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ON IMPROPER INTERVAL EDGE COLOURINGS

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Abstract. We study improper interval edge colourings, defined by the requirement that the edge colours around each vertex form an integer interval. For the corresponding chromatic invariant (being the maximum number of colours in such a colouring), we present upper and lower bounds and discuss their qualities; also, we determine its values and estimates for graphs of various families, like wheels, prisms or complete graphs. The study of this parameter was inspired by the interval colouring, introduced by Asratian, Kamalian (1987). The difference is that we relax the requirement on the original colouring to be proper.

Keywords: edge colouring; interval colouring; improper colouring

MSC 2010: 05C15

1. Introduction

Throughout this paper, we consider simple connected graphs without loops or multiple edges; we use standard graph terminology from book [4].

A proper edge colouring $c\colon V(G) \to \{1, \dots, k\}$ of a graph G which uses each colour from $\{1, \dots, k\}$ at least once is called an *interval colouring* if for each vertex x of G, the set of colours of edges incident with x (the *palette* of x) forms an integer interval; we say that the graph G is *interval k-colourable.* For an interval colourable graph G,

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let $t(G)$ denote the maximum number of colours used in any interval colouring of G. The notion of interval colouring was introduced by Asratian and Kamalian in [1] in connection with specialized scheduling problems and, since then, it was further investigated in many papers, see for example [3], [5], [6], [7], [8], [9], [10]. Not all graphs are proper interval colourable (this concerns, for example, graphs of Class 2); in fact, the problem of determining whether a graph has an interval colouring is NP-complete, even for bipartite graphs, see [2] and [11].

In our paper, we relax the requirement on the above defined colouring to be proper and introduce the parameter $\hat{t}(G)$ being the maximum number of colours in an improper interval colouring of G. Note that $\hat{t}(G)$ is defined for every graph (which is in sharp contrast with $t(G)$, see [3]) and is at least 3 for all graphs of order at least 3 different from K_3. Also, for a graph G which is interval colourable, $t(G) \leqslant \hat{t}(G)$ holds. Compared with the range of results for interval colourings, it seems that improper interval colourings have not been studied yet. Our aim is to contribute to this topic by determining the exact values of $\hat{t}(G)$ or their estimates for graphs of several classic families (this concerns also many graphs which do not possess proper interval colourings), and to establish upper and lower bounds on $\hat{t}(G)$ in terms of the graph diameter and the maximum degree.

2. Properties and results

In the analysis of improper interval colourings, the following observations (which are easy to see) will be useful:

Proposition 2.1. *If a graph G is improperly interval k-colourable with $k \geqslant 3$, then it is also improperly interval k-colourable in such a way that the colours 1 and k are used exactly once.*

Proposition 2.2. *A graph G is improperly interval l-colourable for each $1 \leqslant l \leqslant \hat{t}(G)$.*

First, we present an upper bound on $\hat{t}(G)$ in terms of the maximum degree and the diameter:

Lemma 2.3. *For any connected graph G with maximum degree $\Delta = \Delta(G)$ we have $\hat{t}(G) \leqslant 1 + (\Delta - 1)(\operatorname{diam}(G) + 1)$.*

Proof. Let uv and xy be edges coloured by 1 and k, respectively, in an improper interval colouring of G using $\hat{t}(G)$ colours. Observe that in each such colouring, the colours of each two adjacent edges differ by at most $\Delta - 1$. Now, take the shortest path P between the vertex sets $\{u, v\}$ and $\{x, y\}$. Then P has the length at most $\operatorname{diam}(G)$; note, however, that the edges uv and xy need not belong to P. It follows that the number of colour changes from uv to xy along P is at most $\operatorname{diam}(G) + 1$, which implies the result. □

Note that this lemma generalizes the result of [1] where the right hand side of the above inequality estimates $t(G)$ from above. Using the same arguments, we can prove an analogous inequality with respect to the diameter of the line graph $L(G)$:

Lemma 2.4. *For any connected graph G with maximum degree $\Delta = \Delta(G)$ we have $\hat{t}(G) \leqslant 1 + (\Delta - 1)(\operatorname{diam}(L(G))$; the bound is sharp.*

