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WEAK- AND STRONG-TYPE INEQUALITY FOR THE CONE-LIKE MAXIMAL OPERATOR IN VARIABLE LEBESGUE SPACES

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Abstract. The classical Hardy-Littlewood maximal operator is bounded not only on the classical Lebesgue spaces $L_p(\mathbb{R}^d)$ (in the case $p > 1$), but (in the case when $1/p(\cdot)$ is log-Hölder continuous and $p_- = \inf\{p(x)\colon x \in \mathbb{R}^d\} > 1$) on the variable Lebesgue spaces $L_{p(\cdot)}(\mathbb{R}^d)$, too. Furthermore, the classical Hardy-Littlewood maximal operator is of weak-type $(1,1)$. In the present note we generalize Besicovitch's covering theorem for the so-called γ-rectangles. We introduce a general maximal operator $M_s^{\gamma,\delta}$ and with the help of generalized Φ-functions, the strong- and weak-type inequalities will be proved for this maximal operator. Namely, if the exponent function $1/p(\cdot)$ is log-Hölder continuous and $p_- > s$, where $1 \leqslant s \leqslant \infty$ is arbitrary (or $p_- \geqslant s$), then the maximal operator $M_s^{\gamma,\delta}$ is bounded on the space $L_{p(\cdot)}(\mathbb{R}^d)$ (or the maximal operator is of weak-type $(p(\cdot), p(\cdot))$).

Keywords: variable Lebesgue space; maximal operator; γ-rectangle; Besicovitch's covering theorem; weak-type inequality; strong-type inequality

MSC 2010: 42B25, 42B35, 52C17

1. Introduction

Maximal operators are playing a central role in approximation theory and in Fourier analysis (see Stein and Weiss [18], Stein [17], Weisz [20], [22]). The classical Hardy-Littlewood maximal operator is defined by

$$Mf(x) := \sup\left\{\frac{1}{|Q|}\int_Q |f|\,\mathrm{d}\lambda\colon\ x \in Q\right\}, \quad x \in \mathbb{R}^d,$$

where f is a locally integrable function and the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ with sides parallel to the axis. It is well known that the classical Hardy-

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Littlewood maximal operator is bounded on the classical L_p spaces for any $p > 1$ and it is of weak type $(1, 1)$, i.e.,

$$\sup_{\tau>0} \|\tau\chi_{\{Mf>\tau\}}\|_1 \leqslant C\|f\|_1, \quad f \in L_1(\mathbb{R}^d).$$

If we take the supremum over rectangles $I = I_1 \times \ldots \times I_d$ with $\delta^{-1} \leqslant |I_i|/|I_j| \leqslant \delta$, $i, j = 1, \ldots, d$, where $\delta \geqslant 1$, then the previous result remains true (see e.g. Weisz [22]). The set $\mathbb{R}^d_\delta := \{x \in \mathbb{R}^d\colon\ \delta^{-1}x_j \leqslant x_i \leqslant \delta x_j,\ i, j = 1, \ldots, d\}$ defines a cone in $\mathbb{R}^d$.

Gát in [12] introduced the following cone-like set. Given the functions γ_i and the numbers $\delta_i \geqslant 1$, the set $\mathbb{R}^d_{\gamma,\delta} := \{x \in \mathbb{R}^d\colon\ \delta_i^{-1}\gamma_i(x_1) \leqslant x_i \leqslant \delta_i\gamma_i(x_1),\ i = 1, \ldots, d\}$ is called a cone-like set. The second author in [21] generalized the Hardy-Littlewood maximal operator for cone-like sets, i.e., he took the supremum over all rectangles $I = I_1 \times \ldots \times I_d$ with $\delta_i^{-1}\gamma_i(|I_1|) \leqslant |I_i| \leqslant \delta_i\gamma_i(|I_1|)$, $i = 1, \ldots, d$. He proved that the maximal operator $M^{\gamma,\delta}$ is bounded on the classical L_p spaces in the case $p > 1$ and it is of weak type $(1, 1)$.

The topic of variable Lebesgue spaces is a new chapter of mathematics and is studied intensively nowadays (see Cruz-Uribe, Diening and Fiorenza [4], Cruz-Uribe, Diening and Hästö [5], Diening et al. [10], Cruz-Uribe, Fiorenza and Neugebauer [9], Almeida and Drihem [1], Kopaliani [13]). The variable $L_{p(\cdot)}$-norm is defined by

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0\colon\ \int_{\mathbb{R}^d} \left|\frac{f(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leqslant 1\right\},$$

where $p(x) < \infty$ for all $x \in \mathbb{R}^d$. Variable $L_{p(\cdot)}$ spaces contain all measurable functions f for which $\|f\|_{p(\cdot)} < \infty$. Variable Lebesgue spaces have a lot of common properties with the classical Lebesgue spaces (see Kováčik and Rákosník [14], Cruz-Uribe and Fiorenza [6], Diening et al. [11], Cruz-Uribe, Fiorenza and Neugebauer [8], Cruz-Uribe et al. [7]). For example if $p_- := \inf\{p(x)\colon\ x \in \mathbb{R}^d\} > 1$, then the classical Hardy-Littlewood maximal operator is bounded on the variable $L_{p(\cdot)}$ spaces and if $p_- \geqslant 1$, then it is of weak type $(p(\cdot), p(\cdot))$ (see Cruz-Uribe and Fiorenza [6], Diening et al. [11]).

In this paper, we will investigate the operator $M^{\gamma,\delta}$ for variable Lebesgue spaces. We will prove that if $p_- > 1$, then the maximal operator $M^{\gamma,\delta}$ is bounded on the variable $L_{p(\cdot)}$ spaces and, in the case $p_- \geqslant 1$, we obtain that it is of weak type $(p(\cdot), p(\cdot))$, namely,

$$\sup_{\tau>0} \|\tau\chi_{\{M^{\gamma,\delta}f>\tau\}}\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

In [19] we investigate the θ-summation of the Fourier transform of functions from the variable Lebesgue spaces over cone-like sets. To this end we need the inequalities with respect to the maximal operator $M^{\gamma,\delta}$ proved in this paper. More exactly, in [19] we estimate pointwise the maximal operator of the θ-means of the Fourier transforms by the maximal operator $M^{\gamma,\delta}$. This implies the almost everywhere convergence of the θ-means of f to the function f from the variable Lebesgue spaces. This result is a generalization of the classical result due to Marcinkiewicz and Zygmund, see [15], concerning the almost everywhere convergence of the Fejér means of two-dimensional Fourier series.

2. The variable Lebesgue spaces

A function $p(\cdot)$ belongs to $\mathcal{P}(\mathbb{R}^d)$ if $p\colon\ \mathbb{R}^d \to [1,\infty]$ and $p(\cdot)$ is measurable. Then we say that $p(\cdot)$ is an exponent function. Let

$$p_- := \inf\{p(x)\colon\ x \in \mathbb{R}^d\} \quad \text{and} \quad p_+ := \sup\{p(x)\colon\ x \in \mathbb{R}^d\}.$$

Set

$$\Omega_\infty := \{x \in \mathbb{R}^d\colon\ p(x) = \infty\}.$$

Let us define the modular

$$\varrho_{\mathrm{KR}}(f) := \int_{\mathbb{R}^d \setminus \Omega_\infty} |f(x)|^{p(x)}\,\mathrm{d}x + \|f\|_{L_\infty(\Omega_\infty)}.$$

We can define the $L_{p(\cdot)}(\mathbb{R}^d)$ space with the help of this modular. A measurable function f belongs to the space $L_{p(\cdot)}(\mathbb{R}^d)$ if there exists $\lambda > 0$ such that $\varrho_{\mathrm{KR}}(f/\lambda) < \infty$. This modular generates a norm

$$\|f\|_{\mathrm{KR}} := \inf\Big\{\lambda > 0\colon\ \varrho_{\mathrm{KR}}\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\}.$$

Equipping the space $L_{p(\cdot)}(\mathbb{R}^d)$ with this norm we get a Banach space. In the case when $p(\cdot) = p$ is a constant, we get back the usual $L_p(\mathbb{R}^d)$ spaces. For some technical reasons we will consider another modular and another norm, but we will get the same space with an equivalent norm.

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and let $\varphi_{p(\cdot)}\colon\ \mathbb{R}^d \times [0,\infty] \to \mathbb{R}$ be the function

$$\varphi_{p(\cdot)}(x,t) := \varphi_{p(x)}(t) := \begin{cases} t^{p(x)} & \text{if } p(x) < \infty,\ t \geqslant 0,\\ 0 & \text{if } \ p(x) = \infty \text{ and } t \in [0,1],\\ \infty & \text{if } p(x) = \infty \text{ and } t > 1,\end{cases} \qquad x \in \mathbb{R}^d.$$

The modular generated by the function $\varphi_{p(\cdot)}$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^d} \varphi_{p(\cdot)}(x, |f(x)|)\,\mathrm{d}x := \int_{\mathbb{R}^d} \varphi_{p(x)}(|f(x)|)\,\mathrm{d}x.$$

A measurable function f belongs to the $L_{p(\cdot)}(\mathbb{R}^d)$ space if there exists $\lambda > 0$ such that $\varrho_{p(\cdot)}(f/\lambda) < \infty$. We can see that the modular $\varrho_{p(\cdot)}$ is not a norm. The $L_{p(\cdot)}(\mathbb{R}^d)$-norm can also be defined by

$$\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0\colon\ \varrho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\}.$$

The norms $\|\cdot\|_{\mathrm{KR}}$ and $\|\cdot\|_{p(\cdot)}$ are equivalent (see Diening et al. [11], pages 72–73).

