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# FINITE-TIME OUTER SYNCHRONIZATION BETWEEN TWO COMPLEX DYNAMICAL NETWORKS WITH TIME DELAY AND NOISE PERTURBATION

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In this paper, the finite-time stochastic outer synchronization and generalized outer synchronization between two complex dynamic networks with time delay and noise perturbation are studied. Based on the finite-time stability theory, sufficient conditions for the finite-time outer synchronization are obtained. Numerical examples are examined to illustrate the effectiveness of the analytical results. The effect of time delay and noise perturbation on the convergence time are also numerically demonstrated.

*Keywords:* complex dynamic networks, synchronization, time delay, noise perturbation

*Classification:* 65L99, 70K99

## 1. INTRODUCTION

Complex networks widely exist in our life, from the internet to the world wide web, from communication networks to social organizations, from food webs to ecological communities and so forth [3, 4, 34]. The structure of many real systems in nature can be described by complex networks. Therefore, exploring the network topology and dynamical activities of complex networks is of fundamental importance to understanding the functions of real-world systems. In recent years, the dynamics of complex networks have been extensively investigated. As a typical kind of dynamics, synchronization of complex networks has attracted more and more attention. This is partly due to its broad applications in secure communication, neural networks, biological systems, information science, etc [1].

Synchronization is one of the basic motions in nature where many connected system evolving in synchrony. Therefore, as a typical kind of dynamics on complex network, synchronization is an important research topic [6, 11, 14, 17, 21, 24, 29, 36, 37, 38]. In the past decades, most of the works in network synchronization has paid attention to the inner synchronization, which is concerned with the synchronization among the nodes within a network. On the hand, there exists another type of network synchronization, i. e.. "outer synchronization" between two or more complex networks. Recently, there are some researches who have studied the outer synchronization between two complex

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networks [2, 18, 22, 23, 27, 28]. In Ref. [28], the stochastic LaSalle invariance principle is employed to theoretically prove the outer synchronization between two complex networks with noise perturbation. In Ref. [23], the problem of generalized outer synchronization between two completely different complex dynamical networks is investigated. With a nonlinear control scheme, a sufficient criterion for this generalized outer synchronization and two corollaries are derived based on Barbalat's lemma. Generalized outer synchronization between two uncertain dynamical networks with a novel feature that the couplings of each network are unknown functions was investigated in Ref. [22]. The mixed outer synchronization of coupled complex networks with time-varying coupling delay was investigated in Ref. [2]. Most of above works primarily focus on the asymptotical or exponential synchronization of networks. However, in reality, the networks might always be expected to achieve synchronization as quickly as possible. An effective method to achieve faster convergence rate in complex networks is finite-time synchronization control techniques. In Ref. [35], Yang and Cao investigated the finite-time stochastic synchronization of complex networks by using finite-time stability theorem, inequality techniques. In Ref. [10], Huang et al. investigated the global finite-time stabilization of a class of uncertain nonlinear systems. In Ref. [31], Wang and Han et al. investigated the problem of finite-time chaos control via nonsingular terminal sliding model control. Moreover, the finite-time control techniques have demonstrated better disturbance rejection and robustness against uncertainties [5].

Between the nodes of complex network, the transmission of information is always exist, and a lot of information is private, so secure communication is very important. The principle of signal transmission is select one network as the launch system, sent the useful signal and the output signal out at the same time, select another network as receive system, by using the chaos synchronization of networks, one can recovery the useful information in the receiver channel in finite time. However, in many large scale networks, time delays are unavoidable, due to the finite information transmission and processing speeds among the network nodes. Time delay coupling extensively exists in many biological and physical systems such as gene regulatory networks, communication networks, neural networks and electrical power grids, etc. It has also been discovered that time delays frequently have great influence on the behavior of dynamical systems [9, 15, 19, 33]. In addition, noise is ubiquitous in the real systems, the synchronization of coupled systems or networks is unavoidably affected by different kinds of noise. Therefore, the effect of noise on synchronization has been well studied [8, 13, 16, 20, 25, 26, 32]. Noise is commonly regarded as a persistent disturbance which usually inhibits synchronization. However, recent researchers have reported that noise could also play a constructive role in nonlinear systems [16, 20, 26, 32]. However, to the best of our knowledge, there have not been any general results for the finite-time outer synchronization of complex dynamic networks with time delay and noise perturbation.

Inspired by the above analysis, the question which we address in our present study is: Can finite-time generalized synchronization between two different chaotic systems be achieved with the perturbation of noise? Besides the numerical evidences, are there any analytical arguments illustrating this phenomenon? Utilizing the finite-time stability theory of stochastic differential equations, we analytically show that two systems can realize finite-time generalized synchronization if two chaotic systems have different dynamical behaviors. Finally, some numerical examples are examined to illustrate the effectiveness of the analytical results.

The rest of this paper is organized as follows. In Section 2, we introduce two networks with time-delay and noise perturbation. Sufficient conditions for the finite-time stochastic outer synchronization and finite-time outer synchronization are respectively derived in Section 3 and Section 4. Numerical examples are shown in Section 5. Finally, some conclusions are drawn in Section 6.

*Notations:* Throughout this paper unless specified we let $I_n$ be an $n \times n$ identity matrix. $E[\cdot]$ denotes the expected value of a stochastic process. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. $\|\cdot\|$ be Euclidean norm, for vector $x \in R^n, \|x\| = x^T x$, for matrix $A \in R^{nn}, \|A\| = \sqrt{\lambda_{\max}(A^T A)}$, where $\lambda_{\max}(\cdot)$ means the largest eigenvalue of the matrix.

## 2. PROBLEM STATEMENT AND PRELIMINARIES

In this paper, we consider the dynamical networks described by:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij} P x_j(t-\tau), \qquad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ is state vector of the $i$th node, $f : R^n \to R^n$ is continuously differentiable nonlinear vector function, $\tau$ is the time delay, $P$ is the inner connection matrix between two connected nodes and $C = (c_{ij})_{N\times N}$ represents the coupling configurations of the network, whose entries $c_{ij}$ are defined as follow: if there is a link from node $j$ to node $i (i \neq j)$ then set $c_{ij} > 0$, otherwise $c_{ij} = 0 (i \neq j)$. The diagonal elements of matrix C are defined as

$$c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}, \quad i = 1, 2, \ldots, N.$$

In order to achieve the finite-time outer synchronization between two complex networks, we refer to network (1) as the drive network, and the response network is given by the following equations:

$$\dot{y}_i(t) = f(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t-\tau) + u_i(t) + \sigma_i(e_i(t), e_i(t-\tau), t)\dot{W}, \quad i = 1, 2, \ldots, N, \tag{2}$$

where $y_i(t) = (y_{i1}, y_{i2}, \ldots, y_{in})^T \in R^n$ is the state vector of node $i$; $f$ has the same meaning of network (1), $Q$ is the inner connection matrix between two connected nodes. $D = (d_{ij})_{N\times N}$ has the seem meaning as $C$ in network (1), $e_i(t) = y_i(t) - x_i(t) (i = 1, 2, \ldots, N)$ are the synchronization errors between the drive system (1) and the respond network (2), $u_i(t) (i = 1, 2 \ldots, N)$ are the controllers will be designed. The noise term in system (2) is mostly utilized to describe the coupling process influenced by environmental fluctuation, inaccurate design of coupling strength, etc. where $\sigma_i : R^n \to R^{n\times m}$ is called the noisy intensity matrix, $W = (w_1, w_2, \ldots, w_m)^T$ is an $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Accordingly, $\dot{W}$ is an $m$-dimensional white noise.

