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STRONG LAWS OF LARGE NUMBERS FOR SEQUENCES OF BLOCKWISE AND PAIRWISE m-DEPENDENT RANDOM VARIABLES IN METRIS SPACES

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Abstract. The aim of the paper is to establish strong laws of large numbers for sequences of blockwise and pairwise m-dependent random variables in a convex combination space with or without compactly uniformly integrable condition. Some of our results are even new in the case of real random variables.

Keywords: strong law of large numbers; convex combination space; pairwise m-dependent; blockwise m-dependent; compactly uniformly integrable

MSC 2010: 60F15, 60B05, 60G50

1. Introduction

Móricz [5] extended the classical strong law of large numbers (SLLN) of Kolmogorov to the blockwise m-dependence case. Gaposhkin [4] considered arbitrary blocks and derived SLLN for blockwise independent random variables. Bo [1] studied the Marcinkiewicz-Zygmund law for sequences of blockwise m-dependent random variables with respect to arbitrary blocks. Thanh [8] extended the result of Choi and Sung [2] to the blockwise and pairwise m-dependence case and the result of Etemadi [3] to the pairwise m-dependence case with respect to the blocks $[2^k, 2^{k+1})$, $k \in \mathbb{N}_0$.

Terán and Molchanov [7] introduced the concept of a convex combination space, which is a metric space endowed with a convex combination operation. The class of these metric spaces is not only larger than the class of Banach spaces but also larger than the class of hyperspaces of compact subsets, as well as the class of upper

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semicontinuous functions (fuzzy sets) with compact support in a Banach space. Some very basic sets, such as singletons and balls, may fail to be convex in the convex combination space. Terán and Molchanov [7] also stated various basic properties of the convex combination operation and used these properties to derive the SLLN for pairwise i.i.d. random variables, which extended Theorem 1 of Etemadi [3]. Some more limit theorems for random variables taking values in a convex combination space can be found in Quang and Thuan [6], Thuan et al. [9].

Continuing in this direction, in this study we establish some results on SLLN for sequences of blockwise and pairwise m-dependent random variables in a convex combination space with respect to the blocks $[2^k, 2^{k+1})$ and with or without compactly uniformly integrable condition. Our results are more general than some previously reported ones. The paper is organized as follows. In Section 2, we state and summarize basic results in a convex combination space $\mathfrak{X}$ and some related concepts. In Section 3, some results on SLLN for sequences of blockwise and pairwise m-dependent, blockwise m-dependent, pairwise m-dependent independent identically distributed random variables are established.

2. Preliminaries

Throughout this paper, $(\Omega, \mathcal{A}, P)$ is a complete probability space. For $A \in \mathcal{A}$, the notation $I\{A\}$ (or I_A) is the indicator function of A. At first, we present a short introduction to the approach given by Terán and Molchanov [7]. Let $(\mathfrak{X}, d)$ be a metric space. Based on $\mathfrak{X}$, we introduce a *convex combination operation* which for all $n \geqslant 2$, numbers $\lambda_1, \ldots, \lambda_n > 0$ satisfying $\sum\limits_{i=1}^{n} \lambda_i = 1$, and all $u_1, \ldots, u_n \in \mathfrak{X}$, produces an element of $\mathfrak{X}$, which is denoted by $[\lambda_i, u_i]_{i=1}^n$ or $[\lambda_1, u_1; \ldots; \lambda_n, u_n]$. Assume that $[1, u] = u$ for every $u \in \mathfrak{X}$ and that the following axioms are satisfied:

(i) (Commutativity)

$$[\lambda_i, u_i]_{i=1}^n = [\lambda_{\sigma(i)}, u_{\sigma(i)}]_{i=1}^n \text{ for every permutation } \sigma \text{ of } \{1, \ldots, n\}.$$

(ii) (Associativity)

$$[\lambda_i, u_i]_{i=1}^{n+2} = \Big[\lambda_1, u_1; \ldots; \lambda_n, u_n; \lambda_{n+1} + \lambda_{n+2}, \Big[\frac{\lambda_{n+j}}{\lambda_{n+1} + \lambda_{n+2}}, u_{n+j}\Big]_{j=1}^2\Big].$$

(iii) (Continuity) If $u, v \in \mathfrak{X}$ and $\lambda^{(k)} \to \lambda \in (0,1)$ as $k \to \infty$, then

$$[\lambda^{(k)}, u; 1 - \lambda^{(k)}, v] \to [\lambda, u; 1 - \lambda, v].$$

(iv) (Negative curvature) If $u_1, u_2, v_1, v_2 \in \mathfrak{X}$ and $\lambda \in (0,1)$, then

$$d([\lambda, u_1; 1-\lambda, u_2], [\lambda, v_1; 1-\lambda, v_2]) \leqslant \lambda d(u_1, v_1) + (1-\lambda) d(u_2, v_2).$$

Based on the inductive method and (ii), this axiom can be extended to convex combinations of n element as follows: if $u_i, v_i \in \mathfrak{X}$, $\lambda_i \in (0;1)$, $\sum\limits_{i=1}^{n} \lambda_i = 1$, then $d([\lambda_i, u_i]_{i=1}^n, [\lambda_i, v_i]_{i=1}^n) \leqslant \sum\limits_{i=1}^{n} \lambda_i d(u_i, v_i)$.

(v) (Convexification) For each $u \in \mathfrak{X}$, there exists $\lim\limits_{n\to\infty} [n^{-1}, u]_{i=1}^n$, which will be denoted by $K_{\mathfrak{X}} u$ (or Ku if no confusion can arise), and K is called the *convexification operator*.