We say that $r(\cdot)$ is locally log-Hölder continuous if there exists a constant C_0 such that for all $x, y \in \mathbb{R}^d$, $0 < |x-y| < 1/2$,

$$|r(x) - r(y)| \leqslant \frac{C_0}{-\log(|x-y|)},$$

where $|x| = \|x\|_2$, $x \in \mathbb{R}^d$. We denote this set by $LH_0(\mathbb{R}^d)$.

We say that $r(\cdot)$ is log-Hölder continuous at infinity if there exist constants C_∞ and r_∞ such that for all $x \in \mathbb{R}^d$

$$|r(x) - r_\infty| \leqslant \frac{C_\infty}{\log(\mathrm{e} + |x|)}.$$

We write briefly $r(\cdot) \in LH_\infty(\mathbb{R}^d)$. Let

$$LH(\mathbb{R}^d) := LH_0(\mathbb{R}^d) \cap LH_\infty(\mathbb{R}^d).$$

It is easy to see that if $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, then

$$\varrho_{p(\cdot)}(\lambda f) = \varrho_{p(\cdot)}(|\lambda| f) \leqslant |\lambda| \varrho_{p(\cdot)}(f), \quad |\lambda| \leqslant 1 \tag{2.1}$$

for all measurable functions f. The following result can be found in Diening et al. [11], page 83. If $p(\cdot), q(\cdot), r(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $p \leqslant q \leqslant r$ almost everywhere, then

$$L_{q(\cdot)}(\mathbb{R}^d) \hookrightarrow L_{p(\cdot)}(\mathbb{R}^d) + L_{r(\cdot)}(\mathbb{R}^d). \tag{2.2}$$

Moreover, if $g \in L_{q(\cdot)}(\mathbb{R}^d)$, then $\|g\|_{L_{p(\cdot)}(\mathbb{R}^d) + L_{r(\cdot)}(\mathbb{R}^d)} \leqslant 2\|g\|_{q(\cdot)}$.

3. Besicovitch's covering theorem for γ-rectangles

Now let us define the function $\gamma \in \mathbb{R} \to \mathbb{R}^d$. Let $\gamma := (\gamma_1, \ldots, \gamma_d)$, where $\gamma_1(x) := x$, $x > 0$, $\gamma_i\colon (0,\infty) \to (0,\infty)$, γ_i is strictly increasing, continuous and $\gamma_i(1) = 1$, $\lim\limits_{x\to\infty} \gamma_i(x) = \infty$, $\lim\limits_{x\to 0+} \gamma_i(x) = 0$, $i = 1, \ldots, d$. Suppose, that there exist $c_{1,i}$, $c_{2,i}$, $\xi > 1$, for which

$$c_{1,i}\gamma_i(x) \leqslant \gamma_i(\xi x) \leqslant c_{2,i}\gamma_i(x), \quad x > 0,\ i = 1, \ldots, d.$$

Note that, for example, if $\gamma(x) := x^n$ (or $\gamma(x) := \sqrt[n]{x}$) for an arbitrary $1 \leqslant n \in \mathbb{N}$, then the above assumptions are satisfied. We can see easily that

$$c_{1,i}^n\gamma_i(x) \leqslant \gamma_i(\xi^n x) \leqslant c_{2,i}^n\gamma_i(x), \quad x > 0$$

for all $n \in \mathbb{N}$ and

$$c_{2,i}^l\gamma_i(x) \leqslant \gamma_i(\xi^l x) \leqslant c_{1,i}^l\gamma_i(x), \quad x > 0$$

for all $0 > l \in \mathbb{Z}$.

Let $I_i^\gamma \subset \mathbb{R}$, $i = 1, \ldots, d$, be intervals. Denote the Lebesgue measure of I_i^γ by $|I_i^\gamma|$. The set $\mathcal{I}^\gamma$ contains all rectangles $I^\gamma = I_1^\gamma \times \ldots \times I_d^\gamma \subset \mathbb{R}^d$ for which $|I_i^\gamma| = \gamma_i(|I_1^\gamma|)$, $i = 1, \ldots, d$. $I^\gamma \in \mathcal{I}^\gamma$ is called γ-rectangle. The point $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ is the center of the rectangle $I = I_1 \times \ldots \times I_d$, if $I = [(x_1 - a_1, x_1 + a_1)] \times \ldots \times [(x_d - a_d, x_d + a_d)]$, where $a_i > 0$, $i = 1, \ldots, d$. Let us denote by $I_x^\gamma \in \mathcal{I}^\gamma$ a rectangle with center x.

Now we will define the enlargement of the γ-rectangles. Let $\alpha > 0$ and let I be a γ-rectangle which has a center x and its sides are $\gamma_i(a)$, $i = 1, \ldots, d$. Then denote by αI the rectangle which has the same center x but its sides are $\alpha\gamma_i(a)$, $i = 1, \ldots, d$. Now we will prove two simple lemmas.

Lemma 3.1. *Let $1 \leqslant k \in \mathbb{N}$ and $I_{x_j}^\gamma$, $j = 1, \ldots, k$, be γ-rectangles having centers $x_j \in \mathbb{R}^d$ and sides $\gamma_i(a_j)$, $i = 1, \ldots, d$, $j = 1, \ldots, k$. Suppose that*

$$x_j \notin \bigcup_{l=1, l\neq j}^k I_{x_l}^\gamma \quad \text{and} \quad \bigcap_{j=1}^k I_{x_j}^\gamma \neq \emptyset.$$

Then $k \leqslant 2^d$.

P r o o f. Let $x_j := (x_{j,1}, \ldots, x_{j,d})$, $j = 1, \ldots, k$. We can suppose that $x_j \neq 0$, $j = 1, \ldots, k$, and $0 \in \bigcap\limits_{j=1}^k I_{x_j}^\gamma$. Therefore $|x_{j,i}| \leqslant \gamma_i(a_j)/2$, $i = 1, \ldots, d$, $j = 1, \ldots, k$. Let $l, j \in \{1, \ldots, k\}$ be arbitrary and $j \neq l$. Since $x_l \notin I_{x_j}^\gamma$, there exists $i_0 \in \{1, \ldots, d\}$

such that $|x_{l,i_0} - x_{j,i_0}| > \gamma_{i_0}(a_j)/2$. We claim that there exists $i' \in \{1,\dots,d\}$ such that $x_{j,i'}x_{l,i'} < 0$. For contradiction, suppose that $x_{j,i}x_{l,i} \geqslant 0$, $i = 1,\dots,d$. We can suppose that $x_{j,i} \geqslant 0$ and $x_{l,i} \geqslant 0$, $i = 1,\dots,d$. Since $0 \in I^{\gamma}_{x_j}$, we have $x_{l,i_0} < 0$ or $x_{l,i_0} > x_{j,i_0} + \gamma_{i_0}(a_j)/2$. We have supposed that $x_{l,i} \geqslant 0$, $i = 1,\dots,d$, thus we get that $x_{l,i_0} > x_{j,i_0} + \gamma_{i_0}(a_j)/2$.

At the same time $x_{j,i_0} \geqslant 0$ and $0 \in I^{\gamma}_{x_l}$, thus

$$\frac{1}{2}\gamma_{i_0}(a_j) \leqslant \frac{1}{2}\gamma_{i_0}(a_j) + x_{j,i_0} < x_{l,i_0} \leqslant \frac{1}{2}\gamma_{i_0}(a_l) \Rightarrow a_j < a_l \Rightarrow \gamma_i(a_j) < \gamma_i(a_l)$$

for $i = 1,\dots,d$. Using this and the fact that $0 \in I^{\gamma}_{x_j} \cap I^{\gamma}_{x_l}$, we get

$$x_{j,i} \leqslant \frac{1}{2}\gamma_i(a_j) < \frac{1}{2}\gamma_i(a_l), \quad i = 1,\dots,d \quad \Rightarrow \quad x_j \in I^{\gamma}_{x_l},$$

which is a contradiction. Hence $k \leqslant 2^d$. □

Lemma 3.2. *Let $1 \leqslant m \in \mathbb{N}$, $A \subset \mathbb{R}^d$ be a rectangle with sides $a_j > 0$, $j = 1,\dots,d$, and $B_k \subset \mathbb{R}^d$, $k = 1,\dots,m$, be rectangles with sides $b_{k,j} \geqslant a_j$, $j = 1,\dots,d$. If $A \cap B_k \neq \emptyset$, $k = 1,\dots,m$, then there exist rectangle C_k, $k = 1,\dots,m$, with sides $c_{k,j} = a_j$ and $C_k \subset (3A \cap B_k)$.*

P r o o f. Since $A \cap B_k \neq \emptyset$, there are two cases:

1. $B_k \subset 3A$. Then $B_k \cap 3A = B_k$ and due to $a_j \leqslant b_{k,j}$, $j = 1,\dots,d$, we can draw a rectangle C_k in the rectangle B_k with sides $c_{k,j} := a_j$, $j = 1,\dots,d$.
2. $B_k \not\subseteq 3A$. Then take the rectangle $D_k := 3A \cap B_k$ with sides $d_{k,j}$, $j = 1,\dots,d$. $A \cap B_k \neq \emptyset$, therefore

$$d_{k,j} \geqslant \frac{3}{2}a_j - \frac{1}{2}a_j = a_j, \quad j = 1,\dots,d,$$

so we can draw a rectangle C_k in the rectangle D_k with sides $c_{k,j} := a_j$, $j = 1,\dots,d$, which proves the lemma.