To realized the outer synchronization in a finite-time, we designed the feedback controllers as follow:

$$\begin{aligned} u_i(t) &= -ke_i(t) - \lambda \text{sign}(e_i(t))|e_i(t)|^\theta + \sum_{j=1}^{N}(c_{ij}P - d_{ij}Q)x_j(t-\tau) \\ &\quad -\lambda \left( \int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right), \end{aligned} \tag{3}$$

where $|e_i(t)|^\theta = (|e_{i1}(t)|^\theta, \ldots, |e_{in}(t)|^\theta)^T$, $\text{sign}(e_i(t)) = \text{diag}(\text{sign}(e_1(t)), \ldots, \text{sign}(e_n(t)))$, the parameter $\theta$ is a constant been in interval $(0,1)$, $k$ and $\lambda$ are positive constants to be determined.

**Remark 2.1.** In this paper, the configuration matrices $C$ and $D$ of network (1) and (2) are not assumed to be symmetric or irreducible, which means that network (1) and (2) can be undirected or directed networks, and they may also contain nodes and cluster.

Throughout this paper, we make the following assumptions:

**Assumption 2.2.** For function $f(x)$ there exists a positive constant $l$ such that

$$[x(t) - y(t)]^T[f(x(t)) - f(y(t))] \le [x(t) - y(t)]^T l[x(t) - y(t)], \quad \forall t \ge 0, \forall x, y \in R^n. \tag{4}$$

**Assumption 2.3.** The noise intensity function $\sigma_i(e_i(t), e_i(t-\tau))$ satisfies the Lipschitz condition and there exists two positive constants $\xi$ and $\eta$ such that

$$\text{trace}(\sigma_i(e_i(t))\sigma_i(e_i(t-\tau))) \le \xi e_i^T(t)e_i(t) + \eta e_i^T(t-\tau)e_i(t-\tau), \quad i = 1, 2, \ldots, N. \tag{5}$$

Moreover, $\sigma(0) \equiv 0$.

**Definition 2.4.** Systems (1) and (2) are said to achieve finite-time stochastic complete synchronization, if for any initial states $x_i(0), y_i(0)$, there exists a finite time function $T_0$ such that

$$E\|y_i(t, y_i(0)) - x_i(t, x_i(0))\| = 0,$$

for all $t \ge T_0$, where $T_0 = \inf\{T : x(t) = y(t), \forall t \ge T\}$ is called the settling time.

**Remark 2.5.** The stochastic settling time function $T_0$ is not only a function of $x_i(0), y_i(0)$, but a stochastic variable for fixed $x_i(0)$ and $y_i(0)$. Hence, the finite-time property of $T_0$ is evaluated by $0 < E(T_0) < +\infty$.

Consider the following $n$-dimensional stochastic differential delay equation:

$$dz(t) = \varphi(t, z(t), z(t-\tau))dt + \psi(t, z(t), z(t-\tau))dW, \tag{6}$$

on $t \ge 0$ with initial data $\zeta \in C^{\mu}_{\mathcal{F}_0}([-\tau, 0], R^n)$. Here, $\zeta \in C^{\mu}_{\mathcal{F}_0}([-\tau, 0], R^n)$ represents the family of all $\mathcal{F}_0$-measurable bounded $C^{\mu}_{\mathcal{F}_0}([-\tau, 0], R^n)$-valued random variables.

$\varphi, \psi : [0, +\infty] \times R^n \times R^n \to R^n$ are locally Lipschitz continuous and satisfy linear growth condition. For each $V \in C^{2,1}(R^n \times R_+, R_+)$, the operator $\mathcal{LV}$ associated to Eq. (6) is defined by

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial z} \cdot \varphi + \frac{1}{2}\text{trace}\left[\psi^T \cdot \frac{\partial^2 V}{\partial^2 z} \cdot \psi\right], \tag{7}$$

where $\partial V/\partial z = (\partial V/\partial z_1, \partial V/\partial z_2, \ldots, \partial V/\partial z_n)$, $\partial^2 V/\partial^2 z = (\partial^2 V/\partial z_i \partial z_j)_{n\times n}$.

**Lemma 2.6.** (Wang et al. [30]) Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:

$$\dot{V}(t) \leq -\kappa V^\rho(t),$$

where $\kappa, \rho$ are two constants. Then, for any given $t_0, V(t)$ satisfies the following inequality:

$$V^{1-\rho}(t) \leq V^{1-\rho}(t_0) - \kappa(1-\rho)(t-t_0), \quad t_0 \leq t \leq t_1,$$

and

$$V(t) \equiv 0, \quad t \geq t_1,$$

with $t_1$ is given by

$$t_1 = t_0 + \frac{(V(t_0))^{1-\rho}}{\kappa(1-\rho)}.$$

**Lemma 2.7.** (Jensen inequality, Hardy et al. [7]) Let $a_1, a_2, \ldots, a_n > 0$ and $0 < r < p$. Then

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n a_i^r\right)^{\frac{1}{r}}.$$

**Lemma 2.8.** (Itô formula) Assume that $V(t,x) \in C^{1,2}(R_+ \times R^d)$,

$$x(t) = x(t_0) + \int_{t_0}^t f(s)ds + \int_{t_0}^t g(s)\,\mathrm{d}w(s), \quad t \in J,$$

where $J = [t_0, T] \in R_+$ is the fixed interval, $f \in L^1(J, R^d), g \in L^2(J, R^{d\times m})$, then $V(t, x(t))$ is a Itô process, and

$$\mathrm{d}V(t,x(t)) = \mathcal{L}V(t,x(t))dt + V_x(t,x(t))g(t,x(t))dw(t),$$

where

$$\mathcal{L}V(t,x(t)) = V_t(t,x(t)) + V_x(t,x(t))f(t) + (1/2)\text{trace}[g^T(t)V_{xx}(t,x(t))g(t)].$$

## 3. SUFFICIENT CONDITIONS FOR FINITE-TIME STOCHASTIC SYNCHRONIZATION

In this section, we investigate the finite-time stochastic outer synchronization with time delay and noise perturbation between networks (1) and (2), and the main results are drawn in the following theorem and corollaries.

**Theorem 3.1.** Suppose that Assumption 2.2 and 2.3 hold and there exist positive constants $k, p$ and $\xi, \eta$ such that

$$\begin{cases} 2l - 2k + p + \xi + \lambda_{\max}(\mathcal{Q}^s) \leq 0, \\ \eta - p + 1 \leq 0, \end{cases}$$

where $\mathcal{Q} = D \otimes Q$, $\mathcal{Q}^s = \mathcal{Q}\mathcal{Q}^T$. Then, under the controller (3), networks (1) and (2) can achieve finite-time stochastic outer synchronization.

