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Mathematica Bohemica, Vol. 141 (2016), No. 3, 297–313

Persistent URL: http://dml.cz/dmlcz/145894

141 (2016) MATHEMATICA BOHEMICA No. 3, 297–313

UNIQUENESS AND DIFFERENTIAL POLYNOMIALS OF MEROMORPHIC FUNCTIONS SHARING A NONZERO POLYNOMIAL

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Received February 14, 2014. First published June 15, 2016.
Communicated by Stanisława Kanas

Abstract. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$ then we say that f and g share the value a with weight k. Using this idea of sharing values we study the uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a nonzero polynomial with finite weight. The results of the paper improve and generalize the related results due to Xia and Xu (2011) and the results of Li and Yi (2011).

Keywords: uniqueness; meromorphic function; differential polynomial; weighted sharing

MSC 2010: 30D35

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We assume the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [6] and [18]). For a nonconstant meromorphic function f and positive real number r, we denote by $T(r, f)$ the Nevanlinna characteristic of f and by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside of an exceptional set of finite linear measure. We denote by $T(r)$ the maximum of $T(r, f)$ and $T(r, g)$. The symbol $S(r)$ denotes any quantity satisfying $S(r) = o\{T(r)\}$ as $r \to \infty$.

Let f and g be two nonconstant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If $f - a$ and $g - a$ have the same zeros, we say that f and g share the value a IM (ignoring

The author is thankful to DST-PURSE programme for financial assistance.

DOI: 10.21136/MB.2016.0018-14

multiplicities). If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). In addition, we need the following definitions.

Definition 1 ([8]). Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;f|=1)$ the counting function of simple a points of f. For a positive integer p we denote by $N(r,a;f|\leqslant p)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r,a;f|\leqslant p)$ we denote the corresponding reduced counting function. Analogously we can define $N(r,a;f|\geqslant p)$ and $\overline{N}(r,a;f|\geqslant p)$.

Definition 2 ([9]). Let k be a positive integer or infinity. We denote by $N_k(r,a;f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and k times if $m > k$. Then

$$N_k(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f|\geqslant 2) + \ldots + \overline{N}(r,a;f|\geqslant k).$$

Clearly $N_1(r,a;f) = \overline{N}(r,a;f)$.

Definition 3. Let a be a value in the extended complex plane and k an arbitrary nonnegative integer. We define

$$\Theta(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

and

$$\Theta_{k)}(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,a;f|\leqslant k)}{T(r,f)}.$$

In 1999 Lahiri [10] studied the uniqueness problems of meromorphic functions when two linear differential polynomials share the same 1-points. In the same paper regarding the nonlinear differential polynomials Lahiri asked the following question.

Question 1. What can be said about the relationship between two meromorphic functions f and g when two nonlinear differential polynomials generated by them share certain values?

Afterwards, research works concerning Question 1 have been done by many mathematicians and continuous efforts are being put in to relax the hypothesis of the results, cf. [2]–[5], [8], [13].

In 2004 Lin and Yi [13] proved the following result which dealt with Question 1.

Theorem A. *Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > 2/(n+1)$, $n \geqslant 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value* 1 CM, *then $f \equiv g$.*

A new trend in this direction is to consider the uniqueness of a meromorphic function concerning the value sharing of the k-th derivatives of a linear expression of a meromorphic function. For the last couple of years a number of astonishing results have been obtained regarding the value sharing of nonlinear differential polynomials which are mainly the k-th derivative of some linear expressions of f and g (see [1], [3], [12], [14] and [16], for example). In 2007 Bhoosnurmath and Dyavanal [3] proved the following result which extends Theorem A.

Theorem B. *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty, f) > 3/(n+1)$, and let n, k be two positive integers satisfying $n \geqslant 3k+13$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share* 1 CM, *then $f = g$.*

A recent development to the uniqueness theory has been to consider weighted sharing instead of sharing IM or CM; this implies a gradual change from sharing IM to sharing CM. This notion of weighted sharing was introduced by Lahiri around 2000. It measures how close a shared value is to being shared CM or to being shared IM. The definition is as follows.