Then the metric space $\mathfrak{X}$ endowed with a convex combination operation is referred to as the *convex combination space.* The above axioms imply the following properties:

(2.1) For every $u_{11}, \ldots, u_{mn} \in \mathfrak{X}$ and $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n > 0$ with $\sum\limits_{i=1}^{m} \alpha_i = \sum\limits_{j=1}^{n} \beta_j = 1$, we have

$$[\alpha_i, [\beta_j, u_{ij}]_{j=1}^n]_{i=1}^m = [\alpha_i \beta_j, u_{ij}]_{i=1,j=1}^{i=m,j=n}.$$

(2.2) The convex combination operation is jointly continuous in its $2n$ arguments.

(2.3) The convexification operator K is linear, i.e. $K([\lambda_j, u_j]_{j=1}^n) = [\lambda_j, Ku_j]_{j=1}^n$.

(2.4) If $u \in \mathfrak{X}$ and $\lambda_1, \ldots, \lambda_n > 0$ with $\sum\limits_{j=1}^{n} \lambda_j = 1$, then $K([\lambda_j, u]_{j=1}^n) = Ku = [\lambda_j, Ku]_{j=1}^n$. Hence, K is an idempotent operator in $\mathfrak{X}$.

(2.5) For every $\lambda_1, \lambda_2, \lambda_3 > 0$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $u, v \in \mathfrak{X}$,

$$[\lambda_1, u; \lambda_2, Kv; \lambda_3, Kv] = [\lambda_1 u; (\lambda_2 + \lambda_3), Kv].$$

(2.6) The mapping K is non-expansive with respect to the metric d, which means that $d(Ku, Kv) \leqslant d(u, v)$.

Let $\{\lambda_k \colon k \geqslant 1\} \subset (0;1), \lambda_k \to 0$ as $k \to \infty$, and $u, v \in \mathfrak{X}$. By (iv) and property (2.4), we have

$$\begin{aligned} d([\lambda_k, Ku; 1-\lambda_k, Kv], Kv) &= d([\lambda_k, Ku; 1-\lambda_k, Kv], [\lambda_k, Kv; 1-\lambda_k, Kv]) \\ &\leqslant \lambda_k d(Ku, Kv) \to 0 \quad \text{as } k \to \infty. \end{aligned}$$

It follows that $[\lambda_k, Ku; 1-\lambda_k, Kv] \to Kv$ as $k \to \infty$, and this remark makes it possible to extend weights λ_i from $(0;1)$ to $[0;1]$ for elements in $K(\mathfrak{X})$, which means that we can define $[\lambda_i, u_i]_{i\in I} = [\lambda_i, u_i]_{i\in J}$ with $J = \{i \in I \colon \lambda_i > 0\}$, where $\lambda_i \in [0,1], u_i \in K(\mathfrak{X})$ and $\sum\limits_{i\in J} \lambda_i = \sum\limits_{i\in I} \lambda_i = 1$.

A mapping $X\colon\ \Omega \to \mathfrak{X}$ is called an $\mathfrak{X}$-*valued random variable* if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}(\mathfrak{X})$, where $\mathcal{B}(\mathfrak{X})$ is the Borel σ-algebra on $\mathfrak{X}$. The *distribution* P_X of an $\mathfrak{X}$-valued random variable X is defined by $P_X(B) = P\{X^{-1}(B)\}$ for all $B \in \mathcal{B}(\mathfrak{X})$, and two $\mathfrak{X}$-valued random variables X, Y are said to be *identically distributed* if $P_X = P_Y$.

The collection of $\mathfrak{X}$-valued random variables $\{X_i\colon\ i \in I\}$ is said to be *independent* (*pairwise independent*) if the collection of σ-algebras $\{\sigma(X_i)\colon\ i \in I\}$ is independent (pairwise independent, respectively), where $\sigma(X) = \{X^{-1}(B)\colon\ B \in \mathcal{B}(\mathfrak{X})\}$.

Let m be a fixed nonnegative integer. We say that a finite collection $\{X_i\colon\ 1 \leqslant i \leqslant n\}$ of $\mathfrak{X}$-valued random variables is *m-dependent* if either $n \leqslant m+1$, or $n > m+1$ and the random variables $\{X_1, \dots, X_i\}$ are independent of the random variables $\{X_j, \dots, X_n\}$ whenever $j - i > m$. A finite collection of $\mathfrak{X}$-valued random variables $\{X_i\colon\ 1 \leqslant i \leqslant n\}$ is said to be *pairwise m-dependent* if either $n \leqslant m+1$, or $n > m+1$ and X_i and X_j are independent whenever $j - i > m$. A sequence $\{X_n\colon\ n \geqslant 1\}$ of $\mathfrak{X}$-valued random variables is said to be *pairwise m-dependent* if X_i and X_j are independent whenever $j - i > m$. A sequence $\{X_n\colon\ n \geqslant 1\}$ of $\mathfrak{X}$-valued random variables is said to be *blockwise m-dependent* (*blockwise and pairwise m-dependent*) if for each $k \in \mathbb{N}_0$, the collection $\{X_n\colon\ 2^k \leqslant n < 2^{k+1}\}$ is m-dependent (pairwise m-dependent, respectively).