P r o o f. From the networks (1) and (2), we can get the following error system:

$$\begin{aligned} \dot{e}_i(t) &= f(y_i(t)) - f(x_i(t)) + \sum_{j=1}^{N} d_{ij} Q e_j(t-\tau) + \sigma_i(e_i(t), e_i(t-\tau))\dot{W} \\ &\quad -ke_i(t) - \lambda \mathrm{sign}(e_i(t))|e_i(t)|^{\theta} - \lambda \left( \int_{t-\tau}^{t} p e_i^T(s) e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right) \end{aligned} \tag{8}$$

Consider the following Lyapunov function:

$$V(t) = \sum_{i=1}^{N} e_i^T(t) e_i(t) + \sum_{i=1}^{N} \int_{t-\tau}^{t} p e_i^T(s) e_i(s)\,\mathrm{d}s. \tag{9}$$

Thus, the diffusion operator $\mathcal{L}$ fined in Eq. (7) onto the function $V$ along the trajectory of system (8) is calculated and estimated as follows:

$$\begin{aligned} \mathcal{L}V(t) &= 2\sum_{i=1}^{N} e_i^T(t)[f(y_i(t)) - f(x_i(t)) + \sum_{j=1}^{N} d_{ij} Q e_j(t-\tau) - ke_i(t) \\ &\quad -\lambda \left( \int_{t-\tau}^{t} p e_i^T(s) e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right)] + \sum_{i=1}^{N} \mathrm{trace}(\sigma_i(e_i(t)\sigma_i(e_i(t-\tau)) \\ &\quad -\lambda \mathrm{sign}(e_i(t))|e_i(t)|^{\theta} + p\sum_{i=1}^{N} [e_i^T(t) e_i(t) - e_i^T(t-\tau) e_i(t-\tau)]. \end{aligned} \tag{10}$$

From Assumptions 2.2 and 2.3, we can get

$$\begin{aligned}\mathcal{L}V(t) \quad \leq \quad & 2l\sum_{i=1}^{N} e_i^T(t)e_i(t) + 2\sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} d_{ij}Qe_j(t-\tau) - 2k\sum_{i=1}^{N} e_i^T(t)e_i(t) \\ & -2\lambda\sum_{i=1}^{N} e_i^T(t)\mathrm{sign}(e_i(t))|e_i(t)|^\theta \\ & -2\lambda\sum_{i=1}^{N} e_i^T(t)\left(\int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s\right)^{\frac{1+\theta}{2}}\left(\frac{e_i(t)}{\|e_i(t)\|^2}\right) \\ & +\sum_{i=1}^{N}[\xi e_i^T(t)e_i(t) + \eta e_i^T(t-\tau)e_i(t-\tau)] \\ & +p\sum_{i=1}^{N}[e_i^T(t)e_i(t) - e_i^T(t-\tau)e_i(t-\tau)]. \end{aligned} \tag{11}$$

Simplifying the formula (11), we obtain

$$\begin{aligned}\mathcal{L}V(t) \quad \leq \quad & (2l-2k+p+\xi)\sum_{i=1}^{N} e_i^T(t)e_i(t) + (\eta - p)\sum_{i=1}^{N} e_i^T(t-\tau)e_i(t-\tau) \\ & +2\sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} d_{ij}Qe_j(t-\tau) - 2\lambda\sum_{i=1}^{N} e_i^T(t)\mathrm{sign}(e_i(t))|e_i(t)|^\theta \\ & -2\lambda\sum_{i=1}^{N} e_i^T(t)\left(\int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s\right)^{\frac{1+\theta}{2}}\left(\frac{e_i(t)}{\|e_i(t)\|^2}\right). \end{aligned} \tag{12}$$

Note that

$$\begin{aligned}\sum_{i=1}^{N} e_i^T(t)\mathrm{sign}(e_i(t))|e_i(t)|^\theta \quad & = \quad \sum_{i=1}^{N}(|e_i(t)|^\theta)^T\mathrm{sign}(e_i(t))e_i(t) \\ & = \quad \sum_{i=1}^{N}(|e_i(t)|^\theta)^T|e_i(t)| \\ & = \quad \sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}|^{\theta+1}. \end{aligned}$$

From Lemma 2.7, we have

$$\left(\sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}|^{\theta+1}\right)^{\frac{1}{\theta+1}} \geq \left(\sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}|^2\right)^{\frac{1}{2}}.$$

Hence, we can get

$$\sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}|^{\theta+1} \geq \left(\sum_{i=1}^{N}\sum_{j=1}^{N}|e_{ij}|^2\right)^{\frac{\theta+1}{2}} = \left(\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{\frac{\theta+1}{2}}. \tag{13}$$

Moreover

$$\sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} d_{ij}Qe_j(t-\tau) = e^T(t)D\otimes Qe(t-\tau). \tag{14}$$

From the elementary inequality that

$$2e^T(t)D\otimes Qe(t-\tau) \leq e^T(t)\mathcal{Q}\mathcal{Q}^Te(t) + e^T(t-\tau)e(t-\tau).$$

Thus, we can obtain the following inequality

$$\begin{aligned}\mathcal{L}V(t) \;\; \leq \;\; & (2l-2k+p+\xi+\lambda_{\max}(\mathcal{Q}^s))\sum_{i=1}^{N} e_i^T(t)e_i(t) \\ & +(\eta-p+1)\sum_{i=1}^{N} e_i^T(t-\tau)e_i(t-\tau) \\ & -2\lambda\left[\left(\sum_{i=1}^{N}\int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s\right)^{\frac{1+\theta}{2}} + \left(\sum_{i=1}^{N} e_i^T(t)e_i(t)\right)^{\frac{1+\theta}{2}}\right].\end{aligned} \tag{15}$$

If

$$\begin{cases} 2l-2k+p+\xi+\lambda_{\max}(\mathcal{Q}^s) \leq 0, \\ \eta-p+1 \leq 0, \end{cases} \tag{16}$$

then we can get

$$\mathcal{L}V(t) \leq -2\lambda\left[\sum_{i=1}^{N}\int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s + \sum_{i=1}^{N} e_i^T(t)e_i(t)\right]^{\frac{1+\theta}{2}}.$$

Thus

$$\mathcal{L}V(t) \leq -2\lambda(V(t))^{\frac{1+\theta}{2}}. \tag{17}$$

According to the Lemma 2.8, we have

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + 2\sum_{i=1}^{N} e_i^T(t)\sigma_i(e_i, e_i(t-\tau))\mathrm{d}W(t). \tag{18}$$

Taking the expectations on both sides of (18), we obtain from (17) that

$$E[\dot{V}(t)] \leq -2\lambda(E[V(t)])^{\frac{1+\theta}{2}}. \tag{19}$$

By Lemma 2.6, $E[V(t)]$ converges to zero in a finite time, and the finite time is estimated by

$$t_1 = \frac{(2V(0))^{\frac{1+\theta}{2}}}{\lambda(1-\theta)}, \tag{20}$$

where $t_0 = 0, V(0) = \sum_{i=1}^{N} e_i^T(0)e_i(0) + \sum_{i=1}^{N} \int_{-\tau}^{0} pe_i^T(s)e_i(s)\,\mathrm{d}s$. This means that complete outer synchronization between systems (1) and (2) could be achieved in a finite time for almost every initial data. The proof is completed. □

**Remark 3.2.** We can see that, for any high level noise, there exits sufficiently large positive constant $k$ such that the finite-time stochastic outer synchronization is realized in probability. Hence the synchronization is robust to the noise perturbation.