Definition 4 ([9]). Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity m ($\leqslant k$) if and only if it is an a-point of g with multiplicity m ($\leqslant k$), and z_0 is an a-point of f with multiplicity m ($> k$) if and only if it is an a-point of g with multiplicity n ($> k$), where m is not necessarily equal to n.

We write f, g *share* (a, k) meaning that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leqslant p < k$. Also we note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Using the notion of weighted value sharing, Banerjee [1] proved the following result in 2011 which improves and generalizes Theorem B.

Theorem C. *Let f and g be two transcendental meromorphic functions and $n \geqslant 1$, $k \geqslant 1$, $l \geqslant 0$ three integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Suppose that $(f^n(af+b))^{(k)}$ and $(g^n(ag+b))^{(k)}$ share $(1, l)$ where a and b are any two nonzero constants. If $l \geqslant 2$ and $n \geqslant 3k+9$ or if $l = 1$ and $n \geqslant 4k+10$ or if $l = 0$ and $n \geqslant 9k+18$, then either $(f^n(af+b))^{(k)}(g^n(ag+b))^{(k)} = 1$ or $f = g$. The possibility $(f^n(af+b))^{(k)}(g^n(ag+b))^{(k)} = 1$ does not occur for $k = 1$.*

In 2011 the present author studied the uniqueness problem of meromorphic functions concerning some general differential polynomials and proved the following result which improves and extends Theorem C.

Theorem D ([14])**.** *Let f and g be two transcendental meromorphic functions, and let $n \geqslant 1$, $k \geqslant 1$, $m \geqslant 1$ and $l \geqslant 0$ be four integers. Let $P(z) = a_m z^m + \ldots + a_1 z + a_0$, where a_0 $(\neq 0)$, $a_1, \ldots, a_m$ $(\neq 0)$ are complex constants. Suppose that $(f^n P(f))^{(k)}$ and $(g^n P(g))^{(k)}$ share $(1, l)$ and one of the following conditions holds:*

(a) $l \geqslant 2$ *and* $n > 3k + m + 8$;
(b) $l = 1$ *and* $n > 4k + \frac{3}{2}m + 9$;
(c) $l = 0$ *and* $n > 9k + 4m + 14$.

Then either $(f^n P(f))^{(k)}(g^n P(g))^{(k)} = 1$ or $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m, \ldots, n+m-i, \ldots, n+1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g) = f^n P(f) - g^n P(g)$. The possibility $(f^n P(f))^{(k)}(g^n P(g))^{(k)} = 1$ does not arise for $k = 1$.

In view of Theorems C and D one may ask the following question.

Question 2. Is it possible in any way to remove the conclusion $(f^n P(f))^{(k)} \times (g^n P(g))^{(k)} = 1$ in Theorems C and D?

In this direction Xia and Xu [16] proved the following results, which dealt with Question 2.

Theorem E. *Let n, m, k be three positive integers, and let f and g be two nonconstant meromorphic functions such that $(f^n(f-1)^m)^{(k)}$ and $(g^n(g-1)^m)^{(k)}$ share 1 CM. If $m > k$ and $n \geqslant 3k+m+8$, and $\Theta(\infty, f) > 2m(n+m)/((n+m)^2 - 4k^2)$ or $\Theta(\infty, g) > 2m(n+m)/((n+m)^2 - 4k^2)$, then either $f = g$, or f and g satisfy the algebraic equation $R(f, g) = 0$, where*