□

Besicovitch's covering theorem for cubes is the main point of the proof of the weak-type inequality for the classical Hardy-Littlewood maximal operator in variable $L_{p(\cdot)}(\mathbb{R}^d)$ spaces. Now we will prove Besicovitch's covering theorem for γ-rectangles. The proof of Besicovitch's covering theorem for cubes can be found in [2] and [3] (see also [16]). Our proof is similar.

Theorem 3.1 (Besicovitch's covering theorem for γ-rectangles)**.** *Let $A \subset \mathbb{R}^d$ be a bounded set, $\mathcal{A} := \{I_x^\gamma \in \mathcal{I}^\gamma\colon\ x \in A\}$. Then there exists finite or countable set $\mathcal{B} \subset \mathcal{A}$ such that*

(1) *A can be covered by the rectangles from $\mathcal{B}$, i.e.,*

$$A \subset \bigcup_{I \in \mathcal{B}} I.$$

(2) *There exists a constant $K > 0$ such that*

$$\sum_{I \in \mathcal{B}} \chi_I \leqslant K.$$

(3) *There exist families $\Delta_1, \Delta_2, ..., \Delta_M \subset \mathcal{B}$ such that*

$$A \subset \bigcup_{k=1}^{M} \bigcup_{I \in \Delta_k} I, \quad \text{where } I_{k,i} \cap I_{k,j} = \emptyset,\ i \neq j,\ I_{k,i}, I_{k,j} \in \Delta_k,\ k = 1, \dots, M.$$

Here $M > 0$ is independent of the γ-rectangles.

P r o o f. (1) Let $I_x^\gamma \in \mathcal{A}$ be a γ-rectangle having center x and sides $\gamma_i(a_x)$, $i = 1, \dots, d$ and

$$\Omega := \{a_x > 0\colon\ I_x^\gamma \in \mathcal{A},\ x \in A\}, \quad M_1 := \sup \Omega.$$

Since A is bounded, we can assume that $M_1 < \infty$. Therefore we can choose a γ-rectangle $I_{x_1}^\gamma \in \mathcal{A}$ such that $a_{x_1} \geqslant M_1/2$. Let $I_{x_1}^\gamma \in \mathcal{B}$. Inductively, if

$$x_{j+1} \in A \setminus \bigcup_{i=1}^{j} I_{x_i}^\gamma \quad \text{and} \quad a_{x_{j+1}} \geqslant \frac{1}{2} M_1,$$

then let $I_{x_{j+1}}^\gamma \in \mathcal{B}$. If there is no $x \in A$ such that $x \notin \bigcup_{i=1}^{k_1} I_{x_i}^\gamma$, then we have covered the set A. If there exists $x \in A$ such that $x \notin \bigcup_{i=1}^{k_1} I_{x_i}^\gamma$ but for all $x \in A \setminus \bigcup_{i=1}^{k_1} I_{x_i}^\gamma$, $a_x < M_1/2$, then let

$$M_2 := \sup\left\{ a_x > 0\colon\ x \in A \setminus \bigcup_{i=1}^{k_1} I_{x_i}^\gamma \right\}.$$

We can choose $x_{k_1+1} \in A \setminus \bigcup_{i=1}^{k_1} I^\gamma_{x_i}$ such that $a_{x_{k_1+1}} \geqslant M_2/2$. Let $I^\gamma_{x_{k_1+1}} \in \mathcal{B}$. Inductively again, if

$$x_{j+1} \in A \setminus \bigcup_{i=1}^{j} I^\gamma_{x_i} \qquad \text{and} \qquad a_{x_{j+1}} \geqslant \frac{1}{2} M_2,$$

then let $I^\gamma_{x_{j+1}} \in \mathcal{B}$. Continuing this process we get a strictly increasing sequence (k_n), a strictly decreasing sequence of positive numbers (M_n) with $2M_{n+1} \leqslant M_n$ and a countable collection of γ-rectangles $\mathcal{B}$. Let

$$\begin{aligned} &\Gamma_1 := \{1, 2, \ldots, k_1\}, \quad \Gamma_2 := \{k_1+1, k_1+2, \ldots, k_2\}, \\ &\Gamma_j := \{k_{j-1}+1, k_{j-1}+2, \ldots, k_j\}, \ \ldots \end{aligned}$$

Then the following properties hold:

(a) $M_j/2 \leqslant a_{x_i} \leqslant M_j$, $i \in \Gamma_j$, $1 \leqslant j \in \mathbb{N}$,

(b) $x_{j+1} \notin \bigcup_{i=1}^{j} I^\gamma_{x_i}$, $1 \leqslant j \in \mathbb{N}$,

(c) $x_i \in A \setminus \bigcup_{m \neq k} \bigcup_{j \in \Gamma_m} I^\gamma_{x_j}$, $i \in \Gamma_k$.

The statements (a) and (b) follow from the construction. Let us prove (c). Suppose that $m \neq k$, $j \in \Gamma_m$, $i \in \Gamma_k$. If $m < k$, then for all $\alpha \in \Gamma_m$, $\alpha < \min \Gamma_k$, thus $j < i$ and $x_i \notin I^\gamma_{x_j}$. If $k < m$, then $i < j$ and by the construction $a_{x_i} > a_{x_j}$ and $x_j \notin I^\gamma_{x_i}$, i.e., there exists $i_0 \in \{1, \ldots, d\}$ such that $|x_{j,i_0} - x_{i,i_0}| > \gamma_{i_0}(a_{x_i})/2 > \gamma_{i_0}(a_{x_j})/2$. We obtain that $x_i \notin I^\gamma_{x_j}$.

Due to $\lim_{n\to\infty} M_n = 0$ and to the construction, we have

$$A \subset \bigcup_{i=1}^{\infty} I^\gamma_{x_i} =: \bigcup_{I \in \mathcal{B}} I,$$

which proves statement (1).

Let us consider the statement (2). Suppose that

$$x \in \bigcap_{i=1}^{p} I^\gamma_{x_{m_i}}.$$

We will show that $p \leqslant K$ for a suitable $K > 0$. Let us define the set

$$B := \{1 \leqslant j \in \mathbb{N} \colon \ \Gamma_j \cap \{m_i \colon \ i = 1, \ldots, p\} \neq \emptyset\}.$$

Suppose that $j, l \in B$, $j \neq l$, $k_j \in \Gamma_j$, $k_l \in \Gamma_l$. Then by proposition (c) $x_{k_j} \notin I^\gamma_{x_{k_l}}$ and $x_{k_l} \notin I^\gamma_{x_{k_j}}$. At the same time, since $B \subset \{m_i \colon \ i = 1, \ldots, p\}$, we obtain $x \in I^\gamma_{x_\alpha}$, $\alpha \in B$, therefore by Lemma 3.1, $|B| \leqslant 2^d$.

Fix $1 \leqslant l \in \mathbb{N}$ and let us consider the set

$$C_l := \Gamma_l \cap \{m_i\colon\ i = 1, \ldots, p\}.$$

Since Γ_l is finite, we can suppose that $C_l = \{l_1, l_2, \ldots, l_q\}$. Then the γ-rectangles determined by the set C_l are $I^\gamma_{x_{l_k}}$, $k = 1, \ldots, q$, having center x_{l_k} and sides $\gamma_i(a_{x_{l_k}})$, $i = 1, \ldots, d$, $k = 1, \ldots, q$. Let $1 \leqslant s \in \mathbb{N}$ such that $\xi^{s-1} < 2 \leqslant \xi^s$ and $c := \max\{c_{2,i}\colon\ i = 1, \ldots, d\}$, $\beta := 1/(1 + c^s)$ $(< 1/2)$. The rectangles enlarged by this β have the property that

$$\beta I^\gamma_{x_{l_k}} \cap \beta I^\gamma_{x_{l_j}} = \emptyset, \quad k \neq j = 1, \ldots, q.$$

Indeed, we can suppose that $l_k < l_j$. Then by case (b) $x_{l_j} \notin I^\gamma_{x_{l_k}}$. Therefore there exists $i_0 \in \{1, \ldots, d\}$: $|x_{l_k,i_0} - x_{l_j,i_0}| > \gamma_{i_0}(a_{x_{l_k}})/2$. Since $l_k, l_j \in \Gamma_l$, we have $M_l/2 \leqslant a_{x_{l_k}}, a_{x_{l_j}} \leqslant M_l$, thus $a_{x_{l_j}} \leqslant 2a_{x_{l_k}}$. If there exists $z \in \beta I^\gamma_{x_{l_k}} \cap \beta I^\gamma_{x_{l_j}}$, then

$$\begin{aligned}
\frac{1}{2}\gamma_{i_0}(a_{x_{l_k}}) &< |x_{l_k,i_0} - x_{l_j,i_0}| \leqslant |x_{l_k,i_0} - z_{i_0}| + |z_{i_0} - x_{l_j,i_0}| \\
&\leqslant \frac{\beta}{2}\gamma_{i_0}(a_{x_{l_k}}) + \frac{\beta}{2}\gamma_{i_0}(a_{x_{l_j}}) = \frac{1}{1+c^s}\Big(\frac{1}{2}\gamma_{i_0}(a_{x_{l_k}}) + \frac{1}{2}\gamma_{i_0}(a_{x_{l_j}})\Big).
\end{aligned}$$

Here $\gamma_{i_0}(a_{x_{l_j}}) \leqslant \gamma_{i_0}(2a_{x_{l_k}}) \leqslant \gamma_{i_0}(\xi^s a_{x_{l_k}}) \leqslant c^s_{2,i_0}\gamma_{i_0}(a_{x_{l_k}}) \leqslant c^s\gamma_{i_0}(a_{x_{l_k}})$, therefore

$$\begin{aligned}
\frac{1}{1+c^s}\Big(\frac{1}{2}\gamma_{i_0}(a_{x_{l_k}}) + \frac{1}{2}\gamma_{i_0}(a_{x_{l_j}})\Big) &\leqslant \frac{1}{1+c^s}\Big(\frac{1}{2}\gamma_{i_0}(a_{x_{l_k}}) + \frac{c^s}{2}\gamma_{i_0}(a_{x_{l_k}})\Big) \\
&\leqslant \frac{1}{2}\gamma_{i_0}(a_{x_{l_k}}),
\end{aligned}$$

i.e., $\gamma_{i_0}(a_{x_{l_k}}) < \gamma_{i_0}(a_{x_{l_k}})$, which is a contradiction, so $\beta I^\gamma_{x_{l_k}} \cap \beta I^\gamma_{x_{l_j}} = \emptyset$.