From now on, we assume that $(\mathfrak{X}, d)$ is a separable and complete metric space. We fix $u_0 \in K(\mathfrak{X})$ and consider u_0 to be a special element of $\mathfrak{X}$. A random variable $X\colon\ \Omega \to \mathfrak{X}$ is said to be *integrable* if $Ed(u_0, X) < \infty$. Note that this definition does not depend on the selection of the element u_0. The space of all integrable $\mathfrak{X}$-valued random variables will be denoted by $\mathcal{L}^1_{\mathfrak{X}}$, and the metric on $\mathcal{L}^1_{\mathfrak{X}}$ is defined by $\Delta(X, Y) = Ed(X, Y)$.

If X is a simple function that takes a distinct value $x_i \in \mathfrak{X}$ for each non-null set Ω_i, $i = 1, \dots, n$, the *expectation* of X is defined by $EX = [P(\Omega_i), Kx_i]_{i=1}^n$. By continuity of all Borel functions $X \in \mathcal{L}^1_{\mathfrak{X}}$, then for $X \in \mathcal{L}^1_{\mathfrak{X}}$, the *expectation* of X is defined as the limit of the expectations sequence of simple random variables. Note that if $X, Y \in \mathcal{L}^1_{\mathfrak{X}}$ then $d(EX, EY) \leqslant Ed(X, Y)$.

An example for a convex combination space is the space of nonempty compact subsets. Let $\mathfrak{X}$ be a convex combination space and $k(\mathfrak{X})$ the set of nonempty compact subsets of $\mathfrak{X}$. A set $A \subset \mathfrak{X}$ is called *convex* if $[\lambda_i, u_i]_{i=1}^n \in A$ for all $u_i \in A$ and $\sum\limits_{i=1}^n \lambda_i = 1$. We denote as $\operatorname{co} A$ the *convex hull* of $A \subset \mathfrak{X}$, and $\overline{\operatorname{co}}\, A$ is the closed convex hull of A. Let d_H be the Hausdorff metric on $k(\mathfrak{X})$, that is $d_H(A, B) = \max\Bigl\{\sup\limits_{a \in A} \inf\limits_{b \in B} d(a, b), \sup\limits_{b \in B} \inf\limits_{a \in A} d(b, a)\Bigr\}$ for $A, B \in k(\mathfrak{X})$. It follows from Theorem 6.2 [7] that the space $k(\mathfrak{X})$ with the convex combination $[\lambda_i, A_i]_{i=1}^n = \{[\lambda_i, u_i]_{i=1}^n\colon\ u_i \in A_i$ for all $i\}$ and the metric d_H is a separable complete convex combination space. We also have $K_{k(\mathfrak{X})}A = \overline{\mathrm{co}}\, K_{\mathfrak{X}}A = \overline{\mathrm{co}}\{K_{\mathfrak{X}}u\colon\ u \in A\}$. Further examples for convex combination spaces can be found in [7].

Now we introduce the concept of compact uniform integrability in Cesàro sense for a sequence of random variables taking values in a metric space, which is naturally extended from a Banach space to a metric space. A sequence $\{X_n\colon\ n \geqslant 1\}$ of $\mathfrak{X}$-valued ($k(\mathfrak{X})$-valued) random variables is said to be *compactly uniformly integrable in Cesàro sense* (Cesàro CUI for short) if for every $\varepsilon > 0$, there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathfrak{X}$ ($k(\mathfrak{X})$, respectively) such that

$$\sup_{n\geqslant 1} n^{-1} \sum_{i=1}^{n} E(d(u_0, X_i) I\{X_i \notin \mathcal{K}_\varepsilon\}) \leqslant \varepsilon.$$

$$(\text{resp. } \sup_{n\geqslant 1} n^{-1} \sum_{i=1}^{n} E(d_H(\{u_0\}, X_i) I\{X_i \notin \mathcal{K}_\varepsilon\}) \leqslant \varepsilon).$$

By using a method similar to that used in the proof of Proposition 2.1 of [9], it is easy to show that the concept of Cesàro CUI does not depend on the selected element u_0. A sequence $\{X_n\colon\ n \geqslant 1\}$ of real-valued random variables is said to be *stochastically dominated* by a real-valued random variable X if there exists a constant C $(0 < C < \infty)$ such that

$$P(|X_n| > t) \leqslant CP(|X| > t), \quad n \geqslant 1,\ t \geqslant 0.$$

We complete this summary by a lemma which will be used in the proof of Proposition 3.1 in the next section.

Lemma 2.1 ([6], Lemma 3.3). *Let $\{a_i, b_i\colon\ 1 \leqslant i \leqslant n\} \subset [0,1]$ be a collection of nonnegative constants with $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i = 1$. Then*

$$d([a_i, Kx_i]_{i=1}^n, [b_i, Kx_i]_{i=1}^n) \leqslant \sum_{i=1}^{n} |a_i - b_i| d(u_0, x_i),$$

where $x_1, \ldots, x_n \in \mathfrak{X}$ are arbitrary.

3. SLLN for sequences of blockwise and pairwise m-dependent $\mathfrak{X}$-valued random variables

Let u_0 be a fixed element of $K(\mathfrak{X})$, which is mentioned in Section 2. We denote $\|x\|_{u_0} := d(x, u_0)$ for all $x \in \mathfrak{X}$. In the first theorem, we will establish a result on the SLLN for sequences of blockwise m-dependent and Cesàro CUI $\mathfrak{X}$-valued random variables in a convex combination space. To do that, we need the following proposition.