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m - w_2^n(w_2 - 1)^m.$$

Theorem F. *Let n, m, k be three positive integers, and let f and g be two nonconstant meromorphic functions such that $(f^n(f-1)^m)^{(k)}$ and $(g^n(g-1)^m)^{(k)}$ share* 1 CM. *If $m \leqslant k$ and $n \geqslant 3k+m+8$, and*

$$\Theta(\infty, f) + \Theta_{[k/m])}(1, f) > 1 + \frac{2m(n+m)}{(n+m)^2 - 4k^2} \tag{1.1}$$

or

$$\Theta(\infty, g) + \Theta_{[k/m])}(1, g) > \frac{2m(n+m)}{(n+m)^2 - 4k^2}, \tag{1.2}$$

then the conclusions of Theorem E *hold.*

The following question arises:

Question 3. What can be said if the sharing value 1 in the above theorems is replaced by a nonzero polynomial?

In 2011 Li and Yi [12] answered the above question by proving the following theorems.

Theorem G. *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying $n > 3k+11$ and $\max\{\chi_1, \chi_2\} < 0$, where*

$$\chi_1 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \Theta_{k)}(1, f) - \Theta_{k-1)}(1, f) \tag{1.3}$$

and

$$\chi_2 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \Theta_{k)}(1, g) - \Theta_{k-1)}(1, g). \tag{1.4}$$

If $\Theta(\infty, f) > 2/n$ and if $(f^n(f-1))^{(k)} - P_1$ and $(g^n(g-1))^{(k)} - P_1$ share 0 CM, *where P_1 is a nonzero polynomial, then $f = g$.*

Theorem H. *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying $n > 9k+20$ and $\max\{\chi_1, \chi_2\} < 0$, where χ_1 and χ_2 are defined as in* (1.3) *and* (1.4), *respectively. If $\Theta(\infty, f) > 2/n$ and if $(f^n(f-1))^{(k)} - P_1$ and $(g^n(g-1))^{(k)} - P_1$ share* 0 IM, *where P_1 is a nonzero polynomial, then $f = g$.*

Regarding Theorems G and H, it is natural to ask the following questions which are the motive of the present author.

Question 4. Is it possible in any way to further reduce the lower bound of n in Theorems G and H?

Question 5. What can be said about the relationship between two transcendental meromorphic functions f and g if one replaces the differential polynomials $(f^n(f-1))^{(k)}$ and $(f^n(f-1)^m)^{(k)}$ by $(f^nP(f))^{(k)}$ in Theorems E–H where $P(z)$ is defined as in Theorem D?

In the paper, our main concern is to find the possible answer to the above questions. We prove two theorems which not only give a compact form of Theorems G and H, but at the same time improve and generalize them. We now state the main results of the paper.

Theorem 1. *Let f and g be two transcendental meromorphic functions, and let $n \geqslant 1$, $k \geqslant 1$, $m \geqslant 1$ and $l \geqslant 0$ be four integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Let $P(z)$ be defined as in Theorem* D. *Suppose that $(f^nP(f))^{(k)} - P_1$ and $(g^nP(g))^{(k)} - P_1$ share $(0, l)$, where P_1 is a nonzero polynomial. If $l_i \leqslant k$, and*

$$p + \Theta(\infty, f) + \sum_{i=1}^{p} \Theta_{[k/l_i])}(0, f - c_i) > 2 + \frac{2m(n+m)}{(n+m+2k)(n+m-2k)} \tag{1.5}$$

or

$$p + \Theta(\infty, g) + \sum_{i=1}^{p} \Theta_{[k/l_i])}(0, g - c_i) > 2 + \frac{2m(n+m)}{(n+m+2k)(n+m-2k)} \tag{1.6}$$

where p is the number of distinct roots of $P(z) = 0$, c_i is a zero of $P(z)$ of multiplicity l_i, $i = 1, 2, \ldots, p$, and one of $l \geqslant 2$, $n \geqslant 3k+m+8$; $l = 1$, $n \geqslant 4k + \frac{3}{2}m + 9$; $l = 0$, $n \geqslant 9k + 4m + 14$ is satisfied, then either $f = tg$ for a constant t such that $t^d = 1$, where $d = \gcd(n+m, \ldots, n+m-j, \ldots, n+1, n)$, $a_{m-j} \neq 0$ for some $j = 0, 1, \ldots, m$, or f, g satisfy the equation

$$f^nP(f) - g^nP(g) = 0.$$

In particular, $f = g$ when $m = 1$.