Let $a := \max\{a_{x_{l_k}}\colon\ k = 1, \ldots, q\}$ and let us define the rectangle I_x having center x and sides $2\gamma_i(a)$, $i = 1, \ldots, d$. Then $2\gamma_i(a) \geqslant 2\gamma_i(a_{x_{l_k}})$, $i = 1, \ldots, d$, $k = 1, \ldots, q$. We claim that

$$\bigcup_{k=1}^{q} \beta I^\gamma_{x_{l_k}} \subset I_x.$$

Indeed, suppose that $z \in \beta I^\gamma_{x_{l_k}}$ for a suitable $k \in \{1, \ldots, q\}$, i.e., $|z_i - x_{l_k,i}| \leqslant \beta\gamma_i(a_{x_{l_k}})/2$, $i = 1, \ldots, d$. Since $l_k \in C_l \subset \{m_i\colon\ i = 1, \ldots, p\}$, thus $x \in I^\gamma_{x_{l_k}}$. Due to $\beta < 1/2$ we get

$$\begin{aligned}
|z_i - x_i| \leqslant |z_i - x_{l_k,i}| + |x_{l_k,i} - x_i| &\leqslant \frac{\beta}{2}\gamma_i(a_{x_{l_k}}) + \frac{1}{2}\gamma_i(a_{x_{l_k}}) \\
&< \gamma_i(a_{x_{l_k}}) \leqslant \frac{1}{2}2\gamma_i(a), \quad i = 1, \ldots, d,
\end{aligned}$$

i.e., $z \in I_x$.

Since the rectangles $\beta I^{\gamma}_{x_{l_k}}$, $k = 1, \dots, q$, are pairwise disjoint and the rectangle I_x covers these rectangles, we obtain

$$\sum_{k=1}^{q} |\beta I^{\gamma}_{x_{l_k}}| \leqslant |I_x| = \prod_{i=1}^{d} 2\gamma_i(a) = 2^d \prod_{i=1}^{d} \gamma_i(a).$$

Let $0 > r \in \mathbb{Z}$ such that $\xi^r \leqslant 1/2 < \xi^{r+1}$, $c := \max\{c_{2,i}\colon\ i = 1, \dots, d\}$. Then $c_{2,i}^r \geqslant c^r$, $i = 1, \dots, d$, and by $a_{x_{l_k}} \geqslant M_l/2$

$$\begin{aligned}\sum_{k=1}^{q} |\beta I^{\gamma}_{x_{l_k}}| &= \sum_{k=1}^{q} \prod_{i=1}^{d} \beta\gamma_i(a_{x_{l_k}}) \geqslant \beta^d \sum_{k=1}^{q} \prod_{i=1}^{d} \gamma_i\Big(\frac{1}{2}M_l\Big) \geqslant q\beta^d \prod_{i=1}^{d} \gamma_i(\xi^r M_l) \\ &\geqslant q\beta^d \prod_{i=1}^{d} c^r \gamma_i(M_l) = q(\beta c^r)^d \prod_{i=1}^{d} \gamma_i(M_l) = q\Big(\frac{c^r}{1+c^s}\Big)^d \prod_{i=1}^{d} \gamma_i(M_l).\end{aligned}$$

At the same time, since $a = \max\{a_{x_{l_k}}\colon\ k = 1, \dots, q\} \leqslant M_l$ we get

$$2^d \prod_{i=1}^{d} \gamma_i(a) \leqslant 2^d \prod_{i=1}^{d} \gamma_i(M_l),$$

namely,

$$q\Big(\frac{c^r}{1+c^s}\Big)^d \prod_{i=1}^{d} \gamma_i(M_l) \leqslant 2^d \prod_{i=1}^{d} \gamma_i(M_l) \Leftrightarrow q \leqslant \Big(\frac{2(1+c^s)}{c^r}\Big)^d.$$

Here the constants c, s and r are independent of the rectangles, they only depend on γ. We obtain that

$$p \leqslant |B|q \leqslant \Big(\frac{4(1+c^s)}{c^r}\Big)^d \leqslant \Big\lfloor \Big(\frac{4(1+c^s)}{c^r}\Big)^d \Big\rfloor + 1 =: K,$$

thus (2) is proved.

Finally let us consider (3). For simplicity, denote $I_i := I^{\gamma}_{x_i}$, $a_i := a_{x_i}$, $1 \leqslant i \in \mathbb{N}$, and let the chosen rectangles be $\mathcal{B} := \{I_i\colon\ 1 \leqslant i \in \mathbb{N}\}$ with $A \subset \bigcup_{I \in \mathcal{B}} I$. For any $\varepsilon > 0$ there are only finitely many rectangles I_i with $a_i \geqslant \varepsilon$. Suppose that $I_1, \dots, I_N$ are rectangles such that $a_1 \geqslant \ldots \geqslant a_N \geqslant \varepsilon$ for a suitable $1 \leqslant N \in \mathbb{N}$. Let $I_{1,1} := I_1$ and $I_{1,1} \in \Delta_1$. If there exists a rectangle I_i such that $I_i \cap I_{1,1} = \emptyset$, then let $k_{1,2} := \min\{i \in \{1, \dots, N\}\colon\ I_i \cap I_{1,1} = \emptyset\}$. Choose this rectangle and let $I_{1,2} := I_{k_{1,2}}$ and $I_{1,2} \in \Delta_1$. Inductively, suppose that we have chosen the rectangles $I_{1,1}, \dots, I_{1,j}$ and collected them the set Δ_1. If there exists a rectangle I_i such that $I_i \cap \Big(\bigcup_{l=1}^{j} I_{1,l}\Big) = \emptyset$, then let $k_{1,j+1} := \min\Big\{i \in \{1, \dots, N\}\colon\ I_i \cap \Big(\bigcup_{l=1}^{j} I_{1,l}\Big) = \emptyset\Big\}$ and let $I_{1,j+1} := I_{k_{1,j+1}}$ and $I_{1,j+1} \in \Delta_1$. If for any rectangle I_i, $I_i \cap \Big(\bigcup\limits_{l=1}^{j} I_{1,l}\Big) \neq \emptyset$, $i = 1, \ldots, N$, then let $k_{2,1} := \min\{i \in \{1, \ldots, N\} \colon I_i \notin \Delta_1\}$ and let $I_{2,1} := I_{k_{2,1}}$ and $I_{2,1} \in \Delta_2$. (If the set $\{i \in \{1, \ldots, N\} \colon I_i \notin \Delta_1\}$ is empty, then instead of ε choose $\varepsilon/2$. Then there are only finitely many rectangles I_i with $\varepsilon/2 \leqslant a_i < \varepsilon$.) Continuing this process we obtain families of pairwise disjoint rectangles $\Delta_1, \Delta_2, \ldots$

We claim that there is $M > 0$ such that

$$A \subset \bigcup_{k=1}^{M} \bigcup_{I \in \Delta_k} I, \quad \text{where } M = \Big\lfloor \Big(\frac{12(1+c^s)}{c^r}\Big)^d \Big\rfloor + 1.$$

$\lfloor (12(1+c^s)/c^r)^d \rfloor + 1$ is enough for sure, but it is possible that a lower number is good as well. If M is such that there exists $x \in A \setminus \Big(\bigcup\limits_{k=1}^{M} \bigcup\limits_{I \in \Delta_k} I\Big)$, then $M \leqslant \lfloor (12(1+c^s)/c^r)^d \rfloor$. Since $A \subset \Big(\bigcup\limits_{I \in \mathcal{B}} I\Big)$, there is $I_j \in \mathcal{B}$ such that $x \in I_j$, where the rectangle I_j has center x_j and sides $\gamma_i(a_j)$, $i = 1, \ldots, d$. Then $I_j \notin \Delta_k$ $k = 1, \ldots, M$, otherwise due to $x \in I_j \subset \Big(\bigcup\limits_{I \in \Delta_k} I\Big)$, we get $x \in \Big(\bigcup\limits_{k=1}^{M} \bigcup\limits_{I \in \Delta_k} I\Big)$, which is a contradiction. At the same time for all $k \in \{1, \ldots, M\}$ there exists j_k such that $I_{j_k} \in \Delta_k$ and $I_j \cap I_{j_k} \neq \emptyset$, or else $I_j \in \Delta_k$, which is a contradiction, too. Let the center of I_{j_k} be x_{j_k} with sides $\gamma_i(a_{j_k})$, $i = 1, \ldots, d$, $k = 1, \ldots, M$. Then $a_j \leqslant a_{j_k}$, $k = 1, \ldots, M$, otherwise we would have chosen the rectangle I_j in Δ_k instead of I_{j_k}. By Lemma 3.2, there are rectangles J_k with sides $\gamma_i(a_j)$, $k = 1, \ldots, M$, $i = 1, \ldots, d$, and $J_k \subset (3I_j \cap I_{j_k})$, $k = 1, \ldots, M$. For all $x \in \mathbb{R}^d$: $\sum\limits_{I \in \mathcal{B}} \chi_I(x) \leqslant (4(1+c^s)/c^r)^d =: K$ and due to $J_k \subset I_{j_k} \in \mathcal{B}$ we obtain the same for the rectangles J_k. Therefore $\sum\limits_{k=1}^{M} \chi_{J_k} \leqslant K\, \chi_{\bigcup_{k=1}^{M} J_k}$, i.e., $\chi_{\bigcup_{k=1}^{M} J_k} \geqslant K^{-1} \sum\limits_{k=1}^{M} \chi_{J_k}$. Using this and the fact that $\bigcup\limits_{k=1}^{M} J_k \subset 3I_j$, we obtain