Proposition 3.1. *Let $\mathcal{K}$ be a compact subset of $\mathfrak{X}$. If $\{X_n \colon n \geqslant 1\}$ is a sequence of blockwise and pairwise m-dependent $\mathfrak{X}$-valued random variables satisfying $P(X_n \in \mathcal{K}) = 1$ for all n, then*

$$d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \to 0 \quad \text{a.s. as } n \to \infty. \tag{1}$$

P r o o f. For $\varepsilon > 0$, by the compactness of $\mathcal{K}$, there exists $\{c_1, c_2, \ldots, c_p\} \subset \mathcal{K}$ such that

$$\mathcal{K} \subset \bigcup_{t=1}^{p} B(c_t, \varepsilon), \quad \text{where } B(c_t, \varepsilon) = \{x \in \mathfrak{X} \colon d(x, c_t) < \varepsilon\}.$$

For $n \geqslant 1$, we define the $\mathfrak{X}$-valued random variables as follows:

$$Y_n(\omega) = \begin{cases} c_0 := u_0 & \text{if } X_n(\omega) \notin \mathcal{K}, \\ c_1 & \text{if } X_n(\omega) \in B(c_1, \varepsilon) \cap \mathcal{K}, \\ c_t & \text{if } X_n(\omega) \in B(c_t, \varepsilon) \cap \Big\{ \bigcup\limits_{k=1}^{t-1} B(c_k, \varepsilon) \Big\}^c \cap \mathcal{K}, \ t = 2, \ldots, p. \end{cases}$$

It is obvious that the sequence $\{Y_n \colon n \geqslant 1\}$ is also blockwise and pairwise m-dependent. By the triangular inequality, we have

$$\begin{aligned} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) &\leqslant d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Y_i]_{i=1}^n) \\ &\quad + d([n^{-1}, Y_i]_{i=1}^n, [n^{-1}, KY_i]_{i=1}^n) \\ &\quad + d([n^{-1}, KY_i]_{i=1}^n, [n^{-1}, EY_i]_{i=1}^n) \\ &\quad + d([n^{-1}, EY_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \\ &:= (A_1) + (A_2) + (A_3) + (A_4). \end{aligned}$$

For (A_1), by the construction of Y_n, we have

$$(A_1) = d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Y_i]_{i=1}^n) \leqslant \frac{1}{n} \sum_{i=1}^{n} d(X_i, Y_i) \leqslant \varepsilon \quad \text{a.s.}$$

For (A_2), we put

$$Q_n^t(\omega) = \operatorname{card}\{i\colon\ 1 \leqslant i \leqslant n, Y_i(\omega) = c_t\} = \sum_{i=1}^{n} I\{Y_i = c_t\}(\omega),$$

$$\mathcal{T}_n(\omega) = \{t\colon\ 0 \leqslant t \leqslant p, Q_n^t(\omega) > 0\}, \quad n \geqslant 1.$$

By properties (2.1) and (2.4), we have

$$[n^{-1}, Y_i]_{i=1}^{n} = [n^{-1}Q_n^t, [(Q_n^t)^{-1}, c_t]_{i=1}^{Q_n^t}]_{t\in\mathcal{T}_n}$$

and

$$[n^{-1}, KY_i]_{i=1}^{n} = \big[n^{-1}Q_n^t, [(Q_n^t)^{-1}, Kc_t]_{i=1}^{Q_n^t}\big]_{t\in\mathcal{T}_n} = [n^{-1}Q_n^t, Kc_t]_{t\in\mathcal{T}_n}.$$

Therefore,

$$(A_2) = d([n^{-1}, Y_i]_{i=1}^{n}, [n^{-1}, KY_i]_{i=1}^{n}) \leqslant \sum_{t\in\mathcal{T}_n} \frac{Q_n^t}{n} d([(Q_n^t)^{-1}, c_t]_{i=1}^{Q_n^t}, Kc_t).$$

We will prove that $(A_2) \leqslant \varepsilon$ for all $\omega \in \Omega$ when n is sufficiently large.

Indeed, we consider each $t = 0, 1, \ldots, p$. By the definition of K, we have

$$\lim_{n\to\infty} d([n^{-1}, c_t]_{i=1}^{n}, Kc_t) = 0.$$

Thus, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that

$$d([n^{-1}, c_t]_{i=1}^{n}, Kc_t) < \frac{\varepsilon}{p+1} \quad \text{for all } n \geqslant n_1(\varepsilon) \text{ and for all } t = 0, 1, \ldots, p.$$

We put

$$M_t(\varepsilon) = \max_{1\leqslant k<n_1(\varepsilon)} d([k^{-1}, c_t]_{i=1}^{k}, Kc_t), \quad M(\varepsilon) = \max_{0\leqslant t\leqslant p} M_t(\varepsilon)$$

and choose the smallest integer number $n(\varepsilon)$ such that $n(\varepsilon) \geqslant \varepsilon^{-1}(p+1)M(\varepsilon)n_1(\varepsilon)$.