We also present strengthenings of two theorems from [1]:

Theorem 2.5. *For each triangle-free graph G on n vertices, $\hat{t}(G) \leqslant n - 1$; the bound is sharp.*

Proof. We follow the same reasoning as in the proof of Theorem 1 from [1]; the difference is only in the estimate of the number of elements of the set $A(i)$ (see the original proof, page 38): we obtain that for an improper interval t-colouring of G the inequality $|A(i)| \geqslant f(e_i) - f(e_{i+1}) - 1$, $i = 1, \ldots, k-1$, holds (instead of equality). Hence, the last argument of the original proof rephrases as

$$\begin{aligned} n &\geqslant k + 1 + \sum_{i=1}^{k-1} |A(i)| \geqslant k + 1 + \sum_{i=1}^{k-1} (f(e_i) - f(e_{i+1}) - 1) \\ &= k + 1 + t - 1 - (k - 1) = 1 + t, \end{aligned}$$

implying $t \leqslant n - 1$.

To show the sharpness of the bound, consider the graph of the path on n vertices P_n, $n \geqslant 2$. It is easy to see that $\hat{t}(P_n) = n - 1$. □

Since the original proof of Proposition 4 of [1], page 39, does not require the considered interval colourings to be proper, we obtain the following theorem.

Theorem 2.6. *For each graph G on n vertices, $\hat{t}(G) \leqslant 2n - 1$.*

For the lower bound on $\hat{t}(G)$, we have the following estimate:

Theorem 2.7. *For each graph G, $\hat{t}(G) \geqslant 1 + \operatorname{diam}(L(G))$; the bound is sharp.*

P r o o f. Consider the line graph $L(G)$ of G and let x be a vertex of maximum eccentricity in $L(G)$. Then colour the vertex x with colour 1 and each vertex $y \in V(L(G))$, $y \neq x$ with the colour equal to $1 + \mathrm{dist}_{L(G)}(x, y)$. This vertex colouring of $L(G)$ induces an edge colouring of G. The way the colouring of the vertices of $L(G)$ was constructed gives that in G, the palette of each vertex consists either of two consecutive colours or of a single colour; thus, it is an improper interval colouring of G having the highest colour equal to $\mathrm{diam}(L(G)) + 1$.

To show the sharpness of the lower bound, consider for an integer $k \geqslant 6$ the graph DB_k obtained from a k-vertex path $x_1x_2 \ldots x_k$ by adding new edges x_1x_3 and $x_{k-2}x_k$. Then it is easy to check that $\hat{t}(DB_k) = k - 1$ and $\mathrm{diam}(L(DB_k)) = k - 2$. □

The difference between $t(G)$ and $\hat{t}(G)$ can be arbitrarily large. This can be seen on the graph SN_k formed from a chain of k copies of the graph K_4^- where both chain ends are closed with a different triangle, see Figure 1. It is easy to see that the graph SN_k has a proper interval colouring; observe that in each proper interval colouring of SN_k, the difference of colours of two consecutive bridges incident with the same copy of K_4^- is 0 or 3 while it is possible to construct an improper colouring of SN_k such that the colour difference on consecutive bridges is 4. Thus, we obtain that $\hat{t}(SN_k) - t(SN_k) \geqslant k$. A similar construction can be used also for triangle-free graphs, where instead of copies of K_4^-, the 5-cycle with pendant edges incident with two nonadjacent vertices is used: the difference of colours on bridges in a proper interval colouring is at most 3 whereas it is possible to assign the colours in such a way that the difference is 4 in an improper interval colouring, see Figure 2.

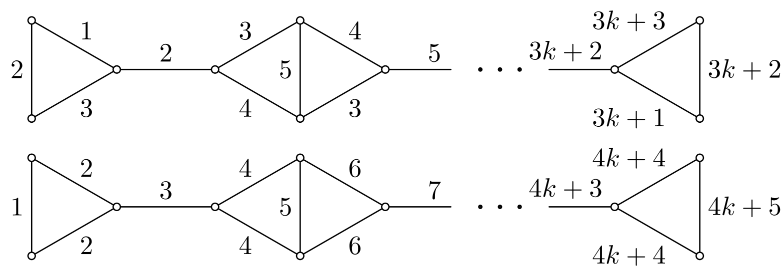

Figure 1. The graph SN_k and its proper and improper interval colourings.