$$\begin{aligned} 3^d |I_j| = |3I_j| \geqslant \Big|\bigcup_{k=1}^{M} J_k\Big| = \int \chi_{\bigcup_{k=1}^{M} J_k} \,\mathrm{d}\lambda \geqslant \frac{1}{K} \sum_{k=1}^{M} \int \chi_{J_k} \,\mathrm{d}\lambda = \\ = \frac{1}{K} \sum_{k=1}^{M} |J_k| = \frac{1}{K} \sum_{k=1}^{M} |I_j| = \frac{1}{K} M |I_j|, \end{aligned}$$

i.e., $M \leqslant 3^d K = (12(1+c^s)/c^r)^d$, which means $M \leqslant \lfloor (12(1+c^s)/c^r)^d \rfloor$ and the proof is complete. $\square$

4. Weak-type inequality for the cone-like maximal operator

Let $\delta := (\delta_1, \dots, \delta_d)$, where $\delta_1 = 1$, $\delta_i \geqslant 1$, $i = 2, \dots, d$, and let us define the set

$$\mathbb{R}^d_{\gamma,\delta} := \{x = (x_1, \dots, x_d) \in \mathbb{R}^d\colon\ \delta_i^{-1}\gamma_i(x_1) \leqslant x_i \leqslant \delta_i\gamma_i(x_1),\ i = 1, \dots, d\}.$$

With the help of this set we can introduce the Hardy-Littlewood maximal operator on cone-like sets. Let $1 \leqslant s < \infty$, $f \in L_s^{\mathrm{loc}}(\mathbb{R}^d)$ and define the maximal operator by

$$M_s^{\gamma,\delta} f(x) := \sup\biggl\{\biggl(\frac{1}{|I|}\int_I |f|^s \,\mathrm{d}\lambda\biggr)^{1/s}\colon\ x \in I,\ (|I_1|, \dots, |I_d|) \in \mathbb{R}^d_{\gamma,\delta}\biggr\}, \quad x \in \mathbb{R}^d.$$

Here $I = I_1 \times \dots \times I_d \subset \mathbb{R}^d$ are rectangles whose sides are parallel to the axes. If $\delta = 1$, then $\mathbb{R}^d_{\gamma,\delta} = \mathrm{graph}(\gamma)$ and the maximal operator on this set is denoted by M_s^γ. If we choose $s = 1$, then we write simply M^γ or $M^{\gamma,\delta}$. It is clear that $M_s^{\gamma,\delta} f = (M^{\gamma,\delta}(|f|^s))^{1/s}$. Weisz proved in [21] that

$$M_s^\gamma f \leqslant M_s^{\gamma,\delta} f \leqslant C M_s^\gamma f.$$

The following lemma plays a central role in the proof of the weak-type and strong-type inequality for the maximal operator $M^{\gamma,\delta}$. An analogous version of this lemma for cubes can be found in Cruz-Uribe and Fiorenza [6], page 95, and in Diening et al. [11], page 99.

Lemma 4.1. *If $p(\cdot)\colon\ \mathbb{R}^d \to [0, \infty)$, $p_+ < \infty$, then the following statements are equivalent:*

(1) *$p(\cdot) \in LH_0(\mathbb{R}^d)$, i.e., there exists a constant $C_0 > 0$ constant such that*

$$|p(x) - p(y)| < \frac{C_0}{-\log(|x - y|)}, \quad x, y \in \mathbb{R}^d,\ 0 < |x - y| < 1/2.$$

(2) *There exists a constant $C > 0$ (which depends on d, γ and $p(\cdot)$ but is independent of the γ-rectangles) such that*

$$|I^\gamma|^{p(x) - p_+(I^\gamma)} \leqslant C \quad \text{and} \quad |I^\gamma|^{p_-(I^\gamma) - p(x)} \leqslant C, \quad x \in I^\gamma$$

for all γ-rectangles I^γ.

P r o o f. We begin the proof with (1) $\Rightarrow$ (2). We will prove the first inequality of (2), the second one is similar. First, suppose that the diagonal of I^γ is $d(I^\gamma) := \Big(\sum\limits_{i=1}^d \gamma_i^2(a)\Big)^{1/2} < 1/2$. Then for such a γ-rectangle I^γ containing x, $|x-y| \leqslant d(I^\gamma) < 1/2$ ($y \in I^\gamma$). Let $f_1(x) := \min\{\gamma_i(x)\colon\ i = 1,\dots,d\}$, $f_2(x) := \max\{\gamma_i(x)\colon\ i = 1,\dots,d\}$, $x \in (0,1)$. Then $f_1(a) \leqslant \gamma_i(a) \leqslant f_2(a)$, $a \in (0,1)$, $i = 1,\dots,d$ and

$$|x-y| \leqslant d(I^\gamma) = \Big(\sum_{i=1}^d \gamma_i^2(a)\Big)^{1/2} \leqslant \sqrt{d}\, f_2(a).$$

We claim that there exists $1 \leqslant k \in \mathbb{N}$ such that $f_2^k(a) \leqslant C f_1(a)$, $a \in (0,1)$, where the constant C is independent of a. Indeed, let $a \in (0,1)$ be arbitrary and $1 \leqslant k_{i,j} \in \mathbb{N}$, $i,j = 1,\dots,d$, $i \neq j$ be exponents such that $c_{1,i}^{k_{i,j}-1} < c_{2,j} \leqslant c_{1,i}^{k_{i,j}}$, $i,j = 1,\dots,d$, $i \neq j$ and $k := \max\{k_{i,j}\colon\ i,j = 1,\dots,d,\ i \neq j\}$, $C := \max\{c_{2,i}\colon\ i = 1,\dots,d\}$. Let $0 > l \in \mathbb{Z}$ be such that $\xi^{l-1} < a \leqslant \xi^l$. Then by $l < 0$ we obtain

$$\begin{aligned}\gamma_i^k(a) &\leqslant \gamma_i^{k_{i,j}}(a) \leqslant \gamma_i^{k_{i,j}}(\xi^l) \leqslant c_{1,i}^{lk_{i,j}}\gamma_i^{k_{i,j}}(1) = \big(c_{1,i}^{k_{i,j}}\big)^l \leqslant c_{2,j}^l \leqslant \gamma_j(\xi^l)\\ &= \gamma_j(\xi\xi^{l-1}) \leqslant c_{2,j}\gamma_j(\xi^{l-1}) \leqslant c_{2,j}\gamma_j(a) \leqslant C\gamma_j(a).\end{aligned}$$

We obtain that for any $i,j = 1,\dots,d$, $i \neq j$: $\gamma_i^k(a) \leqslant C\gamma_j(a)$, i.e., $f_2^k(a) \leqslant C f_1(a)$. Using this we get

$$|x-y| \leqslant \sqrt{d} f_2(a) \leqslant \sqrt{d}\, C^{1/k} f_1^{1/k}(a) \Leftrightarrow f_1(a) \geqslant \Big(\frac{|x-y|}{\sqrt{d}\, C^{1/k}}\Big)^k,$$

and

$$|I^\gamma| = \prod_{i=1}^d \gamma_i(a) \geqslant f_1^d(a) \geqslant \Big(\frac{|x-y|}{\sqrt{d}\, C^{1/k}}\Big)^{kd}.$$

Since $p(x) - p_+(I^\gamma) \leqslant 0$, we get

$$|I^\gamma|^{p(x)-p_+(I^\gamma)} \leqslant \Big(\frac{|x-y|}{\sqrt{d}\, C^{1/k}}\Big)^{kd(p(x)-p_+(I^\gamma))}.$$

In our hypothesis $p(\cdot) \in LH_0(\mathbb{R}^d)$, i.e., $p(\cdot)$ is necessarily continuous. We may assume that I^γ is closed, therefore there exists $y \in I^\gamma$ such that $p_+(I^\gamma) = p(y)$ and

$$p(x) - p_+(I^\gamma) = p(x) - p(y) = -|p(x) - p(y)| > -\frac{C_0}{-\log(|x-y|)}.$$

Since $(|x-y|/(\sqrt{d}\,C^{1/k}))^{kd} < 1$, we obtain

$$
\begin{aligned}
\text{(4.1)}\qquad \Big(\frac{|x-y|}{\sqrt{d}\,C^{1/k}}\Big)^{kd(p(x)-p_+(I^\gamma))} &\leqslant \Big(\frac{|x-y|}{\sqrt{d}\,C^{1/k}}\Big)^{-kdC_0/-\log(|x-y|)} \\
&= \exp\Big(\frac{kdC_0}{\log(|x-y|)}\log\Big(\frac{|x-y|}{\sqrt{d}\,C^{1/k}}\Big)\Big) \\
&= \exp\Big(kdC_0 - kdC_0\log(\sqrt{d}\,C^{1/k})\frac{1}{\log(|x-y|)}\Big).
\end{aligned}
$$