Now, for all $n \geqslant n(\varepsilon)$:

If $Q_n^t(\omega) \geqslant n_1(\varepsilon)$, then

$$\frac{Q_n^t(\omega)}{n} d([(Q_n^t(\omega))^{-1}, c_t]_{i=1}^{Q_n^t(\omega)}, Kc_t) < \frac{\varepsilon}{p+1} \quad (\text{since } n^{-1}Q_n^t(\omega) \leqslant 1).$$

If $0 < Q_n^t(\omega) < n_1(\varepsilon)$, then

$$\frac{Q_n^t(\omega)}{n} d([(Q_n^t(\omega))^{-1}, c_t]_{i=1}^{Q_n^t(\omega)}, Kc_t) \leqslant \frac{n_1(\varepsilon)}{n(\varepsilon)} M(\varepsilon) \leqslant \frac{\varepsilon}{p+1}.$$

Hence, for $n \geqslant n(\varepsilon)$ and for all $\omega \in \Omega$,

$$\frac{Q_n^t(\omega)}{n} d([(Q_n^t(\omega))^{-1}, c_t]_{i=1}^{Q_n^t(\omega)}, Kc_t) \leqslant \frac{\varepsilon}{p+1}.$$

This implies that

$$(A_2) \leqslant \sum_{t \in \mathcal{T}_n} \frac{Q_n^t}{n} d\big([(Q_n^t)^{-1}, c_t]_{i=1}^{Q_n^t}, Kc_t\big) \leqslant \sum_{t=0}^{p} \frac{\varepsilon}{p+1} = \varepsilon$$

for values of n that are sufficiently large.

For (A_3), we have $KY_i = [I\{Y_i = c_t\}, Kc_t]_{t=0}^p$, $EY_i = [P\{Y_i = c_t\}, Kc_t]_{t=0}^p$. Then, by properties (2.1) and (2.5), and Lemma 2.1

$$\begin{aligned}
(A_3) &= d([n^{-1}, [I\{Y_i = c_t\}, Kc_t]_{t=0}^p]_{i=1}^n, [n^{-1}, [P\{Y_i = c_t\}, Kc_t]_{t=0}^p]_{i=1}^n) \\
&= d([n^{-1} I\{Y_i = c_t\}, Kc_t]_{t=0,i=1}^{t=p,i=n}, [n^{-1} P\{Y_i = c_t\}, Kc_t]_{t=0,i=1}^{t=p,i=n}) \\
&= d\bigg(\bigg[\frac{1}{n}\sum_{i=1}^n I\{Y_i = c_t\}, Kc_t\bigg]_{t=0}^p, \bigg[\frac{1}{n}\sum_{i=1}^n P\{Y_i = c_t\}, Kc_t\bigg]_{t=0}^p\bigg) \\
&\leqslant \sum_{t=0}^p \bigg|\frac{1}{n}\sum_{i=1}^n (I\{Y_i = c_t\} - P\{Y_i = c_t\})\bigg| \|c_t\|_{u_0} \\
&= \sum_{t=1}^p \bigg|\frac{1}{n}\sum_{i=1}^n (I\{Y_i = c_t\} - P\{Y_i = c_t\})\bigg| \|c_t\|_{u_0}.
\end{aligned}$$

Using the fact that $\{I\{Y_i = c_t\} - P\{Y_i = c_t\} \colon\ n \geqslant 1\}$ is the sequence of blockwise and pairwise m-dependent real-valued random variables that is stochastically dominated by random variable $X = 2$ and applying Theorem 1 of [8], we have

$$\frac{1}{n}\sum_{i=1}^n (I\{Y_i = c_t\} - P\{Y_i = c_t\}) \to 0 \quad \text{a.s. as } n \to \infty.$$

Thus $(A_3) \to 0$ a.s. as $n \to \infty$.

For (A_4), by the definition of Y_n, we have

$$\begin{aligned}
(A_4) &= d([n^{-1}, EY_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \\
&\leqslant \frac{1}{n}\sum_{i=1}^n d(EY_i, EX_i) \leqslant \frac{1}{n}\sum_{i=1}^n Ed(Y_i, X_i) < \varepsilon.
\end{aligned}$$

Combining the above parts, we obtain

$$\limsup_{n\to\infty} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \leqslant 3\varepsilon \quad \text{a.s.}$$

By the arbitrariness of $\varepsilon > 0$, we derive conclusion (1). □

Applying the above proposition, we obtain the SLLN for sequences of blockwise m-dependent and Cesàro CUI $\mathfrak{X}$-valued random variables.

Theorem 3.2. *Let $\{X_n\colon n \geqslant 1\}$ be a sequence of blockwise m-dependent $\mathfrak{X}$-valued random variables which is Cesàro CUI. If*

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|_{u_0}^2}{n^2} < \infty, \tag{2}$$

then (1) *holds.*

Proof. For $\varepsilon > 0$ arbitrarily small, by Cesàro CUI hypothesis there exists a compact subset $\mathcal{K}_\varepsilon$ of $\mathfrak{X}$ such that

$$\frac{1}{n}\sum_{i=1}^{n} E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\}) \leqslant \varepsilon \quad \text{for all } n.$$

For $n \geqslant 1$, we define the $\mathfrak{X}$-valued random variables as follows:

$$Z_n(\omega) = \begin{cases} X_n(\omega) & \text{if } X_n(\omega) \in \mathcal{K}_\varepsilon, \\ u_0 & \text{if } X_n(\omega) \notin \mathcal{K}_\varepsilon. \end{cases}$$

By the triangular inequality, we have

$$\begin{aligned} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) &\leqslant d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Z_i]_{i=1}^n) \\ &\quad + d([n^{-1}, Z_i]_{i=1}^n, [n^{-1}, EZ_i]_{i=1}^n) \\ &\quad + d([n^{-1}, EZ_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \\ &:= (B_1) + (B_2) + (B_3). \end{aligned}$$