Theorem 2. *Let f and g be two transcendental meromorphic functions, and let $n \geqslant 1$, $k \geqslant 1$, $m \geqslant 1$ and $l \geqslant 0$ be four integers such that $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Let $P(z)$ be defined as in Theorem* D. *Suppose that $(f^nP(f))^{(k)} - P_1$ and $(g^nP(g))^{(k)} - P_1$ share $(0, l)$, where P_1 is a nonzero polynomial, $l_i > k$ for $i = 1, 2, \ldots, p$ and one of the following conditions holds:*

(a) $l \geqslant 2$ *and* $n \geqslant \max\{2k+3m, 3k+m+8\}$;
(b) $l = 1$ *and* $n \geqslant \max\{2k+3m, 4k+\frac{3}{2}m+9\}$;
(c) $l = 0$ *and* $n \geqslant \max\{2k+3m, 9k+4m+14\}$.

Then the conclusions of Theorem 1 *hold.*

Remark 1. If $P(z) = 0$ has only one root of multiplicity m then the inequalities (1.5) and (1.6) are the same as (1.1) and (1.2). In this case Theorems 1 and 2 improve Theorems F and E, respectively, by relaxing the nature of sharing.

Remark 2. Taking $P(z) = z - 1$ we see that Theorem 1 improves Theorem G by reducing the lower bound of n as well as by relaxing the nature of sharing. Theorem 1 also improves Theorem H by reducing the lower bound of n.

2. Lemmas

Let F and G be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by H the function

$$H = \Big(\frac{F''}{F'} - \frac{2F'}{F-1}\Big) - \Big(\frac{G''}{G'} - \frac{2G'}{G-1}\Big).$$

Lemma 1 ([17])**.** *Let f be a transcendental meromorphic function, and let $P_n(f)$ be a polynomial in f of the form*

$$P_n(f) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0,$$

where a_n $(\neq 0)$, $a_{n-1}, \ldots, a_1, a_0$ are complex numbers. Then

$$T(r, P_n(f)) = nT(r, f) + O(1).$$

Lemma 2 ([19])**.** *Let f be a nonconstant meromorphic function, and let p, k be positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f), \tag{2.1}$$

$$N_p(r, 0; f^{(k)}) \leqslant k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f). \tag{2.2}$$

Lemma 3 ([11]). *If $N(r,0;f^{(k)}|f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity, then*

$$N(r,0;f^{(k)}|f\neq 0)\leqslant k\overline{N}(r,\infty;f)+N(r,0;f|<k)+k\overline{N}(r,0;f|\geqslant k)+S(r,f).$$

Lemma 4 ([9]). *Let f and g be two nonconstant meromorphic functions sharing $(1,2)$. Then one of the following cases occurs:*

(i) $T(r)\leqslant N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)+S(r)$,

(ii) $f=g$,

(iii) $fg=1$.

Lemma 5 ([2]). *Let F and G be two nonconstant meromorphic functions sharing $(1,1)$ and let $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant{}& N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\&+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 6 ([2]). *Let F and G be two nonconstant meromorphic functions sharing $(1,0)$ and let $H\not\equiv 0$. Then*

$$\begin{aligned}T(r,F)\leqslant{}& N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\&+\overline{N}(r,0;G)+2\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,F)+S(r,G).\end{aligned}$$

Lemma 7. *Let f and g be two transcendental meromorphic functions, and let n, k, m be three positive integers. If $l_i>k$ and $n\geqslant 2k+3m$ or if $l_i\leqslant k$ and (1.5) or (1.6) holds, then*

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)}\not\equiv P_1^2,$$

where P_1 is a nonzero polynomial, $P(z)$ is defined as in Theorem D *and l_i's, $i=1,2,\dots,p$ are positive integers defined as in Theorem* 1.

