+2} &= (0, \dots, 0, r_{k+1}, r_{k+2}, 0, \dots, 0), \\ &\vdots \\ A_{k+3} &= (0, \dots, 0, r_{k+1}, r_{k+2}, r_{k+3}, 0, \dots, 0), \\ A_d &= (0, \dots, 0, r_{k+1}, r_{k+2}, r_{k+3}, \dots, r_d). \end{aligned}$$

Therefore, for  $k = 0$ , we get the path-simplex, and for  $k = d$  the hypercube-corner simplex. Allowing some of the  $r_k$ 's to approach zero with different rates, in general, we arrive at various degenerated simplices still satisfying the semiregularity property.

**Example 3.** Let  $A_0 = (h, 0, 0)$ ,  $A_1 = (0, h^3, h)$ ,  $A_2 = (-h, 0, 0)$ , and  $A_3 = (0, h^3, -h)$  be the vertices of the tetrahedron  $S$  for some  $h \in (0, 1)$ . From Figure 1 we observe that  $S$  is a sliver tetrahedron when  $h \rightarrow 0$ , i.e., the condition (5) is violated. We show that the optimal interpolation order of the linear interpolant  $\pi_S$  is not guaranteed (cf. Theorem 1). Setting

$$v(x_1, x_2, x_3) = x_1^2,$$

we immediately find that

$$v(A_0) - v(A_2) = h^2, \quad v(A_1) = v(A_3) = 0.$$

Using the linearity of  $\pi_S v$ , we obtain

$$\begin{aligned} \frac{\partial}{\partial x_2}(v - \pi_S v) &= -\frac{\partial(\pi_S v)}{\partial x_2} = \frac{(\pi_S v)((A_0 + A_2)/2) - (\pi_S v)((A_1 + A_3)/2)}{h^3} \\ &= \frac{v(A_0) + v(A_2) - v(A_1) - v(A_3)}{2h^3} = h^{-1} \rightarrow \infty \quad \text{as } h \rightarrow 0. \end{aligned}$$

Hence, the estimate (10) does not hold.

**Example 4.** Let  $A_0 = (0, 0, 0)$ ,  $A_1 = (h, 0, 0)$ ,  $A_2 = (h, 0, h^3)$ , and  $A_3 = (-h, h^3, 0)$  be the vertices of the tetrahedron  $S$  for some  $h \in (0, 1)$ . From Figure 1 we see that  $S$  is a spike tetrahedron when  $h \rightarrow 0$ . We again show that the optimal interpolation order of  $\pi_S$  is violated, since (5) is not true. Putting

$$v_1(x_1, x_2, x_3) = x_1^2,$$

we observe that

$$v(A_0) = 0, \quad v(A_1) = v(A_2) = v(A_3) = h^2.$$

Since  $\pi_S v$  is linear, we deduce that

$$\begin{aligned} \frac{\partial}{\partial x_2}(v - \pi_S v) &= -\frac{\partial(\pi_S v)}{\partial x_2} = \frac{(\pi_S v)(A_0) - (\pi_S v)((A_1 + A_3)/2)}{\frac{1}{2}h^3} \\ &= -\frac{v(A_1) + v(A_3)}{h^3} = \frac{2h^{-2}}{h^3} \rightarrow -\infty \end{aligned}$$

whenever  $h \rightarrow 0$ . Consequently, the estimate (10) is not valid.