The construction generalizes to classes of graphs of arbitrarily large girth. Moreover, one can consider several other suitable configurations to show that the colour difference on two selected edges can be greater in an improper version of the colouring rather than in the proper one, and these configurations may be used to form other graphs (for example 2-connected) with arbitrarily large difference between $t(G)$ and $\hat{t}(G)$.

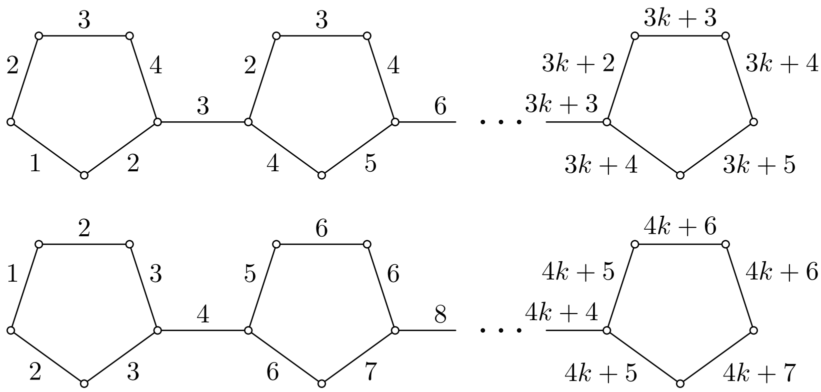

Figure 2. An analogous construction for triangle-free graphs.

There is no known significant upper bound on the difference $\hat{t}(G) - t(G)$ in terms of the number of vertices of G.

In the following, we establish the exact values and estimates of $\hat{t}(G)$ of graphs from several standard graph families.

Theorem 2.8. *For an n-wheel W_n,*

$$\hat{t}(W_n) = \begin{cases} 4 & \text{if } n = 3; \\ n & \text{if } n \geqslant 4. \end{cases}$$

Proof. Suppose first that $n = 3$. If $\hat{t}(W_3) \geqslant 5$, then the unique edge of W_3 which is not adjacent to the edge of maximum colour would have a colour at least 2, so colour 1 is not used, a contradiction; on the other hand, an improper interval 4-colouring of W_3 is easy to find.

Now, let $n \geqslant 4$. An improper interval n-colouring of W_n can be constructed in the following way: if x is the centre of W_n and $x_1, \ldots, x_n$ are its neighbours in counter clockwise order, assign to each edge x_ix_{i+1}, $1 \leqslant i \leqslant \lceil n/2 \rceil$ colour $2i - 1$, to each edge x_ix_{i+1}, $\lceil n/2 \rceil < i \leqslant n$ (indices taken modulo n) colour $2n + 4 - 2i$, and to each edge xx_i, $i \notin \{1, \lceil n/2 \rceil + 1\}$ the colour equal to the arithmetic mean of colours of edges $x_{i-1}x_i, x_ix_{i+1}$, whereas the edge xx_1 receives colour 1 and the edge $xx_{\lceil n/2 \rceil + 1}$ receives colour n. It is easy to check that in this colour assignment, the palette of each vertex forms an integer interval (note that the palette of x_1 is $[1, 2]$ and the palette of $x_{\lceil n/2t \rceil + 1}$ is $[n - 1, n]$). See Figure 3 for illustration.