For (B_1), we have

$$\begin{aligned} &d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Z_i]_{i=1}^n) \\ &\qquad \leqslant \frac{1}{n}\sum_{i=1}^{n} d(X_i, Z_i) = \frac{1}{n}\sum_{i=1}^{n} \|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\} \\ &\qquad = \frac{1}{n}\sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\})) \\ &\qquad\quad + \frac{1}{n}\sum_{i=1}^{n} E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\}) \\ &\qquad \leqslant \frac{1}{n}\sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\})) + \varepsilon. \end{aligned}$$

It is clear that $\{\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\} - E(\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\})\colon\ n \geqslant 1\}$ is a sequence of blockwise m-dependent real-valued random variables. Moreover,

$$\sum_{n=1}^{\infty} \frac{E(\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\} - E(\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\}))^2}{n^2} \\ \leqslant \sum_{n=1}^{\infty} \frac{E\|X_n\|_{u_0}^2}{n^2} < \infty \quad \text{(by (2))}.$$

Then by Theorem 1 of [5], we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\})) = 0 \quad \text{a.s.}$$

Hence,

$$\limsup_{n\to\infty} d\big([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Z_i]_{i=1}^n\big) \leqslant \varepsilon \quad \text{a.s.}$$

For (B_2), it is clear that $\{Z_n\colon\ n \geqslant 1\}$ is a sequence of blockwise and pairwise m-dependent $\mathfrak{X}$-valued random variables and $Z_n \in \mathcal{K}_\varepsilon \cup \{u_0\}$ for all n. By applying Proposition 3.1, we have $(B_2) \to 0$ a.s. as $n \to \infty$.

For (B_3), we have

$$d\big([n^{-1}, EZ_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n\big) \leqslant \frac{1}{n} \sum_{i=1}^{n} d(EZ_i, EX_i) \\ \leqslant \frac{1}{n} \sum_{i=1}^{n} Ed(Z_i, X_i) = \frac{1}{n} \sum_{i=1}^{n} E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\}) \leqslant \varepsilon.$$

Combining the above parts, we obtain

$$\limsup_{n\to\infty} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, EX_i]_{i=1}^n) \leqslant 2\varepsilon \quad \text{a.s.}$$

The proof is completed. □

For sequences of blockwise and pairwise m-dependent and Cesàro CUI $\mathfrak{X}$-valued random variables, we obtain the following result.

Theorem 3.3. *Let $\{X_n\colon n \geqslant 1\}$ be a sequence of blockwise and pairwise m-dependent $\mathfrak{X}$-valued random variables which are Cesàro CUI. Suppose that $\{\|X_n\|_{u_0}\colon n \geqslant 1\}$ is stochastically dominated by a real-valued random variable X. If*

$$E(|X|(\log^+ |X|)^2) < \infty, \tag{3}$$

then (1) *holds.*

P r o o f. We will use the same method and notation as in the proof of Theorem 3.2. We also have $(B_2) \to 0$ a.s. as $n \to \infty$ and $(B_3) \leqslant \varepsilon$. Now we consider (B_1). Note that $\{\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\}\colon n \geqslant 1\}$ is a sequence of blockwise and pairwise m-dependent random variables and for all n, $t \geqslant 0$,

$$P(\|X_n\|_{u_0} I\{X_n \notin \mathcal{K}_\varepsilon\} \geqslant t) \leqslant P(\|X_n\|_{u_0} \geqslant t) \leqslant CP(|X| \geqslant t).$$

Then by Theorem 1 of [8], we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\} - E(\|X_i\|_{u_0} I\{X_i \notin \mathcal{K}_\varepsilon\})) = 0 \quad \text{a.s.}$$

Hence,

$$\limsup_{n\to\infty} d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, Z_i]_{i=1}^n) \leqslant \varepsilon \quad \text{a.s.}$$

The proof is completed. □

R e m a r k 3.4. In general, the compactly uniform integrability is stronger than the uniform integrability. But, they are equivalent for the real case. Furthermore, if $\{\|X_n\|_{u_0}\colon n \geqslant 1\}$ is stochastically dominated by a real-valued random variable X and $E|X| < \infty$, then $\{\|X_n\|_{u_0}\colon n \geqslant 1\}$ is uniformly integrable. Therefore, in the above theorem, if $\mathfrak{X} = \mathbb{R}$ we obtain Theorem 1 of [8] with $r = 1$.

In the next part, we will establish the SLLN for sequences of blockwise m-dependent and identically distributed $\mathfrak{X}$-valued random variables. First, we will prove the following lemma.

Lemma 3.5. *Let $\{X, X_n\colon n \geqslant 1\}$ be a sequence of blockwise m-dependent and identically distributed real-valued random variables. If $E|X| < \infty$, then*

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to EX \quad \text{a.s. as } n \to \infty. \tag{4}$$

P r o o f. Since $\{X_n^+\colon n \geqslant 1\}$ and $\{X_n^-\colon n \geqslant 1\}$ satisfy the hypotheses of the lemma and $X_n = X_n^+ - X_n^-$, without loss of generality we can assume that $X_n \geqslant 0$.