P r o o f. We discuss the following two cases separately.

Case (i). Let $l_i>k$ for $i=1,2,\dots,p$. We may assume that

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)}=P_1^2. \tag{2.3}$$

We write $P(z)$ as

$$P(z) = a_m(z-c_1)^{l_1}(z-c_2)^{l_2}\dots(z-c_i)^{l_i}\dots(z-c_p)^{l_p},$$

where $\sum\limits_{i=1}^{p} l_i = m$, $1 \leqslant p \leqslant m$; $c_i \neq c_j$, $i \neq j$, $1 \leqslant i,j \leqslant p$; c_i's are nonzero constants and l_i's are positive integers, $i = 1,2,\dots,p$. Let $z_0 \notin \{z\colon\ P_1(z) = 0\}$ be a zero of f with multiplicity p_0 ($\geqslant 1$). Then it follows from (2.3) that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0 ($\geqslant 1$). Then we have

$$np_0 - k = (n+m)q_0 + k. \tag{2.4}$$

From (2.4) we get $mq_0 + 2k = n(p_0 - q_0) \geqslant n$, i.e., $q_0 \geqslant (n-2k)/m$. Thus from (2.4) we obtain $np_0 = (n+m)q_0 + 2k$, and so

$$p_0 \geqslant \frac{n+m-2k}{m}.$$

Let $z_1 \notin \{z\colon\ P_1(z) = 0\}$ be a zero of $P(f)$ with multiplicity p_1 and be a zero of $f - c_i$ of order r_i for $i = 1,2,\dots,p$. Then $p_1 = r_i l_i$ for $i = 1,2,\dots,p$. Since $l_i > k$, z_1 is a zero of $(f^nP(f))^{(k)}$ of multiplicity $r_i l_i - k$. Then (2.3) implies that z_1 is a pole of g with multiplicity q_1, say. Therefore from (2.3) we get

$$r_i l_i - k = (n+m)q_1 + k$$

i.e., $r_i \geqslant (n+m+2k)/l_i$ for $i = 1,2,\dots,p$. Let $z_2 \notin \{z\colon\ P_1(z) = 0\}$ be a zero of $(f^nP(f))^{(k)}$ of order p_2 that is not a zero of $f^nP(f)$. Then from (2.3) we see that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2. Then

$$p_2 = (n+m)q_2 + k \geqslant n+m+k.$$

Suppose that $z_3 \notin \{z\colon\ P_1(z) = 0\}$ is a pole of f. Then by virtue of (2.3), z_3 is either a zero of $g^nP(g)$ or a zero of $(g^nP(g))^{(k)}$. Therefore

$$\overline{N}(r,\infty;f) \leqslant \overline{N}(r,0;g) + \overline{N}(r,0;P(g)) + \overline{N}(r,0;h^{(k)}|h \neq 0) + S(r,g), \tag{2.5}$$

where $\overline{N}(r,0;h^{(k)}|h \neq 0)$ denotes the reduced counting function of those zeros of $h^{(k)}$ that are not zeros of h and $h = g^nP(g)$.