**Remark 3.3.** The convergence time of the proposed algorithm is closely related to the protocol parameters $\lambda$, $\theta$ and $\tau$. From (20), one can see that for fixed parameter $\theta$ and $V(0)$, the synchronization time decreases as $\lambda$ increases. In addition, by some straightforward arguments, it can be shown that smaller $\theta$ can lead to a shorter convergence time when initial states of two networks differ a little from each other.

Based on Theorem 3.1, we can easily derive the following corollary:

**Corollary 3.4.** Let Assumptions 2.2 and 2.3 hold. If networks (1) and (2) have the same topological structures and uniform inner-coupling matrices, i. e.., $C = D$, $P = Q$, and

$$\begin{cases} 2l - 2k + p + \xi \leq 0, \\ \eta - p \leq 0. \end{cases}$$

Then networks (1) and (2) can achieve finite-time stochastic outer synchronization under the following control schemes:

$$u_i(t) = -ke_i(t) - \lambda \mathrm{sign}(e_i(t))|e_i(t)|^\theta - \lambda \left( \int_{t-\tau}^{t} pe_i^T(s)e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right), \quad i = 1, 2, \ldots, N.$$

**Corollary 3.5.** Let Assumption 2.2 holds. If $\sigma_i(e_i(t), e_i(t-\tau), t) \equiv 0$ in system (2) and

$$2l - 2k + p + \lambda_{\max}(\mathcal{Q}^s) \leq 0.$$

Then networks (1) and (2) can achieve finite-time stochastic outer synchronization under the controllers (3).

## 4. SUFFICIENT CONDITIONS FOR THE FINITE-TIME GENERALIZED OUTER SYNCHRONIZATION

In this section, we investigate the finite-time generalized outer synchronization between two complex dynamic networks with time delay and noise perturbation. To realize the generalized outer synchronization, we refer to system (1) as the drive network, and the response network is given by following equations:

$$\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^{N} d_{ij} Q y_j(t-\tau) + u_i(t) + \sigma_i(e_i(t), e_i(t-\tau), t)\dot{W}, i = 1, 2, \ldots, N, \quad (21)$$

where $g : R^n \to R^n$ is a continuously differentiable functions which determine the dynamical behavior of the nodes in the network (21). The coupling configuration matrix $D$ and inner connection matrix $Q$ have same meaning as those in Sec. 3.

**Definition 4.1.** Let $\phi_i : R^n \to R^n (i = 1, 2, \ldots, N)$ be continuously differentiable function. Network (21) is said to achieve finite-time generalized outer synchronization with network (1), if for any initial states $x_i(0), y_i(0)$, there exists a finite time function $T_0^*$ such that

$$E\|y_i(t, y_i(0)) - \phi(x_i(t, x_i(0)))\| = 0,$$

for all $t \geq T_0^*$, where $T_0^* = \inf\{T^* : x(t) = y(t), \forall t \geq T^*\}$ is called the settling time.

Applying the above approach, we can obtain the following results for the finite-time generalized outer synchronization between networks (1) and (21). To realized the synchronization in a finite-time, we designed the feedback controllers as follow:

$$\begin{aligned} u_i(t) \;=\; & \mathcal{D}\phi_i(x_i)\dot{x}_i - g(\phi_i(x_i)) - ke_i(t) - \lambda \mathrm{sign}(e_i(t))|e_i(t)|^\theta \\ & - \sum_{j=1}^{N} d_{ij} Q \phi_j x_j(t-\tau) - \lambda \left( \int_{t-\tau}^{t} p e_i^T(s) e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right). \end{aligned} \quad (22)$$

**Theorem 4.2.** Suppose that there exists a constant $l^*$ such that

$$[x(t) - y(t)]^T [g(x(t)) - g(y(t))] \leq [x(t) - y(t)]^T l^* [x(t) - y(t)], \qquad \forall t \geq 0, \forall x, y \in R^n, \quad (23)$$

and Assumption 2.3 hold, if the following condition is satisfied:

$$\begin{cases} 2l^* - 2l + p + \xi + \lambda_{\max}(\mathcal{Q}^s) \leq 0, \\ \eta - p + 1 \leq 0, \end{cases}$$

where $\mathcal{Q}$ and $\mathcal{Q}^s$ have the seem meaning as those in Theorem 3.1. Then, under the controllers (22), networks (1) and (21) can achieve finite-time stochastically generalized outer synchronization.

*Proof.* Letting $e_i(t) = y_i(t) - \phi_i(x_i(t))$, one has the following error system:

$$\begin{aligned} \dot{e}_i(t) \;=\; & g(y_i(t)) - g(\phi_i(x_i(t))) - ke_i(t) - \lambda \mathrm{sign}(e_i(t))|e_i(t)|^\theta - \sum_{j=1}^{N} d_{ij} Q e_j(t-\tau) \\ & - \lambda \left( \int_{t-\tau}^{t} p e_i^T(s) e_i(s)\,\mathrm{d}s \right)^{\frac{1+\theta}{2}} \left( \frac{e_i(t)}{\|e_i(t)\|^2} \right). \end{aligned} \quad (24)$$

Construct the following Lyapunov function

$$V(t)=\sum_{i=1}^{N}e_i^T(t)e_i(t)+\sum_{i=1}^{N}\int_{t-\tau}^{t}pe_i^T(s)e_i(s)\,\mathrm{d}s$$

the diffusion operator $\mathcal{L}$ fined in Eq. (7) onto the function $V$ along the trajectory of system (24) is calculated and estimated as follows:

$$\begin{aligned}\mathcal{L}V(t) \;=\;& 2\sum_{i=1}^{N}e_i^T(t)[g(y_i(t))-g(\phi_i(x_i(t)))+\sum_{j=1}^{N}d_{ij}Qe_j(t-\tau)-ke_i(t)\\ &-\lambda\mathrm{sign}(e_i(t))|e_i(t)|^{\theta}-\lambda\left(\int_{t-\tau}^{t}pe_i^T(s)e_i(s)\,\mathrm{d}s\right)^{\frac{1+\theta}{2}}\left(\frac{e_i(t)}{\|e_i(t)\|^2}\right)]\\ &+\sum_{i=1}^{N}\mathrm{trace}(\sigma_i(e_i(t)\sigma_i(e_i(t-\tau))\\ &+p\sum_{i=1}^{N}[e_i^T(t)e_i(t)-e_i^T(t-\tau)e_i(t-\tau)].\end{aligned}\tag{25}$$

From (23) and Assumption 2.3 we get

$$\begin{aligned}\mathcal{L}V(t) \;\le\;& 2l^*\sum_{i=1}^{N}e_i^T(t)e_i(t)+2\sum_{i=1}^{N}e_i^T(t)\sum_{j=1}^{N}d_{ij}Qe_j(t-\tau)-2k\sum_{i=1}^{N}e_i^T(t)e_i(t)\\ &-2\lambda\sum_{i=1}^{N}e_i^T(t)\mathrm{sign}(e_i(t))|e_i(t)|^{\theta}\\ &-2\lambda\sum_{i=1}^{N}e_i^T(t)\left(\int_{t-\tau}^{t}e_i^T(s)e_i(s)\,\mathrm{d}s\right)^{\frac{1+\theta}{2}}\left(\frac{e_i(t)}{\|e_i(t)\|^2}\right)\\ &+\sum_{i=1}^{N}[\xi e_i^T(t)e_i(t)+\eta e_i^T(t-\tau)e_i(t-\tau)]\\ &+p\sum_{i=1}^{N}[e_i^T(t)e_i(t)-e_i^T(t-\tau)e_i(t-\tau)].\end{aligned}\tag{26}$$

The last proof is omitted as it is similar to that of Theorem 3.1. □

**Corollary 4.3.** Let (23) hold. If $\sigma_i(e_i(t),e_i(t-\tau),t)\equiv 0$ in system (21) and

$$\left\{\begin{array}{l}2l^*-2k+p+\lambda_{\max}(\mathcal{Q}^s)\le 0,\\ 1-p\le 0.\end{array}\right.$$

Then networks (1) and (21) can achieve finite-time generalized outer synchronization under the controllers (3).