Assume now that for some n, $\hat{t}(W_n) \geqslant n + 1$. Consider first the case when n is odd. Then at least one of colours 1 and $n+1$ is used at a rim edge of W_n (otherwise the palette of the central vertex of W_n would not form an integer interval). Since

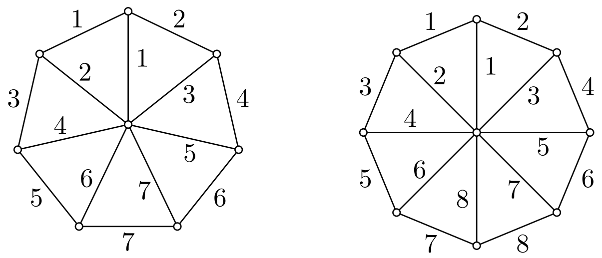

Figure 3. The improper interval n-colouring of a wheel W_n for $n = 7$ and $n = 8$.

replacing each colour c by colour $n + 2 - c$ yields also an improper interval $(n + 1)$-colouring of W_n, we can suppose, without loss of generality, that colour 1 is used on a rim edge uv. Note that the colours of two adjacent rim edges can differ by at most 2; this means that two edges yw, wz which are—taking into account the bidirectional distance on the rim—most distant from uv, have the colour at most $1 + 2(n - 1)/2 = n$. Thus, colour $n + 1$ has to appear at a spoke edge of W_n and, repeating the above colour difference argument, we obtain that this spoke edge is incident with the vertex w. Due to the fact that in the considered improper interval colouring, colour 1 is unique, the edge uv is adjacent with at least two edges coloured by 2 and at least one of them—say, vq—is a rim edge (otherwise again, the palette of the central vertex would not, an integer interval). Then the colour sequence of the rim path P from uv through vq ending by one of the rim edges incident to w is $1, 2, 4, \dots, 2i, 2(i + 1), \dots, n - 1$; this implies that one of yw and wz has colour $n - 1$ and the other one has colour n. Now, if we take the rim path P' starting at uv, but with the opposite direction to P, we get that its colour sequence is $1, 3, 5, \dots, 2i - 1, 2i + 1, \dots, n$. But then colour n does not appear at a spoke edge, hence, the palette of the central vertex is not an integer interval, a contradiction.

Consider now the case when n is even. Rephrasing the above arguments, we can suppose that colour 1 is used on a rim edge uv. According to the possible position of colour $n + 1$, we distinguish two cases:

Case 1: Colour $n + 1$ is on a rim edge. Then the colour sequences of both rim paths starting at uv are $1, 3, 5, \dots, 2i - 1, 2i + 1, \dots, n + 1$, which gives that colours of all spoke edges are even numbers, a contradiction.

Case 2: Colour $n + 1$ is on a spoke edge xw. Let yw and wz be the rim edges adjacent to xw; without loss of generality, let yw be closer to uv than wz to uv. Then the colour sequence of a rim path starting at uv and ending at yw is $1, 3, \dots, 2i - 1$, $2i + 1, \dots, n - 1$, which yields that wz has colour n. By the same argument as for n odd, colour 2 has to be used on a rim edge incident with uv, hence, the colour sequence of the rim path between uv and wz is $2, 4, \dots, 2i, 2(i + 1), \dots, n$. But then again, colour n is missing at spoke edges, a contradiction. $\square$

Note that by [3] only three wheels are proper interval colourable, namely W_3, W_6 and W_9.

Next, we present the exact value for a graph $Y_n = C_n \,\Box\, K_2$, the graph of an n-sided prism.

Theorem 2.9. *For an n-prism graph Y_n with $n \geqslant 3$, $\hat{t}(Y_n) = n + 2$.*

P r o o f. Let $Y_n = C_n \,\Box\, K_2$. In a plane drawing of Y_n, there are two n-gonal faces $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ interconnected by the edges $x_i y_i$, $i = 1, \ldots, n$ (the *side edges*); the edges of type $x_i x_{i+1}$ or $y_i y_{i+1}$ ($i = 1, \ldots, n$, indices are modulo n) will be called *base edges* in the sequel.

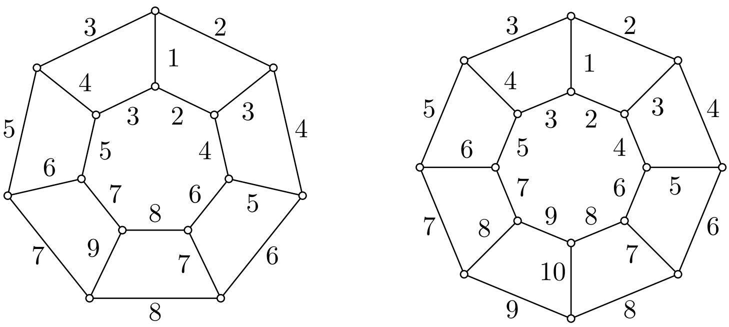

Figure 4. The improper interval $(n+2)$-colouring of a prism Y_n for $n = 7$ and $n = 8$.