Since $0 < |x-y| < 1/2$, we have $\log(|x-y|) < 0$ and

$$
0 > \frac{1}{\log(|x-y|)} \geqslant \frac{1}{\log\left(\frac{1}{2}\right)}, \quad |x-y| \in \Big(0, \frac{1}{2}\Big).
$$

By $kdC_0\log(\sqrt{d}\,C^{1/k})/\log(|x-y|) < 0$, we can estimate (4.1) by

$$
\exp\Big(kdC_0 - kdC_0\log(\sqrt{d}\,C^{1/k})\frac{1}{\log\left(\frac{1}{2}\right)}\Big) =: C(d, p(\cdot), \gamma),
$$

which proves the claim (1) $\Rightarrow$ (2) in the case $d(I^\gamma) < 1/2$. We can split the case $d(I^\gamma) \geqslant 1/2$ into three cases:

(a) $f_1(a) \geqslant 1/(2\sqrt{d})$,
(b) $f_1(a) < 1/(2\sqrt{d}) \leqslant f_2(a)$,
(c) $f_2(a) < 1/(2\sqrt{d})$.

First, let us consider the case (a). Then $\gamma_i(a) \geqslant 1/(2\sqrt{d})$, $i = 1, \dots, d$, and $|I^\gamma| \geqslant 1/(2\sqrt{d})^d$. By $p(x) - p_+(I^\gamma) \leqslant 0$, we get

$$
|I^\gamma|^{p(x)-p_+(I^\gamma)} \leqslant (2\sqrt{d})^{-d(p(x)-p_+(I^\gamma))} \leqslant (2\sqrt{d})^{d(p_+-p_-)} =: C(d, p(\cdot), \gamma).
$$

Let us consider the case (b). Suppose that $l \in \{1, \dots, d\}$ such that $f_2(a) = \gamma_l(a)$, i.e., the l^{th} side of the γ-rectangle I^γ is the longest side. Then $1/(2\sqrt{d}) \leqslant \gamma_l(a)$, therefore $a \geqslant \gamma_l^{-1}(1/(2\sqrt{d}))$. Then

$$
|I^\gamma| = \prod_{i=1}^{d}\gamma_i(a) \geqslant \prod_{i=1}^{d}\gamma_i\Big(\gamma_l^{-1}\Big(\frac{1}{2\sqrt{d}}\Big)\Big) = \frac{1}{2\sqrt{d}}\prod_{i\neq l}\gamma_i\Big(\gamma_l^{-1}\Big(\frac{1}{2\sqrt{d}}\Big)\Big).
$$

Since $p(x) - p_+(I^\gamma) \leqslant 0$, $2\sqrt{d} > 1$ and $\prod_{i\neq l}\gamma_i(\gamma_l^{-1}(1/(2\sqrt{d}))) < 1$, we obtain

$$
\begin{aligned}
|I^\gamma|^{p(x)-p_+(I^\gamma)} &\leqslant \Big(\frac{2\sqrt{d}}{\prod_{i\neq l}\gamma_i\big(\gamma_l^{-1}(1/(2\sqrt{d}))\big)}\Big)^{p_+(I^\gamma)-p(x)} \\
&\leqslant \Big(\frac{2\sqrt{d}}{\prod_{i\neq l}\gamma_i\big(\gamma_l^{-1}(1/(2\sqrt{d}))\big)}\Big)^{p_+-p_-} \\
&\leqslant \max_{l=1,\dots,d}\Big\{\Big(\frac{2\sqrt{d}}{\prod_{i\neq l}\gamma_i\big(\gamma_l^{-1}(1/(2\sqrt{d}))\big)}\Big)^{p_+-p_-}\Big\} =: C(d, p(\cdot), \gamma).
\end{aligned}
$$

Finally, in the case (c), the diagonal of the γ-rectangle I^γ, $d(I^\gamma) < 1/2$, so we have finished the induction (1) $\Rightarrow$ (2).

The other way can be proved analogously as in Cruz-Uribe and Fiorenza [6], page 96. $\square$

The following results can be found in Diening et al. [11], pages 102–105, for cubes. We can prove them analogously for γ-rectangles by the help of Lemma 4.1. For the sake of completeness, Lemmas 4.2–4.5 are presented here, though they are used only for the proof of Theorem 4.1 (see Diening et al. [11], page 115).

Lemma 4.2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then there exists $\beta \in (0,1)$ such that*

$$\varphi_{p(x)}[\beta(\lambda|I^\gamma|^{-1})^{1/p_-(I^\gamma)}] \leqslant \lambda|I^\gamma|^{-1}, \quad x \in I^\gamma$$

for every $\lambda \in [0,1]$ and γ-rectangle I^γ.

Lemma 4.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH_0(\mathbb{R}^d)$ and let $q\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$\frac{1}{q(x,y)} := \max\Big\{\frac{1}{p(x)} - \frac{1}{p(y)}, 0\Big\}, \quad \frac{1}{q(\cdot,\cdot)} \in LH_0(\mathbb{R}^d \times \mathbb{R}^d).$$

Then for any $\eta \in (0,1)$ there exists $\mu \in (0,1)$ such that

$$\begin{aligned}\varphi_{p(x)}\Big(\mu\frac{1}{|I^\gamma|}\int_{I^\gamma}|f(y)|\,\mathrm{d}y\Big) &\leqslant \frac{1}{|I^\gamma|}\int_{I^\gamma}\varphi_{p(y)}(|f(y)|)\,\mathrm{d}y\\ &\quad + \frac{1}{|I^\gamma|}\int_{I^\gamma}\varphi_{q(x,y)}(\eta)\chi_{\{0<|f(y)|\leqslant 1\}}(y)\,\mathrm{d}y,\end{aligned}$$

for every γ-rectangle I^γ, $x \in I^\gamma$ and $f \in L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)$, $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \leqslant 1$.

Lemma 4.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then for any $m > 0$ there exists $\beta \in (0,1)$ depending only on m, $p(\cdot)$, γ and d but independent of the γ-rectangles, such that*

$$\begin{aligned}\varphi_{p(x)}\Big(\beta\frac{1}{|I^\gamma|}\int_{I^\gamma}|f(y)|\,\mathrm{d}y\Big) &\leqslant \frac{1}{|I^\gamma|}\int_{I^\gamma}\varphi_{p(y)}(|f(y)|)\,\mathrm{d}y\\ &\quad + \frac{1}{2}\Big(\frac{1}{|I^\gamma|}\int_{I^\gamma}\Big[\frac{1}{(\mathrm{e}+|x|)^m} + \frac{1}{(\mathrm{e}+|y|)^m}\Big]\chi_{\{0<|f(y)|\leqslant 1\}}(y)\,\mathrm{d}y\Big)^{p_-}\end{aligned}$$

and

$$\begin{aligned}\varphi_{p(x)}\Big(\beta\frac{1}{|I^\gamma|}\int_{I^\gamma}|f(y)|\,\mathrm{d}y\Big) &\leqslant \frac{1}{|I^\gamma|}\int_{I^\gamma}\varphi_{p(y)}(|f(y)|)\,\mathrm{d}y\\ &\quad + \frac{1}{2}\frac{1}{|I^\gamma|}\int_{I^\gamma}\Big[\frac{1}{(\mathrm{e}+|x|)^m} + \frac{1}{(\mathrm{e}+|y|)^m}\Big]\chi_{\{0<|f(y)|\leqslant 1\}}(y)\,\mathrm{d}y\end{aligned}$$

for every γ-rectangle $I^\gamma \subset \mathbb{R}^d$, all $x \in I^\gamma$ and all $f \in L_{p(\cdot)}(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \leqslant 1$.

If we integrate the second estimate over a γ-rectangle I^γ, then we get the following corollary.

Lemma 4.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then for any $m > 0$ there exists $\beta \in (0,1)$ such that*

$$\int_{I^\gamma} \varphi_{p(x)}\Big(\beta \frac{1}{|I^\gamma|}\int_{I^\gamma} |f(y)|\,\mathrm{d}y\Big)\,\mathrm{d}x \leqslant \int_{I^\gamma} \varphi_{p(y)}(|f(y)|)\,\mathrm{d}y + \int_{I^\gamma} \frac{1}{(\mathrm{e}+|y|)^m}\,\mathrm{d}y,$$
$$\int_{I^\gamma} \varphi_{p(x)}\Big(\beta \frac{1}{|I^\gamma|}\int_{I^\gamma} |f(y)|\,\mathrm{d}y\Big)\,\mathrm{d}x \leqslant \int_{I^\gamma} \varphi_{p(y)}(|f(y)|)\,\mathrm{d}y + |\{y \in I^\gamma\colon\ 0 < |f(y)| \leqslant 1\}|$$

for every γ-rectangle $I^\gamma \subset \mathbb{R}^d$ and all $f \in L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)$, $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \leqslant 1$.