By Lemma 3 we have

$$\begin{aligned}\overline{N}(r,0;h^{(k)}|h \neq 0) &\leqslant \frac{1}{n+m+k}N(r,0;h^{(k)}|h \neq 0)\\ &\leqslant \frac{1}{n+m+k}(k\overline{N}(r,\infty;h) + N(r,0;h|<k) + k\overline{N}(r,0;h|\geqslant k))\\ &\leqslant \frac{1}{n+m+k}(k\overline{N}(r,\infty;h) + N_k(r,0;h))\\ &\leqslant \frac{k}{n+m+k}(\overline{N}(r,\infty;g) + \overline{N}(r,0;g) + \overline{N}(r,0;P(g))).\end{aligned}$$

So from (2.5) we obtain

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leqslant \frac{n+m+2k}{n+m+k}(\overline{N}(r,0;g)+\overline{N}(r,0;P(g)))\\
&\quad+\frac{k}{n+m+k}\overline{N}(r,\infty;g)+S(r,g)\\
&\leqslant \frac{n+m+2k}{n+m+k}\Big(\frac{m}{n+m-2k}+\frac{m}{n+m+2k}\Big)T(r,g)\\
&\quad+\frac{k}{n+m+k}\overline{N}(r,\infty;g)+S(r,g)\\
&\leqslant \Big(\frac{2m(n+m)}{(n+m+k)(n+m-2k)}+\frac{k}{n+m+k}\Big)T(r,g)+S(r,g).
\end{aligned}$$

Using the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}
pT(r,f) &\leqslant \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{i=1}^{p}\overline{N}(r,c_i;f)+S(r,f)\\
&\leqslant \Big(\frac{2m(n+m)}{(n+m+k)(n+m-2k)}+\frac{k}{n+m+k}\Big)T(r,g)\\
&\quad+\frac{2m(n+m)}{(n+m+2k)(n+m-2k)}T(r,f)+S(r,f)+S(r,g).
\end{aligned}\tag{2.6}$$

Similarly,

$$\begin{aligned}
pT(r,g) &\leqslant \Big(\frac{2m(n+m)}{(n+m+k)(n+m-2k)}+\frac{k}{n+m+k}\Big)T(r,f)\\
&\quad+\frac{2m(n+m)}{(n+m+2k)(n+m-2k)}T(r,g)+S(r,f)+S(r,g).
\end{aligned}\tag{2.7}$$

From (2.6) and (2.7) we obtain

$$\begin{aligned}
&\Big(p-\frac{k}{n+m+k}-\frac{2m(n+m)}{(n+m+k)(n+m-2k)}\\
&\quad-\frac{2m(n+m)}{(n+m+2k)(n+m-2k)}\Big)(T(r,f)+T(r,g))\leqslant S(r,f)+S(r,g).
\end{aligned}\tag{2.8}$$

Since $n\geqslant 2k+3m$, a simple calculation shows that

$$p-\frac{k}{n+m+k}-\frac{2m(n+m)}{(n+m+k)(n+m-2k)}-\frac{2m(n+m)}{(n+m+2k)(n+m-2k)}>0,$$

which contradicts (2.8).

Case (ii). Let $l_i \leqslant k$ for $i = 1, 2, \ldots, p$. Let $z_4 \notin \{z\colon\ P_1(z) = 0\}$ be a zero of $P(f)$ with multiplicity p_4 and a zero of $f - c_i$ of order $r_i \geqslant [k/l_i] + 1$ for $i = 1, 2, \ldots, p$. Then z_4 is a zero of $(f^n P(f))^{(k)}$ of multiplicity $r_i l_i - k$ $(\geqslant 1)$. Then (2.3) implies that z_4 is a pole of g. Suppose that z_4 is a pole of g of order q_4 $(\geqslant 1)$. Thus we obtain

$$r_i \geqslant \frac{n+m+2k}{l_i}$$

for $i = 1, 2, \ldots, p$. Thus

$$\begin{aligned}
\overline{N}(r, 0; f - c_i) &\leqslant \overline{N}\Big(r, 0; f - c_i| \leqslant \Big[\frac{k}{l_i}\Big]\Big) + \overline{N}\Big(r, 0; f - c_i| \geqslant \Big[\frac{k}{l_i}\Big] + 1\Big) \\
&\leqslant \overline{N}\Big(r, 0; f - c_i| \leqslant \Big[\frac{k}{l_i}\Big]\Big) + \frac{l_i}{n+m+2k} N\Big(r, 0; f - c_i| \geqslant \Big[\frac{k}{l_i}\Big] + 1\Big).
\end{aligned}$$