First, we show that there exists an improper interval colouring of graph Y_n with $n + 2$ colours. It can be constructed as follows: Assign the edge $x_1 y_1$ colour 1, the edges $x_1 x_2$, $y_1 y_2$ colour 2, the edges $x_1 x_n$, $y_1 y_n$ colour 3 and the edge $x_{\lceil n/2 \rceil + 1} y_{\lceil n/2 \rceil + 1}$ colour $n + 2$. Now, assign edges $x_i y_i$ colour $2i - 1$ for $i = 2, \ldots, \lceil n/2 \rceil$. Assign the remaining side edges $x_i y_i$ colour $2(n - i) + 4$ for $i = n, \ldots, \lceil n/2 \rceil + 2$. The colours of the remaining base edges are now determined unambiguously, see Figure 4 for illustration.

Now we prove that the colouring is optimal. Observe that $\operatorname{diam}(L(Y_n)) = \lfloor n/2 \rfloor + 1$, hence Lemma 2.4 gives an upper bound $\hat{t}(Y_n) \leqslant 2\lfloor n/2 \rfloor + 3$. For n odd this meets the lower bound.

Suppose that n is even and there is an interval edge colouring of Y_n with $n + 3$ colours. From the proof of Lemma 2.4 we know that for any pair of edges in G coloured with 1 and $n + 3$, the corresponding vertices in $L(G)$ are at the distance $\operatorname{diam}(L(G))$. This is possible either for a pair of side edges $x_i y_i$ and $x_j y_j$ with $|i - j| = n/2$, or for a pair of base edges $x_i x_{i+1}$ and $y_j y_{j+1}$ with $|i - j| = n/2$. In both the cases all edges incident with the edge assigned colour 1 lie on some path of the optimal length between the edges assigned 1 and $n + 3$, so all these edges have to be assigned colour 3, which is a contradiction, since there is no colour 2 in the palettes of vertices incident with the edge assigned colour 1. □

Finally, we discuss the improper interval colourings of complete graphs, which is of particular interest due to the ongoing intensive research in [9], [10].

Theorem 2.10. *For a complete graph K_n with $n \geqslant 5$, we have $\hat{t}(K_n) \leqslant 2n-5$.*

P r o o f. By contradiction. Assume that there exists a positive integer $n \geqslant 5$ such that K_n is improperly interval $(2n-4)$-colourable. Let xy and uv be edges coloured by 1 and $2n-4$, respectively; note that xy and uv are not adjacent. Then the colour of xu (xv, yu or yv) is either $n-2$ or $n-1$. We claim that the palettes of both x and y contain each of colours $1,\ldots,n-1$ exactly once; similarly, the palettes of both u and v contain each of the colours $n-2,\ldots,2n-4$ exactly once: If both xu and xv had colour $n-2$, then palettes of u and v would contain each of colours $n-2,\ldots,2n-4$ exactly once, hence, yu and yv would have colour $n-1$, which is impossible.

Let z be a vertex of K_n such that zx has colour 2. Then the colour of the edge zu is at least n (because of u); note, however, that the difference of the highest and the lowest colour (which is colour 2) in the palette of z is at most $n-2$, which means that the highest colour in the palette of z is at most n. Therefore, the colour of zu is equal to n. The same argument can be used for the edge zv, obtaining that its colour is also n. But then the palette of z is not an integer interval, a contradiction. □