Set $Y_n = X_n I\{X_n \leqslant n\}$, $n \geqslant 1$. Let $F(x)$ be the distribution function of X. We have

$$\sum_{n=1}^{\infty} \frac{EY_n^2}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^n x^2 \,\mathrm{d}F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \int_k^{k+1} x^2 \,\mathrm{d}F(x)$$
$$= \sum_{k=0}^{\infty} \int_k^{k+1} x^2 \,\mathrm{d}F(x) \sum_{n=k+1}^{\infty} \frac{1}{n^2}.$$

On the other hand, we have

$$\sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqslant \frac{2}{k+1}, \quad k \in \mathbb{N}_0.$$

Indeed, if $k = 0$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leqslant 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 2.$$

If $k \geqslant 1$, then

$$\sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqslant \sum_{n=k+1}^{\infty} \frac{1}{n(n-1)} = \frac{1}{k} \leqslant \frac{2}{k+1}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{EY_n^2}{n^2} \leqslant 2 \sum_{k=0}^{\infty} \frac{1}{k+1} \int_k^{k+1} x^2 \,\mathrm{d}F(x) \leqslant 2 \sum_{k=0}^{\infty} \int_k^{k+1} x \,\mathrm{d}F(x) = 2EX < \infty.$$

This implies that

$$\sum_{n=1}^{\infty} \frac{E(Y_n - EY_n)^2}{n^2} \leqslant \sum_{n=1}^{\infty} \frac{EY_n^2}{n^2} < \infty.$$

Since $\{Y_n \colon n \geqslant 1\}$ is a sequence of blockwise m-dependent random variables, $\{Y_n - EY_n \colon n \geqslant 1\}$ is so as well. By Theorem 1 of [5], we have

$$\frac{1}{n} \sum_{i=1}^{n} (Y_i - EY_i) \to 0 \quad \text{a.s. as } n \to \infty.$$

On the other hand,

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(X_n > n) = \sum_{n=1}^{\infty} \int_n^{\infty} \mathrm{d}F(x)$$
$$= \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \int_i^{i+1} \mathrm{d}F(x) = \sum_{i=1}^{\infty} i \int_i^{i+1} \mathrm{d}F(x)$$
$$\leqslant \sum_{i=1}^{\infty} \int_i^{i+1} x \,\mathrm{d}F(x) \leqslant EX < \infty.$$

It follows from $EY_n \uparrow EX$ as $n \to \infty$ and the Borel-Cantelli lemma that

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to EX \quad \text{a.s. as } n \to \infty.$$

The lemma is proved. $\square$

Now we will establish the SLLN for sequences of blockwise m-dependent and identically distributed $\mathfrak{X}$-valued random variables.

Theorem 3.6. *Let $\{X, X_n\colon\ n \geqslant 1\}$ be a sequence of blockwise m-dependent and identically distributed $\mathfrak{X}$-valued random variables. If*

$$E\|X\|_{u_0} < \infty, \tag{5}$$

then

$$[n^{-1}, X_i]_{i=1}^{n} \to EX \quad \text{a.s. as } n \to \infty. \tag{6}$$

P r o o f. First, assume that X is a simple function taking distinct values $x_1, x_2, \ldots, x_p$ on non-null sets $\Omega_1, \Omega_2, \ldots, \Omega_p$. For each $t = 1, \ldots, p$, we put

$$U_n^t(\omega) = \operatorname{card}\{i\colon\ 1 \leqslant i \leqslant n, X_i(\omega) = x_t\} = \sum_{i=1}^{n} I\{X_i = x_t\}(\omega).$$

By Lemma 3.5, we have $n^{-1}U_n^t \to P(\Omega_t) > 0$ a.s. as $n \to \infty$. Hence, $U_n^t \to \infty$ a.s. as $n \to \infty$ and when n is large enough, $U_n^t > 0$ a.s. For n large enough, by property (2.1) we have

$$[n^{-1}, X_i]_{i=1}^{n} = [n^{-1}U_n^t, [(U_n^t)^{-1}, x_t]_{i=1}^{U_n^t}]_{t=1}^{p} \quad \text{a.s.}$$

On the other hand,

$$[n^{-1}U_n^t, [(U_n^t)^{-1}, x_t]_{i=1}^{U_n^t}]_{t=1}^{p} \to [P(\Omega_t), Kx_t]_{t=1}^{p} = EX \quad \text{a.s. as } n \to \infty.$$

So the conclusion (6) holds for simple random variables.

Let us consider the general case when X is an integrable $\mathfrak{X}$-valued random variable. For each $\varepsilon > 0$, by Proposition 4.1 (d) of [7], there exists a sequence $\{\varphi_k\colon\ k \geqslant 1\}$ of (Borel measurable) maps such that $Ed(\varphi_k(X), X) \to 0$ as $k \to \infty$. Therefore, there exists k_0 such that $Ed(\varphi_{k_0}(X), X) \leqslant \varepsilon$. Furthermore, by the assumption and

Proposition 4.1 (a) of [7], $\{\varphi_{k_0}(X), \varphi_{k_0}(X_n)\colon n \geqslant 1\}$ is a sequence of blockwise m-dependent identically distributed and simple random variables. We have

$$E\|\varphi_{k_0}(X)\|_{u_0} \leqslant E\|X\|_{u_0} + \varepsilon < \infty. \tag{7}$$

By the triangle inequality,

$$\begin{aligned} d([n^{-1}, X_i]_{i=1}^n, EX) &\leqslant d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n) \\ &\quad + d([n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n, E\varphi_{k_0}(X)) \\ &\quad + d(E\varphi_{k_0}(X), EX) := (C_1) + (C_2) + (C_3). \end{aligned}$$

For (C_1) we have

$$d([n^{-1}, X_i]_{i=1}^n, [n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n) \leqslant \frac{1}{n}\sum_{i=1}^n d(X_i, \varphi_{k_0}(X_i)).$$