Then by Nevanlinna's second fundamental theorem, we obtain

$$\begin{aligned}
pT(r, f) &\leqslant \overline{N}(r, \infty; f) + \overline{N}(r, 0; f) + \sum_{i=1}^{p} \overline{N}(r, c_i; f) + S(r, f) \\
&\leqslant \overline{N}(r, \infty; f) + \frac{m}{n+m-2k} N(r, 0; f) + \sum_{i=1}^{p} \overline{N}\Big(r, 0; f - c_i| \leqslant \Big[\frac{k}{l_i}\Big]\Big) \\
&\quad + \sum_{i=1}^{p} \frac{l_i}{n+m+2k} N\Big(r, 0; f - c_i| \geqslant \Big[\frac{k}{l_i}\Big] + 1\Big) + S(r, f).
\end{aligned}$$

This gives

$$\Big(p + \Theta(\infty, f) + \sum_{i=1}^{p} \Theta_{[k/l_i])}(0, f - c_i) - 2 - \frac{2m(n+m)}{(n+m+2k)(n+m-2k)}\Big) T(r, f) \leqslant S(r, f),$$

which contradicts the assumption (1.5). This proves the lemma. □

Lemma 8. *Let f and g be two nonconstant meromorphic functions such that*

$$\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n},$$

where n $(\geqslant 3)$ is an integer. Then

$$f^n(af + b) = g^n(ag + b)$$

implies $f = g$, where a, b are any two nonzero finite complex constants.

P r o o f. We omit the proof since it can be carried out along the lines of the proof of Lemma 6 in [7]. □

The following lemma can be proved in the same manner as Lemma 2.14 in [15].

Lemma 9. *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers. Suppose that $F=(f^nP(f))^{(k)}/P_1$ and $G=(g^nP(g))^{(k)}/P_1$ where P_1 is a nonzero polynomial. If there exist two nonzero constants d_1 and d_2 such that $\overline{N}(r,d_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,d_2;G)=\overline{N}(r,0;F)$, then $n\leqslant 3k+m+3$.*

Lemma 10 ([15])**.** *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers. Suppose that $F_1=(f^nP(f))^{(k)}$ and $G_1=(g^nP(g))^{(k)}$. If there exist two nonzero constants d_3 and d_4 such that $\overline{N}(r,d_3;F_1)=\overline{N}(r,0;G_1)$ and $\overline{N}(r,d_4;G_1)=\overline{N}(r,0;F_1)$, then $n\leqslant 3k+m+3$.*

3. Proof of the theorem

Proof of Theorem 2. Let F and G be defined as in Lemma 9. Then F, G are transcendental meromorphic functions that share $(1,l)$. Then from (2.1) we obtain

$$\begin{aligned}(3.1)\quad & N_2(r,0;F)\\ &\leqslant N_2(r,0;(f^nP(f))^{(k)})+S(r,f)\\ &\leqslant T(r,(f^nP(f))^{(k)})-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))+S(r,f)\\ &\leqslant T(r,F)-(n+m)T(r,f)+N_{k+2}(r,0;f^nP(f))+O\{\log r\}+S(r,f).\end{aligned}$$

Again by (2.2) we have

$$N_2(r,0;F)\leqslant k\overline{N}(r,\infty;f)+N_{k+2}(r,0;f^nP(f))+S(r,f). \tag{3.2}$$

Therefore from (3.1) we get

$$\begin{aligned}(n+m)T(r,f)\leqslant T(r,F)&+N_{k+2}(r,0;f^nP(f))\\ &-N_2(r,0;F)+O\{\log r\}+S(r,f).\end{aligned}\tag{3.3}$$

We now discuss the following three cases separately.