Theorem 2.11. *For each n, $\hat{t}(K_n) < \hat{t}(K_{n+1})$.*

P r o o f. Let c be an improper interval $\hat{t}$-colouring of K_n with $\hat{t} = \hat{t}(G)$, and let $w_1,\ldots,w_n$ be an ordering of vertices of K_n such that w_1 is incident with an edge of colour $\hat{t}$. Now, add to K_n a new vertex u and for each $1 < i \leqslant n$ add a new edge uw_i coloured with colour $c(w_1w_i)+1$; the edge uw_1 will then be coloured with $\hat{t}+1$. □

Theorem 2.12. *For each n, $\hat{t}(K_{n+2}) - \hat{t}(K_n) \geqslant 3$.*

P r o o f. Let c be an improper interval $\hat{t}$-colouring of K_n with $\hat{t} = \hat{t}(G)$, and let $w_1,\ldots,w_n$ be an ordering of vertices of K_n such that w_1 and w_2 are endvertices of an edge of colour $\hat{t}$. Now, add to K_n two new vertices x, y and for each $3 \leqslant i \leqslant n$ add new edges xw_i and yw_i coloured with colour $c(w_iw_1)+1$ and new edges xw_1 and yw_2 coloured with $\hat{t}+1$. In addition, add new edges yw_1 and xw_2 coloured with $\hat{t}+2$ and the new edge xy coloured with $\hat{t}+3$. □

Theorem 2.13. *For each n, $\hat{t}(K_n) \geqslant (7n-17)/4$.*

P r o o f. We prove first that $\hat{t}(K_{4k}) \geqslant 7k - 3$ for every $k \geqslant 1$. The general bound is then implied by Theorems 2.11 and 2.12.

Let $n = 4k$ for $k \geqslant 1$, let $G = K_n$, and let $V(G) = \{0, 1, \ldots, n-1\}$. We define a colouring of the edges of G in the following manner: The colour of the edge joining vertices $4i + r$ and $4j + s$ with $0 \leqslant i < j \leqslant k$ and $r, s \in \{0, 1, 2, 3\}$ is given in Table 1, the colour of the edge joining vertices $4i + r$ and $4i + s$ with $0 \leqslant i \leqslant k$ and $0 \leqslant r < s \leqslant 3$ is given in Table 2. It is easy to observe that this is an (improper) interval colouring of G.

	$4j$	$4j+1$	$4j+2$	$4j+3$
$4i$	$3i+4j$	$3i+4j+1$	$3i+4j+2$	$3i+4j+3$
$4i+1$	$3i+4j+1$	$3i+4j$	$3i+4j+3$	$3i+4j+2$
$4i+2$	$3i+4j+2$	$3i+4j+3$	$3i+4j+1$	$3i+4j+4$
$4i+3$	$3i+4j+3$	$3i+4j+2$	$3i+4j+4$	$3i+4j+1$

Table 1. The colouring of a complete graph on $n = 4k$ vertices: the colour of an edge joining vertices $4i + r$ and $4j + s$ with $0 \leqslant i < j \leqslant k$ and $r, s \in \{0, 1, 2, 3\}$.

	$4i$	$4i+1$	$4i+2$	$4i+3$
$4i$		$7i+1$	$7i+2$	$7i+3$
$4i+1$			$7i+3$	$7i+2$
$4i+2$				$7i+4$
$4i+3$				

Table 2. The colouring of a complete graph on $n = 4k$ vertices: the colour of an edge joining vertices $4i + r$ and $4i + s$ with $0 \leqslant i \leqslant k$ and $0 \leqslant r < s \leqslant 3$.

□

Note that for proper interval colourings it is not known whether the sequence $\{t(K_n)\}_{n=1}^{\infty}$ is monotone; also, any value of n for which $t(K_{n-1}) > t(K_n)$ yields that $\hat{t}(K_n) \geqslant t(K_n) + 2$.

To conclude this part, we list the exact values of $\hat{t}(K_n)$ for some small values of n, see Table 3.

n	2	3	4	5	6	7	8	9	10	11	12
lower bound from Theorem 2.13			4	5	7	8	11	12	14	15	18
$\hat{t}(K_n)$	1	2	4	5	7	8	11	12	14	16	18
upper bound from Theorem 2.10				5	7	9	11	13	15	17	19

Table 3. Bound and exact values of $\hat{t}(K_n)$ for small values of n.

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