Note that $\{d(X, \varphi_{k_0}(X)), d(X_n, \varphi_{k_0}(X_n))\colon n \geqslant 1\}$ is a sequence of blockwise m-dependent and identically distributed real-valued random variables. Thus by Lemma 3.5 again, we have

$$\frac{1}{n}\sum_{i=1}^n d(X_i, \varphi_{k_0}(X_i)) \to Ed(X, \varphi_{k_0}(X)) \leqslant \varepsilon \quad \text{a.s. as } n \to \infty.$$

For (C_2), it follows from the first case and (7) that

$$d([n^{-1}, \varphi_{k_0}(X_i)]_{i=1}^n, E\varphi_{k_0}(X)) \to 0 \quad \text{a.s. as } n \to \infty.$$

For (C_3), we have

$$d(E\varphi_{k_0}(X), EX) \leqslant Ed(X, \varphi_{k_0}(X)) \leqslant \varepsilon.$$

Combining the above parts, we obtain

$$\limsup_{n\to\infty} d([n^{-1}, X_i]_{i=1}^n, EX) \leqslant 2\varepsilon \quad \text{a.s.}$$

By the arbitrariness of $\varepsilon > 0$, we derive the conclusion (6). □

By using Theorem 2 of [8] and a method similar to that used in the proof of Theorem 3.6, we derive SLLN for sequences of pairwise m-dependent and identically distributed $\mathfrak{X}$-valued random variables. Therefore, the proof will be omitted.

Theorem 3.7. *Let $\{X, X_n\colon\ n \geqslant 1\}$ be a sequence of pairwise m-dependent and identically distributed $\mathfrak{X}$-valued random variables. Then condition* (5) *implies* (6).

R e m a r k 3.8. Note that pairwise 0-dependence is equivalent to pairwise independence. Therefore, Theorem 5.1 of [7] is a special case of Theorem 3.7.

As mentioned in Section 2, $(k(\mathfrak{X}), d_H)$ is a separable and complete convex combination space. We denote the expectation of an integrable random variable X in $(k(\mathfrak{X}), d_H)$ by $E_{k(\mathfrak{X})}X$. Then, by applying the above theorems, we obtain the following corollaries immediately:

Corollary 3.9. *Let $\{X, X_n\colon\ n \geqslant 1\}$ be a sequence of blockwise m-dependent (or pairwise m-dependent) and identically distributed $k(\mathfrak{X})$-valued random variables. If $E\|X\|_{\{u_0\}} < \infty$, then*

$$[n^{-1}, X_i]_{i=1}^n \to E_{k(\mathfrak{X})}X \quad \text{a.s. as } n \to \infty. \tag{8}$$

Corollary 3.10. *Let $\{X_n\colon\ n \geqslant 1\}$ be a sequence of blockwise m-dependent $k(\mathfrak{X})$-valued random variables which are Cesàro CUI. If*

$$\sum_{n=1}^{\infty} \frac{E\|X_n\|_{\{u_0\}}^2}{n^2} < \infty, \tag{9}$$

then

$$d_H([n^{-1}, X_i]_{i=1}^n, [n^{-1}, E_{k(\mathfrak{X})}X_i]_{i=1}^n) \to 0 \quad \text{a.s. as } n \to \infty. \tag{10}$$

Corollary 3.11. *Let $\{X_n\colon\ n \geqslant 1\}$ be a sequence of blockwise and pairwise m-dependent $k(\mathfrak{X})$-valued random variables which are Cesàro CUI. Suppose that $\{\|X_n\|_{\{u_0\}}\colon\ n \geqslant 1\}$ is stochastically dominated by a real-valued random variable X. If*

$$E(|X|(\log^+ |X|)^2) < \infty, \tag{11}$$

then (10) *holds.*

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References

[1] *Z. Bo*: Marcinkiewicz-Zygmund law for sequences of blockwise m-dependent random variables. Statist. Probab. Lett. *38* (1998), 83–88. zbl MR

[2] *B. D. Choi, S. H. Sung*: On convergence of $(S_n - ES_n)/n^{1/r}$, $1 < r < 2$, for pairwise independent random variables. Bull. Korean Math. Soc. *22* (1985), 79–82. zbl MR

[3] *N. Etemadi*: An elementary proof of the strong law of large numbers. Z. Wahrscheinlichkeitstheor. Verw. Geb. *55* (1981), 119–122. zbl MR

[4] *V. F. Gaposhkin*: On the strong law of large numbers for blockwise independent and blockwise orthogonal random variables. Theory Probab. Appl. *39* (1994), 677–684; translation from Teor. Veroyatn. Primen. *39* (1994), 804–812. (In Russian.) zbl MR

[5] *F. Móricz*: Strong limit theorems for blockwise m-dependent and blockwise quasi-orthogonal sequences of random variables. Proc. Am. Math. Soc. *101* (1987), 709–715. zbl MR

[6] *N. V. Quang, N. T. Thuan*: On the strong laws of large numbers for double arrays of random variables in convex combination spaces. Acta Math. Hung. *134* (2012), 543–564. zbl MR

[7] *P. Terán, I. Molchanov*: The law of large numbers in a metric space with a convex combination operation. J. Theor. Probab. *19* (2006), 875–898. zbl MR

[8] *L. V. Thanh*: Strong laws of large numbers for sequences of blockwise and pairwise m-dependent random variables. Bull. Inst. Math., Acad. Sin. *33* (2005), 397–405. zbl MR

[9] *N. T. Thuan, N. V. Quang, P. T. Nguyen*: Complete convergence for arrays of rowwise independent random variables and fuzzy random variables in convex combination spaces. Fuzzy Sets Syst. *250* (2014), 52–68. zbl MR

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