## 5. SIMULATION RESULTS

In this section, we take two examples to illustrate the feasibility and effectiveness of the theoretical results obtained in previous sections. In the numerical simulations, we use the Euler-Maruyama numerical scheme [12] to solve all the stochastic differential equations, and all the differential equations are solved with step-size 0.001.

**Example 1.** In this example, we take the Rössler like system as the node dynamics of networks (1) and (2), the system can be described as:

$$\begin{cases} \dot{x}_1 = -\alpha(\vartheta x_1 + \beta x_2 + \varepsilon x_3) \\ \dot{x}_2 = \alpha x_1 + \alpha\gamma x_2 \\ \dot{x}_3 = -\alpha\mu x_3 + \alpha\mu\psi(x_1) \end{cases} \tag{27}$$

where $x = (x_1, x_2, x_3)^T \in R^3$ is the state vector,

$$\psi(s) = \begin{cases} 0, & s < 2.56; \\ \rho(s - 2.56), & s \geq 2.56. \end{cases}$$

As show in Figure 1, the Rössler-like system has a chaotic attractor when $\alpha = 0.03$, $\beta = 1.5$, $\gamma = 0.2$, $\mu = 1.5$, $\varepsilon = 0.75$, $\rho = 21.43$ and $\vartheta = 0.075$. And it is easy to compute that the $\lambda_{\max}(A + A^T) = 0.0208$ and

$$\begin{aligned} (x-y)^T(f(x) - f(y)) &= \alpha\mu(x_3 - y_3)(\psi(x_1) - \psi(y_1)) \\ &\leq \frac{\alpha\mu\rho}{2}(x-y)^T(x-y). \end{aligned}$$

Therefore, the Assumption 2.2 is satisfied with $l = 0.4822$.

Take $\sigma_i(e_i, e_i(t-\tau)) = \sigma_0 \mathrm{diag}(e_{i1}(t) - e_{i1}(t-\tau), e_{i2}(t) - e_{i2}(t-\tau)), e_{i3}(t) - e_{i3}(t-\tau)), i = 1, 2, \ldots, N$. Meanwhile, assume that $W(t) = [w_1(t), w_2(t), w_3(t)]$ is a three-dimensional Brownian motion. Then, $\sigma_i(e_i, e_i(t-\tau), t)$ satisfies the locally Lipschitz condition and the linear growth condition, i. e..,

$$\mathrm{trace}(\sigma_i^T \sigma_i) \leqslant 2\sigma_0^2 e_i^T(t) e_i(t) + 2\sigma_0^2 e_i^T(t-\tau) e_i(t-\tau).$$

The initial state of the $i$th node of drive network $\zeta_i = [1 + \exp(-i), 1 - 0.1 * i, 2 + \cos i](-\tau \leq t \leq 0)$, while the initial state of the $i$th node of the respond network is $\zeta_j = [1 + \cos i, 1 - \sin i, 2 + \exp(-i)](-\tau \leq t \leq 0)$.

In the numerical simulation, for brevity, we always set $P = Q = I_3$.

The configuration matrix is given as follows:

$$C = D = \begin{pmatrix} -3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

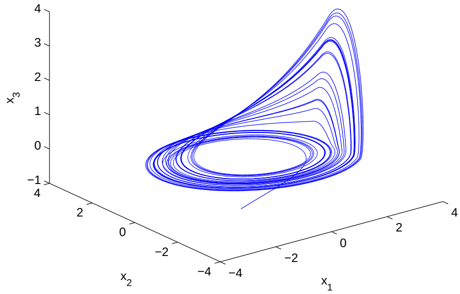



**Fig. 1.** Chaotic attractor generated by the system (27) when $\alpha = 0.03$, $\beta = 1.5$, $\gamma = 0.2$, $\mu = 1.5$, $\varepsilon = 0.75$, $\rho = 21.43$ and $\delta = 0.075$.

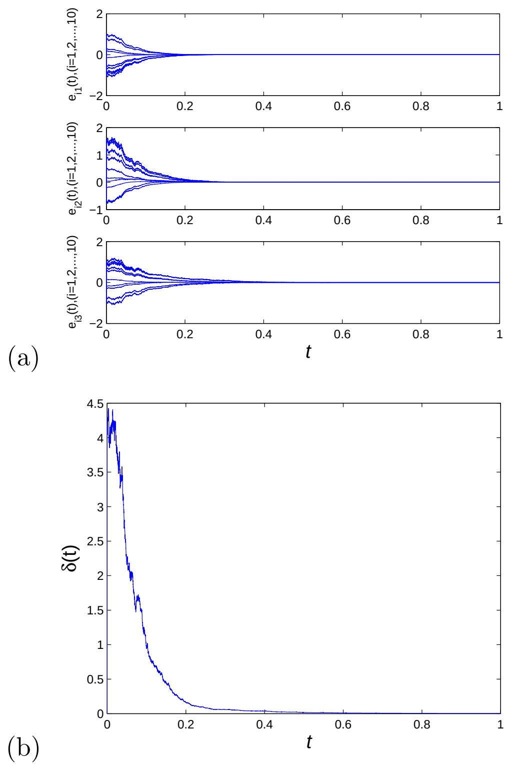



**Fig. 2.** Trajectories of synchronization error (a) and the total synchronization error (b) between network (1) and (2) with $k = 3$, $\lambda = 1.5$, $\theta = 0.6$, $\sigma_0 = 1$, $\tau = 0.3$.

It is easy to compute that $\lambda_{\max}(\mathcal{Q}^s) = 0.503$. Take $k = 6$, $\lambda = 1.5$, $\theta = 0.6$, $\tau = 0.3$, $\sigma_0 = 1.5$, we simulate the evolution of the networks according to the controllers defined in Eq. (3). According to the Theorem 3.1, we take the constants $p = 3$, $\xi = 1$, $\eta = 2$, networks (1) and (2) can reach outer synchronization in a finite time. By computing we get $T_1 = 0.8907$. Figures 2(a) and 2(b) show the trajectories of synchronization errors $e_{ij}(t)(i = 1, 2, \ldots, 10; j = 1, 2, 3)$ and the total synchronization error $\delta(t)$, where $\delta(t) = \|e(t)\|$. From Figure 2, we can find that the outer synchronization is realized after $t \geq 0.45$, and the simulations match the theoretical results perfectly.

To study the effect of the control parameters $\tau$ and $\sigma_0$ on the settling time, we simulate the evolution of two networks with the controllers defined in Eq. (3) through taking different values of $\tau$ and $\sigma_0$. Figure 3 gives the evolutions of the total error values of $\tau$ and shows that networks with small time delay converge faster than those with large time delay. Figure 4 gives the evolutions of the total error values of $\sigma_0$ and shows that the synchronization time decreases when parameter $\sigma_0$ increases.