Let $1 \leqslant N \in \mathbb{N}$. A family $\mathcal{H}$ of measurable sets $U \subset \mathbb{R}^d$ is locally N-finite, if

$$\sum_{U\in\mathcal{H}} \chi_U(x) \leqslant N$$

for almost every $x \in \mathbb{R}^d$. Note that a family $\mathcal{H}$ of sets $U \subset \mathbb{R}^d$ is locally 1-finite if and only if the sets $U \in \mathcal{H}$ are pairwise disjoint. Now let us introduce the set of exponent functions

$$\begin{aligned}\mathcal{A}^\gamma := \{p(\cdot) \in \mathcal{P}(\mathbb{R}^d)\colon &\text{ there exists } K > 0 \text{ for all families } \mathcal{I}^\gamma \text{ of pairwise disjoint}\\ &\gamma\text{-rectangles such that for all } f \in L_{p(\cdot)}(\mathbb{R}^d), \|T_{\mathcal{I}^\gamma} f\|_{p(\cdot)} \leqslant K\|f\|_{p(\cdot)}\},\end{aligned}$$

where

$$T_{\mathcal{I}^\gamma} f = \sum_{I^\gamma\in\mathcal{I}^\gamma} \frac{1}{|I^\gamma|}\int_{I^\gamma} |f(y)|\,\mathrm{d}y \cdot \chi_{I^\gamma} =: \sum_{I^\gamma\in\mathcal{I}^\gamma} A_{I^\gamma}(f)\cdot\chi_{I^\gamma}.$$

The following theorem can be found in Diening et al. [11], page 115, for cubes. Using Lemmas 4.2–4.5 we can prove Theorem 4.1 in the same way for γ-rectangles.

Theorem 4.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then $p(\cdot) \in \mathcal{A}^\gamma$ and*

$$\|T_{\mathcal{I}^\gamma} f\|_{p(\cdot)} \leqslant CN\|f\|_{p(\cdot)}$$

for any locally N-finite family $\mathcal{I}^\gamma$ of γ-rectangles and all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

The following theorem states that in the case $p(\cdot) \in \mathcal{A}^\gamma$, the maximal operator $M^{\gamma,\delta}$ is of weak type $(p(\cdot),p(\cdot))$.

Theorem 4.2. *Let $p(\cdot) \in \mathcal{A}^\gamma$. Then*

$$\sup_{\tau>0} \|\tau \chi_{\{M^{\gamma,\delta} f > \tau\}}\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

Proof. Since $M^{\gamma,\delta} f \leqslant CM^\gamma f$, $f \in L_{p(\cdot)}(\mathbb{R}^d)$, it is enough to prove the inequality for the maximal operator M^γ. Let I_x^γ be γ-rectangles with sides $|I_{x,i}^\gamma| = \gamma_i(|I_{x,1}^\gamma|)$, $i = 1, \ldots, d$, and let x be the center of I_x^γ. Let

$$M_c^\gamma f(x) := \sup\left\{\frac{1}{|I_x^\gamma|}\int_{I_x^\gamma} |f(y)| \,\mathrm{d}y \colon\ x \in I_x^\gamma\right\}, \quad x \in \mathbb{R}^d$$

be the centered maximal operator, where the supremum is taken over all γ-rectangles having center x.

Suppose that $x \in I_z^\gamma$, where I_z^γ is a γ-rectangle with center z and sides $\gamma_i(a)$, $i = 1, \ldots, d$. Let $1 \leqslant l \in \mathbb{N}$ be an exponent for which $\xi^{l-1} < 2 \leqslant \xi^l$ and let $a^* := \max\{\gamma_i^{-1}(\xi^l \gamma_i(a)) \colon\ i = 1, \ldots, d\}$. Consider the γ-rectangle $(I_x^\gamma)^*$ having center x and sides $\gamma_i(a^*)$, $i = 1, \ldots, d$. Then $\gamma_i(a^*) \geqslant \xi^l \gamma_i(a) \geqslant 2\gamma_i(a)$, $i = 1, \ldots, d$. We claim that $I_z^\gamma \subset (I_x^\gamma)^*$. Indeed, suppose that $y \in I_z^\gamma$, i.e., $|z_i - y_i| \leqslant \gamma_i(a)/2$, $i = 1, \ldots, d$. By the definition of a^* and due to $x \in I_z^\gamma$

$$|x_i - y_i| \leqslant |x_i - z_i| + |z_i - y_i| \leqslant \frac{1}{2} 2\gamma_i(a) \leqslant \frac{1}{2}\gamma_i(a^*),$$

thus $y \in (I_x^\gamma)^*$. Let $1 \leqslant r_i \in \mathbb{N}$, $i = 1, \ldots, d$ be exponents for which $c_{1,i}^{r_i - 1} < \xi^l \leqslant c_{1,i}^{r_i}$, $r := \max\{r_i \colon\ i = 1, \ldots, d\}$. Then

$$\xi^l \gamma_i(a) \leqslant c_{1,i}^{r_i} \gamma_i(a) \leqslant \gamma_i(\xi^{r_i} a) \Rightarrow \gamma_i^{-1}(\xi^l \gamma_i(a)) \leqslant \xi^{r_i} a \leqslant \xi^r a, \quad i = 1, \ldots, d.$$

We get that $a^* \leqslant \xi^r a$, therefore $\gamma_i(a^*) \leqslant \gamma_i(\xi^r a) \leqslant c_{2,i}^r \gamma_i(a)$, $i = 1, \ldots, d$. Thus

$$\frac{|(I_x^\gamma)^*|}{|I_z^\gamma|} = \frac{\prod_{i=1}^d \gamma_i(a^*)}{\prod_{i=1}^d \gamma_i(a)} \leqslant \frac{\prod_{i=1}^d c_{2,i}^r \gamma_i(a)}{\prod_{i=1}^d \gamma_i(a)} = \prod_{i=1}^d c_{2,i}^r =: C.$$

Here the constant C is independent of the rectangles, it depends only on d and γ. Therefore we get

$$\frac{1}{|I_z^\gamma|}\int_{I_z^\gamma} |f| \,\mathrm{d}\lambda \leqslant \frac{1}{|I_z^\gamma|}\int_{(I_x^\gamma)^*} |f| \,\mathrm{d}\lambda = \frac{|(I_x^\gamma)^*|}{|I_z^\gamma|} \frac{1}{|(I_x^\gamma)^*|}\int_{(I_x^\gamma)^*} |f| \,\mathrm{d}\lambda \leqslant CM_c^\gamma f(x).$$

Taking the supremum over all γ-rectangles containing x, we obtain

$$M_c^\gamma f \leqslant M^\gamma f \leqslant C M_c^\gamma f, \quad f \in L_1^{\mathrm{loc}}(\mathbb{R}^d),$$

so it is enough to prove the theorem for the maximal operator M_c^γ.

Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$, $\|f\|_{p(\cdot)} \leqslant 1$ and let $\tau > 0$ be arbitrary. Denote $\Omega_\tau := \{M_c^\gamma f > \tau\}$. Then Ω_τ is an open set. Let $K \subset \Omega_\tau$, K compact. Then for all $x \in K$, there exists γ-rectangle I_x^γ with center x, such that

$$A_{I_x^\gamma} f := \frac{1}{|I_x^\gamma|} \int_{I_x^\gamma} |f| \,\mathrm{d}\lambda > \tau.$$

Using Theorem 3.1, from the set $\{I_x^\gamma \colon x \in K\}$ we can choose families $\Delta_1, \Delta_2, \ldots, \Delta_M$ such that $K \subset \bigcup_{k=1}^{M} \bigcup_{I \in \Delta_k} I$ and $I_{k,i} \cap I_{k,j} = \emptyset$, $i \neq j$, $I_{k,i}, I_{k,j} \in \Delta_k$, $k = 1, \ldots, M$. Then for almost every $x \in \mathbb{R}^d$

$$\tau \chi_K(x) \leqslant \sum_{k=1}^{M} \sum_{I_x^\gamma \in \Delta_k} \tau \chi_{I_x^\gamma}(x) < \sum_{k=1}^{M} \sum_{I_x^\gamma \in \Delta_k} A_{I_x^\gamma} f \chi_{I_x^\gamma}(x) = \sum_{k=1}^{M} T_{\Delta_k} f(x).$$

That is,

$$\|\tau \chi_K\|_{p(\cdot)} \leqslant \left\| \sum_{k=1}^{M} T_{\Delta_k} f \right\|_{p(\cdot)} \leqslant \sum_{k=1}^{M} \|T_{\Delta_k} f\|_{p(\cdot)}.$$

Since $p(\cdot) \in \mathcal{A}^\gamma$, there exists a constant $C > 0$ for which $\|T_{\Delta_k} f\|_{p(\cdot)} \leqslant C \|f\|_{p(\cdot)}$, i.e.,

$$\|\tau \chi_K\|_{p(\cdot)} \leqslant \sum_{k=1}^{M} C \|f\|_{p(\cdot)} = CM \|f\|_{p(\cdot)}.$$

Let $K_j \subset \Omega_\tau$, K_j compact, $K_j \subset K_{j+1}$, $j \in \mathbb{N}$, such that $\bigcup_{j \in \mathbb{N}} K_j = \Omega_\tau$. Then due to the monotone convergence theorem

$$\|\tau \chi_{\{M_c^\gamma f > \tau\}}\|_{p(\cdot)} = \|\tau \chi_{\Omega_\tau}\|_{p(\cdot)} = \lim_{j \to \infty} \|\tau \chi_{K_j}\|_{p(\cdot)} \leqslant CM \|f\|_{p(\cdot)},$$

which proves the theorem. $\square$

Since $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ and $1/p(\cdot) \in LH(\mathbb{R}^d)$ implies $p(\cdot) \in \mathcal{A}^\gamma$ (see Theorem 4.1), we can formulate the next theorem.