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## References

- [1] *T. Apel*: Anisotropic Finite Elements: Local Estimates and Applications. Advances in Numerical Mathematics, Teubner, Stuttgart, 1999. [zbl](#) [MR](#)
- [2] *T. Apel, M. Dobrowolski*: Anisotropic interpolation with applications to the finite element method. *Computing* *47* (1992), 277–293. [zbl](#) [MR](#) [doi](#)
- [3] *I. Babuška, A. K. Aziz*: On the angle condition in the finite element method. *SIAM J. Numer. Anal.* *13* (1976), 214–226. [zbl](#) [MR](#) [doi](#)
- [4] *R. E. Barnhill, J. A. Gregory*: Sard kernel theorems on triangular domains with application to finite element error bounds. *Numer. Math.* *25* (1975), 215–229. [zbl](#) [MR](#) [doi](#)
- [5] *P. Bartoš*: The sine theorem for simplexes in  $E_n$ . *Cas. Mat.* *93* (1968), 273–277 (In Czech). [zbl](#) [MR](#)
- [6] *J. Brandts, S. Korotov, M. Křížek*: On the equivalence of regularity criteria for triangular and tetrahedral finite element partitions. *Comput. Math. Appl.* *55* (2008), 2227–2233. [zbl](#) [MR](#) [doi](#)
- [7] *J. Brandts, S. Korotov, M. Křížek*: On the equivalence of ball conditions for simplicial finite elements in  $\mathbb{R}^d$ . *Appl. Math. Lett.* *22* (2009), 1210–1212. [zbl](#) [MR](#) [doi](#)
- [8] *J. Brandts, S. Korotov, M. Křížek*: Generalization of the Zlámal condition for simplicial finite elements in  $\mathbb{R}^d$ . *Appl. Math., Praha* *56* (2011), 417–424. [zbl](#) [MR](#) [doi](#)
- [9] *S.-W. Cheng, T. K. Dey, H. Edelsbrunner, M. A. Facello, S.-H. Teng*: Sliver exudation. Proc. of the Fifteenth Annual Symposium on Computational Geometry, Miami Beach, 1999. ACM, New York, 1999, pp. 1–13. [MR](#) [doi](#)
- [10] *P. G. Ciarlet*: The Finite Element Method for Elliptic Problems. Studies in Mathematics and Its Applications 4, North-Holland Publishing, Amsterdam, 1978. [zbl](#) [MR](#)
- [11] *H. Edelsbrunner*: Triangulations and meshes in computational geometry. *Acta Numerica* (2000), 133–213. [zbl](#) [MR](#) [doi](#)
- [12] *F. Eriksson*: The law of sines for tetrahedra and  $n$ -simplices. *Geom. Dedicata* *7* (1978), 71–80. [zbl](#) [MR](#) [doi](#)
- [13] *A. Hannukainen, S. Korotov, M. Křížek*: The maximum angle condition is not necessary for convergence of the finite element method. *Numer. Math.* *120* (2012), 79–88. [zbl](#) [MR](#) [doi](#)
- [14] *P. Jamet*: Estimations d’erreur pour des éléments finis droits presque dégénérées. *Rev. Franc. Automat. Inform. Rech. Operat.* *10*, Analyse numer., R-1 (1976), 43–60. [zbl](#) [MR](#)
- [15] *K. Kobayashi, T. Tsuchiya*: A priori error estimates for Lagrange interpolation on triangles. *Appl. Math., Praha* *60* (2015), 485–499. [zbl](#) [MR](#) [doi](#)
- [16] *K. Kobayashi, T. Tsuchiya*: On the circumradius condition for piecewise linear triangular elements. *Japan J. Ind. Appl. Math.* *32* (2015), 65–76. [zbl](#) [MR](#) [doi](#)
- [17] *K. Kobayashi, T. Tsuchiya*: Extending Babuška-Aziz’s theorem to higher-order Lagrange interpolation. *Appl. Math., Praha* *61* (2016), 121–133. [zbl](#) [MR](#) [doi](#)
- [18] *M. Křížek*: On semiregular families of triangulations and linear interpolation. *Appl. Math., Praha* *36* (1991), 223–232. [zbl](#) [MR](#)
- [19] *M. Křížek*: On the maximum angle condition for linear tetrahedral elements. *SIAM J. Numer. Anal.* *29* (1992), 513–520. [zbl](#) [MR](#) [doi](#)
- [20] *V. Kučera*: A note on necessary and sufficient conditions for convergence of the finite element method. Proc. Conf. Appl. Math. 2015 (J. Brandts et al., eds.). Institute of Mathematics CAS, Prague, 2015, pp. 132–139.
- [21] *V. Kučera*: On necessary and sufficient conditions for finite element convergence. Available at [arXiv:1601.02942](#) (2016).
- [22] *V. Kučera*: Several notes on the circumradius condition. *Appl. Math., Praha* *61* (2016), 287–298. [zbl](#) [MR](#) [doi](#)
- [23] *S. Mao, Z. Shi*: Error estimates of triangular finite elements under a weak angle condition. *J. Comput. Appl. Math.* *230* (2009), 329–331. [zbl](#) [MR](#) [doi](#)

- [24] *P. Oswald*: Divergence of FEM: Babuška-Aziz triangulations revisited. *Appl. Math., Praha* 60 (2015), 473–484. [zbl](#) [MR](#) [doi](#)
- [25] *K. Rektorys*: Survey of Applicable Mathematics. Vol. I. Mathematics and Its Applications 280, Kluwer Academic Publishers, Dordrecht, 1994. [zbl](#) [MR](#) [doi](#)
- [26] *G. Strang, G. J. Fix*: An Analysis of the Finite Element Method. Prentice-Hall Series in Automatic Computation, Englewood Cliffs, New Jersey, 1973. [zbl](#) [MR](#)
- [27] *J. L. Synge*: The Hypercircle in Mathematical Physics. A Method for the Approximate Solution of Boundary Value Problems. Cambridge University Press, Cambridge, 1957. [zbl](#) [MR](#)
- [28] *A. Ženíšek*: The convergence of the finite element method for boundary value problems of the system of elliptic equations. *Apl. Mat.* 14 (1969), 355–377 (In Czech). [zbl](#) [MR](#)
- [29] *M. Zlámal*: On the finite element method. *Numer. Math.* 12 (1968), 394–409. [zbl](#) [MR](#) [doi](#)

*Authors' addresses:* *Antti Hannukainen*, Department of Mathematics and Systems Analysis, Aalto University, P.O. Box 11100, FI-00076 Espoo, Finland, e-mail: [antti.hannukainen@aalto.fi](mailto:antti.hannukainen@aalto.fi); *Sergey Korotov*, Department of Computing, Mathematics and Physics, Western Norway University of Applied Sciences, Inndalsveien 28, 5020 Bergen, Norway, e-mail: [sergey.korotov@hib.no](mailto:sergey.korotov@hib.no); *Michal Krížek*, Institute of Mathematics, Czech Academy of Sciences, Žitná 25, CZ-115 67 Praha 1, Czech Republic, e-mail: [krizek@math.cas.cz](mailto:krizek@math.cas.cz), [krizek@cesnet.cz](mailto:krizek@cesnet.cz).