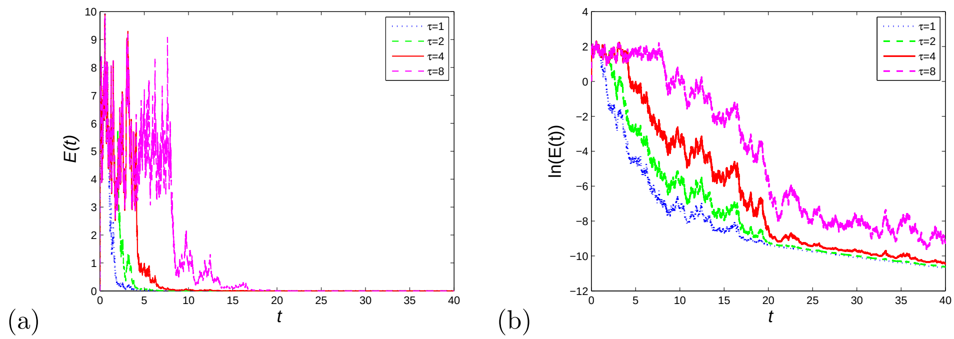



**Fig. 3.** (a) Time evolution of total synchronization error $E(t)$ with time delay $\tau = 1, 2, 4, 8$ and $\sigma = 0.3$; (b) The corresponding logarithmic plot.

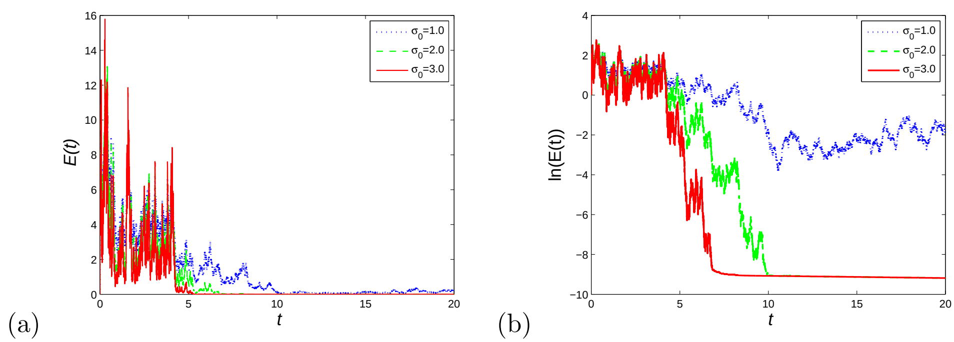



**Fig. 4.** (a) Time evolution of total synchronization error $E(t)$ with time delay $\sigma = 1.0, 2.0, 3.0$ and $\tau = 4$; (b) The corresponding logarithmic plot.

**Example 2.** In this example, we take the Rössler like system as the node dynamics of drive networks (1) and Lorenz system as the node dynamics of response networks (21). The Lorenz system is described by:

$$\begin{cases} \dot{x}_1 = -a(x_1 - x_2) \\ \dot{x}_2 = cx_1 - x_2 - x_1x_3 \\ \dot{x}_3 = -bx_3 + x_1x_2 \end{cases} \tag{28}$$

which has a chaotic attractor when $a = 10, b = 8/3, c = 28$. We can easy see that

$$\begin{aligned} (x-y)^T(f(x)-f(y)) &= (x-y)A(x-y)^T + \alpha\mu(x_3-y_3)(\psi(x_1)-\psi(y_1)) \\ &\leq \left[\lambda_{\max}(\frac{A+A^T}{2}) + \frac{\alpha\mu\rho}{2}\right](x-y)^T(x-y). \end{aligned}$$

Therefore, the Assumption 2.2 is satisfied.

The map $\phi_i$ is defined as

$$\phi_i(x_i) = \left(x_{i1} - x_{i2}, x_{i2}^2 + 1, \frac{1}{2}x_{i3}^2 - x_{i1}\right)^T, \quad i = 1, 2, \ldots, N.$$

Then

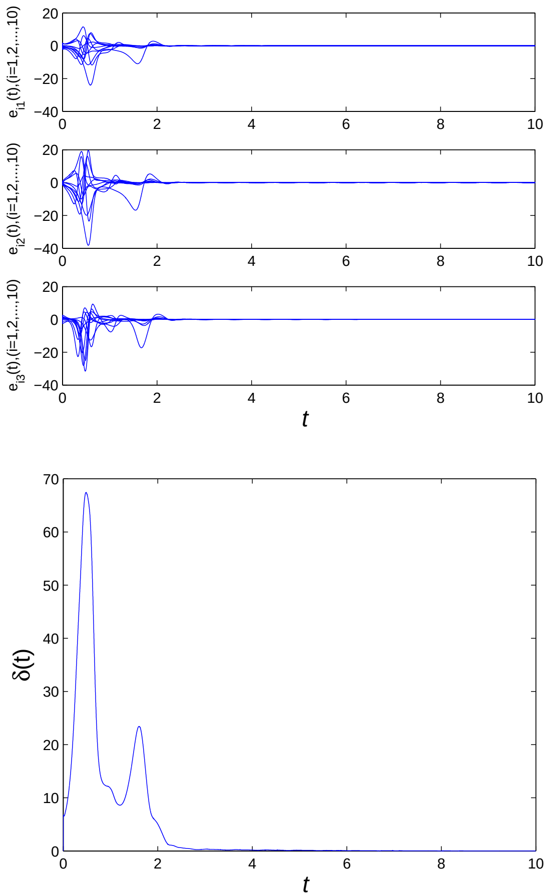
$$\mathcal{D}\phi_i(x_i) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2x_{i2} & 0 \\ -1 & 0 & x_{i3} \end{pmatrix}.$$

In this numerical simulations, for brevity, we set $P = Q = I_3$. The configuration matrix for the drive network is given as follows:

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & -3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}$$

The configuration matrix for the response network is given as follows:

$$D = \begin{pmatrix} -3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$




(a)

(b)

**Fig. 5.** Trajectories of synchronization error (a) and the total synchronization error (b) between networks (1) and (21) with $k = 32$, $\lambda = 3$, $\theta = 0.6$, $\sigma_0 = 0.02$, $\tau = 0.5$.

It is easy to compute that $\lambda_{\max}(\mathcal{Q}) = 29.4769$. Take $k = 32$, $\lambda = 3$, $\theta = 0.6$, $\tau = 0.5$, we simulate the evolution of the networks according to the controllers defined in (22) with $\sigma_0 = 0.02$. According to Theorem 4.2, we take $p = 3$, $\xi = 1$, $\eta = 2$, network (1) and network (21) can achieve generalized outer synchronization in a finite time. By computing we get $T_1 = 1.8576$. Figures 5(a) and 5(b) show the trajectories of synchronization errors $e_{ij}(t)(i = 1, 2, \ldots, 10; j = 1, 2, 3)$ and the total synchronization error $\delta(t)$, where $\delta(t) = \|e(t)\|$. From Figure 5, one can find that the generalized outer synchronization is realized after $t \geq 2.95$, and the simulations match the theoretical results perfectly.