Theorem 4.3. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then*

$$\sup_{\tau>0} \|\tau \chi_{\{M^{\gamma,\delta} f > \tau\}}\|_{p(\cdot)} \leqslant C \|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

We get easily the weak-type inequality for the maximal operator $M_s^{\gamma,\delta}$.

Theorem 4.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. If $p_- \geqslant s$, then*

$$\sup_{\tau>0} \|\tau \chi_{\{M_s^{\gamma,\delta} f > \tau\}}\|_{p(\cdot)} \leqslant C \|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

Proof. First of all, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$, then for any $s > 0$ such that $sp_- \geqslant 1$, we get

$$\||f|^s\|_{p(\cdot)} = \|f\|^s_{sp(\cdot)} \tag{4.2}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$. Indeed,

$$\begin{aligned} \|f\|^s_{sp(\cdot)} &= \Big(\inf\Big\{\lambda > 0\colon\ \varrho_{sp(\cdot)}\Big(\frac{f}{\lambda}\Big) \leqslant 1\Big\}\Big)^s \\ &= \inf\Big\{\lambda^s > 0\colon\ \varrho_{p(\cdot)}\Big(\frac{|f|^s}{\lambda^s}\Big) \leqslant 1\Big\} = \||f|^s\|_{p(\cdot)}. \end{aligned}$$

The more general version of (4.2) can be found in Diening et all [11], page 74. Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ and $\tau > 0$ be arbitrary. Then due to $p_- \geqslant s$, we get that $(p(\cdot)/s)_- \geqslant 1$ and

$$\begin{aligned} \|\tau \chi_{\{M_s^{\gamma,\delta} f > \tau\}}\|_{p(\cdot)} &= \|(\tau^s \chi_{\{M^{\gamma,\delta}(|f|^s) > \tau^s\}})^{1/s}\|_{p(\cdot)} = \|\tau^s \chi_{\{M^{\gamma,\delta}(|f|^s) > \tau^s\}}\|^{1/s}_{p(\cdot)/s} \\ &\leqslant C \||f|^s\|^{1/s}_{p(\cdot)/s} = C\|f\|_{p(\cdot)}, \end{aligned}$$

which proves the theorem. $\square$

5. Strong-type inequality for the cone-like maximal operator

The proof of the next lemma for γ-rectangles is analogous to that of Lemma 4.3.6. in Diening et al. [11], page 110, for cubes.

Lemma 5.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. Then for any $m > 0$ there exists $\beta \in (0,1)$ such that*

$$\varphi_{p(x)}(\beta M^\gamma f(x)) \leqslant M^\gamma(\varphi_{p(\cdot)}(f))(x) + M^\gamma(\mathrm{e} + |\cdot|^{-m})(x), \quad x \in \mathbb{R}^d$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d) + L_\infty(\mathbb{R}^d)$, $\|f\|_{L_{p(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \leqslant 1$.

Now we are ready to prove the strong-type inequality of the maximal operator $M^{\gamma,\delta}$ on the variable $L_{p(\cdot)}(\mathbb{R}^d)$ spaces.

Theorem 5.1. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. If $p_- > 1$, then*

$$\|M^{\gamma,\delta} f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

Proof. It is enough to prove the theorem for the maximal operator M^γ due to $M^{\gamma,\delta} f \leqslant C M^\gamma f$ for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$. Let $q(\cdot) := p(\cdot)/p_-$. Since $1/p(\cdot) \in LH(\mathbb{R}^d)$, thus $1/q(\cdot) = p_-/p(\cdot) \in LH(\mathbb{R}^d)$. It is true that $q_- = (p(\cdot)/p_-)_- = 1$. Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ be arbitrary with $\|f\|_{p(\cdot)} \leqslant 1/2$. We can see easily that $\varphi_{q(x)}(rt) \leqslant r\varphi_{q(x)}(t)$ for all $t > 0$ and $r \in [0,1]$. Since $q(\cdot) \leqslant p(\cdot) \leqslant \infty$, we get (see (2.2)):

$$f \in L_{q(\cdot)(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \quad \text{and} \quad \|f\|_{L_{q(\cdot)}(\mathbb{R}^d)+L_\infty(\mathbb{R}^d)} \leqslant 2\|f\|_{p(\cdot)} \leqslant 1.$$

Consequently, we can apply Lemma 5.1 to obtain

$$\varphi_{q(x)}\Big(\frac{\beta}{2} M^\gamma f(x)\Big) \leqslant \frac{1}{2}\varphi_{q(x)}(\beta M^\gamma f(x)) \leqslant \frac{1}{2} M^\gamma(\varphi_{q(\cdot)}(f))(x) + \frac{1}{2}h(x), \quad x \in \mathbb{R}^d,$$

where $h(x) := M^\gamma\left((\mathrm{e} + |\cdot|^{-m})\right)(x)$. Let $m > d$. It is clear that $\varphi_{p(x)}(t) = (\varphi_{q(x)}(t))^{p_-}$, $t \geqslant 0$, $x \in \mathbb{R}^d$, thus by Jensen's inequality

$$\begin{aligned}\varphi_{p(x)}\Big(\frac{\beta}{2} M^\gamma f(x)\Big) &\leqslant \Big(\frac{1}{2} M^\gamma(\varphi_{q(\cdot)}(f))(x) + \frac{1}{2}h(x)\Big)^{p_-} \\ &\leqslant \frac{1}{2}[M^\gamma(\varphi_{q(\cdot)}(f))(x)]^{p_-} + \frac{1}{2}(h(x))^{p_-}.\end{aligned}$$

If we integrate both sides of this inequality over $\mathbb{R}^d$, we get

$$\begin{aligned}\varrho_{p(\cdot)}\Big(\frac{\beta}{2}M^\gamma f\Big) &= \int_{\mathbb{R}^d}\varphi_{p(x)}\Big(\frac{\beta}{2}M^\gamma f(x)\Big)\,\mathrm{d}x\\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^d}[M^\gamma(\varphi_{q(\cdot)}(f))(x)]^{p_-}\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d}(h(x))^{p_-}\,\mathrm{d}x\\ &= \frac{1}{2}\|M^\gamma(\varphi_{q(\cdot)}(f))\|_{p_-}^{p_-} + \frac{1}{2}\|h\|_{p_-}^{p_-}.\end{aligned}$$

If $\|f\|_{p(\cdot)} \leqslant 1$, then $\varrho_{p(\cdot)}(f) \leqslant 1$, and therefore

$$\|\varphi_{q(\cdot)}(f)\|_{p_-}^{p_-} = \int_{\mathbb{R}^d}\varphi_{q(x)}(|f(x)|)^{p_-}\,\mathrm{d}x = \int_{\mathbb{R}^d}\varphi_{p(x)}(|f(x)|)\,\mathrm{d}x = \varrho_{p(\cdot)}(f) \leqslant 1.$$

Since $p_- > 1$, the maximal operator M^γ is bounded on the space $L_{p_-}(\mathbb{R}^d)$, i.e.,

$$\|M^\gamma(\varphi_{q(\cdot)}(f))\|_{p_-} \leqslant C_1\|\varphi_{q(\cdot)}(f)\|_{p_-} \leqslant C_1.$$

At the same time since, $mp_- > d$, we have $(\mathrm{e}+|\cdot|)^{-mp_-} \in L_1(\mathbb{R}^d)$, i.e., $(\mathrm{e}+|\cdot|)^{-m} \in L_{p_-}(\mathbb{R}^d)$, thus

$$\|h\|_{p_-}^{p_-} = \|M^\gamma((\mathrm{e}+|\cdot|)^{-m})\|_{p_-}^{p_-} \leqslant C_2\|(\mathrm{e}+|\cdot|)^{-m}\|_{p_-}^{p_-} = C_3 < \infty.$$

We see that there exists a constant C (we can assume that $C > 1$) such that $\varrho_{p(\cdot)}(\beta/2M^\gamma f) \leqslant C$, so by inequality (2.1)

$$\varrho_{p(\cdot)}\Big(\frac{\beta}{2C}M^\gamma f\Big) \leqslant \frac{1}{C}\varrho_{p(\cdot)}\Big(\frac{\beta}{2}M^\gamma f\Big) \leqslant 1 \Rightarrow \|M^\gamma f\|_{p(\cdot)} \leqslant \frac{2C}{\beta}.$$

Consequently, $\|M^\gamma f\|_{p(\cdot)} \leqslant K$ for $\|f\|_{p(\cdot)} \leqslant 1/2$. The proof is completed by the scaling argument. $\square$

Using the fact that $M_s^{\gamma,\delta}f = (M^{\gamma,\delta}(|f|^s))^{1/s}$, we get the following theorem.

Theorem 5.2. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, $1/p(\cdot) \in LH(\mathbb{R}^d)$. If $p_- > s$, then*

$$\|M_s^{\gamma,\delta}f\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

for all $f \in L_{p(\cdot)}(\mathbb{R}^d)$.

P r o o f. It is enough to prove the theorem only for the maximal operator M_s^γ. Let $f \in L_{p(\cdot)}(\mathbb{R}^d)$ be arbitrary. Then due to $p_- > s$, $(p(\cdot)/s)_- > 1$. Using (4.2) we get

$$\|M_s^\gamma f\|_{p(\cdot)} = \|M^\gamma(|f|^s)^{1/s}\|_{p(\cdot)} = \|M^\gamma(|f|^s)\|_{p(\cdot)/s}^{1/s} \leqslant C\||f|^s\|_{p(\cdot)/s}^{1/s} = C\|f\|_{p(\cdot)},$$

which proves the theorem. $\square$

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