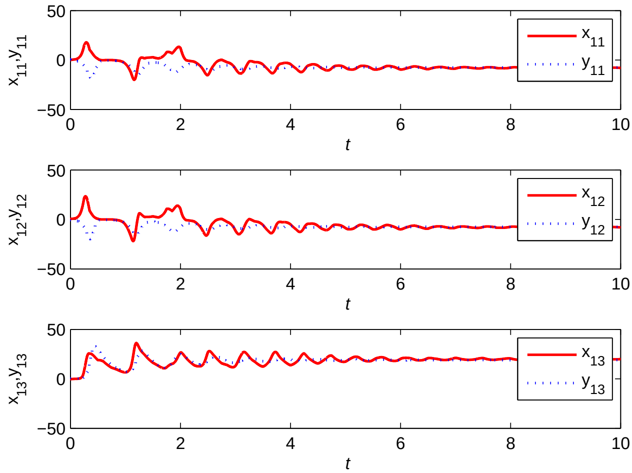



**Fig. 6.** The trajectory of first node $x_{1j}(j = 1, 2, 3)$ of Rössler-like system and the first node $y_{1j}(j = 1, 2, 3)$ of Lorenz system.

Figure 6 shows the trajectory of first node $x_1(j = 1, 2, 3)$ of Rössler-like system and the first node $y_1(j = 1, 2, 3)$ of Lorenz system.

**Example 3.** Some real networks often have complex topology, and the network topology may play a vital role in outer synchronization. In order to demonstrate the effectiveness of the theoretical results on small-world networks, we assume that networks (1) and (2) are small-world networks. The algorithm starts from a regular lattice with $N$ nodes, and then each link is rewired to another node randomly chosen from all possible nodes with a certain probability $p$ (we should avoid self-loops and link duplications). First, we generate a small-world network with $N = 200, p = 0.5$ and the average degree $\langle d \rangle = 10$. Take $\sigma_0 = 1.5$, we simulate the evolution of the networks according to the protocol defined in Eq. (3). The results are exhibited in Figures 7(a) and 7(b). From Figure 7 one can see that the small-world can also realize outer synchronization. To illustrate the effect of the small-word network parameters $p$ on the speed of synchronization, we simulate the evolution of two networks with different values of $p$ in Figure 8.

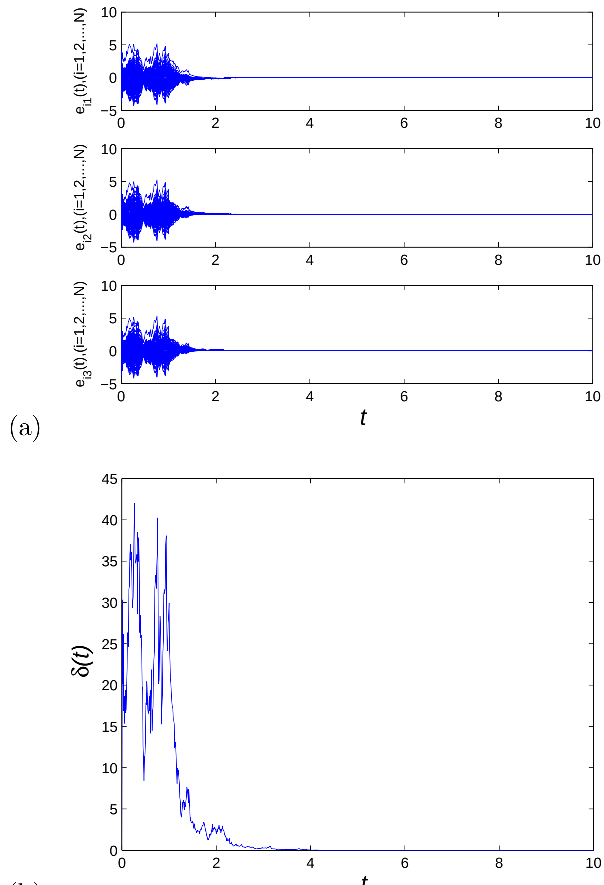



**Fig. 7.** Trajectories of synchronization error (a) and the total synchronization error (b) between networks (1) and (2) with $\sigma_0 = 1.5, \tau = 1, p = 0.5$..

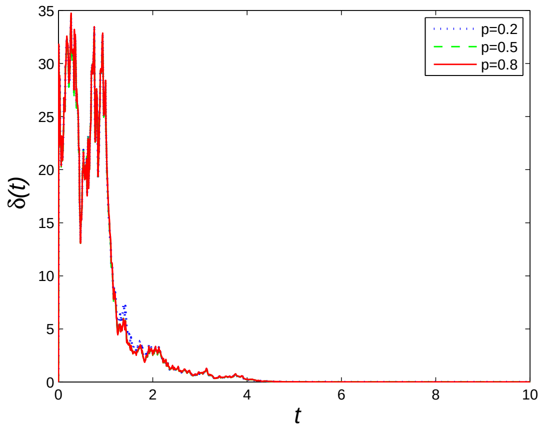



**Fig. 8.** Trajectories of total synchronization error between small-world networks (1) and (2) with $p = 0.2, 0.5, 0.8$..

## 6. CONCLUSIONS

In this paper, we have investigated the finite time outer synchronization between two complex networks with time delay and noise perturbation. First, we considered the stochastic synchronization between two complex networks, sufficient conditions for stochastic synchronization are obtained based on the finite time stability theory of stochastic differential equations. The generalized outer synchronization between two different complex networks is also investigated based on a continuously differentiable function $\phi$. The theoretical results show that two networks can achieve outer synchronization even if the coupling configuration matrix is not symmetric or irreducible. Numerical simulations fully verify our main results. The effect of time delay $\tau$ and the strength of noise $\theta$ on the synchronization speed are also numerically demonstrated. From the simulation results we can see that small time delay $\tau$ converge faster than those with large time delay and the synchronization time decreases when parameter $\sigma$ increases. The controllers designed in this paper are very extensive, they also can be used to solve the stabilization of neural network and the projective synchronization of complex networks. In this paper, we considered the complex networks are continuously. If the networks are not continuously, how to design the controller to achieve synchronization is an interesting problem. Therefore, studying the finite time synchronization of two complex networks with discontinuous coupling is important. This problem is our future research direction.

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## REFERENCES


[1] A. Arenas, A. Daz-Guilera, J. Kurths, Y. Moreno, and C. S. Zhou: Synchronization in complex networks. Phys. Rep. *469* (2008), 93–153. DOI:10.1016/j.physrep.2008.09.002

[2] M. M. Asheghan, J. Míguez, M. T. Hamidi-Beheshti and M. S. Tavazoe: Robust outer synchronization between two complex networks with fractional order dynamics. Chaos *21* (2011), 033121. DOI:10.1063/1.3629986

[3] A. L. Barabasi: Scal-free networks: a decade and beyond. Science *325* (2009), 412–413. DOI:10.1126/science.1173299

[4] A. L. Barabasi and R. Albert: Emergence of scaling in random networks. Science *286* (1999), 509–512. DOI:10.1126/science.286.5439.509

[5] S. P. Blat and D. S. Bernstein: Finite-time stability of continuous autonomous systems. SIAM. J. Control Optim. *38* (2000), 751–766. DOI:10.1137/s0363012997321358

[6] W. L. Guo, F. Austin, and S.H. Chen: Global synchronization of nonlinearly coupled complex networks with non-delayed and delayed coupling. Commun. Nonlinear Sci. Numer. Simulat. *15* (2010), 1631–1639. DOI:10.1016/j.cnsns.2009.06.016

[7] G. Hardy, J. Littlewood, and G. Polya: Inequalities. Cambridge University Press 1988.

[8] B. Hauschildt, N. B. Jason, A. Balanov, and E. Scholl: Noise-induced cooperative dynamics and its control in coupled neuron models. Phys. Rev. E *74* (2006), 051906. DOI:10.1103/physreve.74.051906

[9] J. J. Huang, C. D. Li, T. W. Huang, and X. He: Finite-time lag synchronization of delayed neural networks. Neurocomputing *139* (2013), 145–149. DOI:10.1016/j.neucom.2014.02.050

[10] X. Q. Huang, W. Lin, and B. Yang: Global finite-time synchronization of a class of uncertain nonlinear systems. Automatica *41* (2005), 881–888. DOI:10.1016/j.automatica.2004.11.036

[11] Y. B. Kazanovich and R. M. Borisyuk: Synchronization in a neural network of phase oscillators with the central element. Biological Cybernetics *71* (1994), 177–185. DOI:10.1007/s004220050080

[12] P. E. Kloeden and E. Platen: Numerical Solution of Stochastic Differential Equations. Springer, Heidelberg 1992. DOI:10.1007/978-3-662-12616-5

[13] G. Korniss: Synchronization in weighted unccorrelated complex networks in a noisy environment: optimization and connections with transport efficiency. Phys. Rev. E *75* (2007), 051121. DOI:10.1103/physreve.75.051121

[14] H. Y. Li, Y. A. Hu, and R. Q. Wang: Adaptive finite-time synchronization of cross-strict feedback hyperchaotic systems with parameter uncertainties. Kybernetika *49* (2013), 554–567.

[15] L. Li, J. Kurths, H. Peng, Y. Yang, and Q. Luo: Exponentially asymptotic synchronization of uncertain complex time-delay dynamical networks. The European Physical Journal B *86* (2013), 1–9. DOI:10.1140/epjb/e2013-30517-6

[16] W. Lin and G. R. Chen: Using white noise to enhance synchronization of coupled chaotic systems. Chaos *16* (2006), 013134. DOI:10.1063/1.2183734

[17] J. H. Lü, X. H. Yu and G. R. Chen: Chaos synchronization of general complex dynamical networks. Physica A *334* (2004), 281–302. DOI:10.1016/j.physa.2003.10.052

[18] V. Lynnyk and S. Čelikovský: On the anti-synchronization detection for the generalized Lorenz system and its applications to secure encryption. Kybernetika *46* (2010), 1–18.

[19] J. Mei, M. H. Jiang, W. M. Xu and B. Wang: Finite-time synchronization control of complex dynamical networks with time delay. Commun. Nonlinear Sci. Numer. Simulat. *18* (2013), 2462–2478. DOI:10.1016/j.cnsns.2012.11.009

[20] K. H. Nagail and H. Kori: Noise-induced synchronization of a large population of globally coupled nonidentical oscillators. Phys. Rev.E *81* (2010), 065202. DOI:10.1103/physreve.81.065202

[21] L. M. Pecora and T. L. Carrol: Master stability functions for synchronized coupled system. Phys. Rev. Lett. *80* (1998), 2109–2112. DOI:10.1103/physrevlett.80.2109

[22] W. G. Sun and S. X. Li: Generalized outer synchronization between two uncertain dynamical networks. Nonlinear Dyn. *77* (2014), 481–489. DOI:10.1007/s11071-014-1311-7

[23] Y. Z. Sun, W. Li, and J. Ruan: Generalized outer synchronization between complex dynamic networks with time delay and noise perturbation. Commun. Nonliear Sci. Numer. Simul. *18* (2013), 989–998. DOI:10.1016/j.cnsns.2012.08.040

[24] F. Sun, H. Peng, Q. Luo, L. Li, and Y. Yang: Parameter identification and projective synchronization between different chaotic systems. Chaos *19* (2009), 023109. DOI:10.1063/1.3127599

[25] Y. Z. Sun and J. Ruan: Synchronization in coupled time-delayed systems with parameter mismath and noise perturbation. Chaos *19* (2009), 043113. DOI:10.1063/1.3262488

[26] Y. Z. Sun, H. J. Shi, E. A. Bakare, and Q. X. Meng: Noise-induced outer synchronization between two different complex dynamical networks. Nonlinear Dyn. *76* (2014), 519–528. DOI:10.1007/s11071-013-1145-8

[27] H.W. Tang, L. Chen, J. A. Lu, and C.K. Tse: Adaptive synchronization between two nonidentical topological structures. Physica A *387* (2008), 5623–5630. DOI:10.1016/j.physa.2008.05.047

[28] G. J. Wang, J.D. Cao, and J. Q. Lu: Outer synchronization between two nonidentical networks with circumstance noise. Physica A *389* (2010), 1480–1488. DOI:10.1016/j.physa.2009.12.014

[29] X. F. Wang and G. R. Chen: Synchronization in scal-free dynamical networks: robustness and fragility. IEEE Trans. Circuits Syst. I *49* (2002), 54–62. DOI:10.1109/81.974874

[30] H. Wang, Z. Z. Han, Q. Y. Xie, and W. Zhang: Finite-time synchronization of uncertain unified chaotic systems based on CLF. Nonlinear Anal.: Real World Appl. *10* (2009), 2842–2849. DOI:10.1016/j.nonrwa.2008.08.010

[31] H. Wang, Z. Z. Han, Q. Y. Xie, and W. Zhang: Finite-time chaos control via nonsingular terminal sliding model control. Commun. Nonlinear Sci. Numer. Simulat. *14* (2012), 2728–2733. DOI:10.1016/j.cnsns.2008.08.013

[32] W. Wang, L. Li, H. Peng, J. Xiao, and Y. Yang: Synchronization control of memristor-based recurrent neural networks with perturbations. Neural Networks. *53* (2014), 8–14. DOI:10.1016/j.neunet.2014.01.010

[33] W. P. Wang, H. P. Peng, L. X. Li, J. H. Xiao, and Y. X. Yang: Finite-time function projective synchronization in complex multi-links networks with time-varying delay. Neural Process. Lett. *41* (2015), 71–88. DOI:10.1007/s11063-013-9335-4

[34] D. J. Watts and S. H. Strogatz: Collective dynamics of small-world networks. Nature *393* (1998), 440–442. DOI:10.1038/30918

[35] X. S. Yang and J. D. Cao: Finite-time stochastic synchronization of complex networks. App. Math. Modeling. *34* (2010), 3631–3641. DOI:10.1016/j.apm.2010.03.012

[36] W. W. Yu, G. R. Chen, and J. H. Lü: On pinning synchronization of complex dynamical networks. Automatica *45* (2009), 429–435. DOI:10.1016/j.automatica.2008.07.016

[37] X. B. Zhou, M. R. Jiang, and Y. Q. Huang: Switched modified function projective synchronization between two complex nonlinear hyperchaotic systems based on adaptive control and parameter identification. Kybernetika *50* (2014), 632–642. DOI:10.14736/kyb-2014-4-0632

[38] C. S. Zhou, A. E. Motter, and J. Kurths: Universality in the synchronization of weighted random networks. Phys. Rev. Lett. *96* (2006), 034101. DOI:10.1103/physrevlett.96.034101

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