Case 1. Let $l\geqslant 2$. We assume that (i) of Lemma 4 holds. Then using (3.2) we obtain from (3.3)

$$\begin{aligned}(n+m)T(r,f)&\leqslant N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\ &\quad+N_{k+2}(r,0;f^nP(f))+O\{\log r\}+S(r,f)+S(r,g)\\ &\leqslant N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+2\overline{N}(r,\infty;f)\\ &\quad+(k+2)\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g)\end{aligned}\tag{3.4}$$

$$
\begin{aligned}
&\leqslant (k+m+2)(T(r,f)+T(r,g)) + 2\overline{N}(r,\infty;f) \\
&\quad + (k+2)\overline{N}(r,\infty;g) + O\{\log r\} + S(r,f) + S(r,g) \\
&\leqslant (k+m+4-2\Theta(\infty,f)+\varepsilon)T(r,f) + \big(2k+m+4 \\
&\quad - (k+2)\Theta(\infty,g)+\varepsilon\big)T(r,g) + S(r,f) + S(r,g) \\
&\leqslant \big(3k+2m+8-2\Theta(\infty,f)-2\Theta(\infty,g) \\
&\quad - k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\varepsilon\big)T(r) + S(r).
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(n+m)T(r,g) \leqslant \big(&3k+2m+8-2\Theta(\infty,f)-2\Theta(\infty,g) \\
&- k\min\{\Theta(\infty,f),\Theta(\infty,g)\} + 2\varepsilon\big)T(r) + S(r).
\end{aligned}
\tag{3.5}
$$

From (3.4) and (3.5) we obtain

$$
(n-3k-m-8+2\Theta(\infty,f)+2\Theta(\infty,g)+k\min\{\Theta(\infty,f),\Theta(\infty,g)\}-2\varepsilon)T(r) \leqslant S(r),
$$

contradicting the fact that $n \geqslant \max\{2k+3m, 3k+m+8\}$, $\Theta(\infty,f)+\Theta(\infty,g) > 4/n$ and $\varepsilon > 0$ is arbitrary. Therefore by Lemma 4 and Lemma 7 we conclude that $F = G$. Then

$$
(f^nP(f))^{(k)} = (g^nP(g))^{(k)}.
$$

Integrating both sides we obtain

$$
(f^nP(f))^{(k-1)} = (g^nP(g))^{(k-1)} + d_{k-1},
$$

where d_{k-1} is a constant. We assume that $d_{k-1} \neq 0$. Then from Lemma 10 we obtain $n \leqslant 3k+m$, a contradiction. Hence $d_{k-1} = 0$. Repeating k-times, we obtain

$$
f^nP(f) = g^nP(g). \tag{3.6}
$$

Let $h = f/g$. If h is a constant, by putting $f = gh$ in (3.6) we get

$$
\begin{aligned}
a_mg^{n+m}(h^{n+m}-1) + a_{m-1}g^{n+m-1}(h^{n+m-1}-1) + \dots \\
+ a_1g^{n+1}(h^{n+1}-1) + a_0g^n(h^n-1) = 0,
\end{aligned}
$$

which implies $h^d = 1$, where $d = \gcd(n+m,\dots,n+m-j,\dots,n+1,n)$ for some $j = 0,1,\dots,m$. Thus $f = tg$ for a constant t such that $t^d = 1$, $d = \gcd(n+m,\dots,n+m-j,\dots,n+1,n)$, for some $j = 0,1,\dots,m$.

If h is not a constant, then from (3.6) we see that f and g satisfy the algebraic equation $R(f,g)=0$, where

$$R(f,g)=f^nP(f)-g^nP(g).$$

Case 2. Let $l=1$ and $H\not\equiv 0$. Using Lemma 5 and (3.2) we obtain from (3.3)

$$\begin{aligned}(3.7)\quad (n+m)T(r,f)\\
&\leqslant N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\\
&\quad+\frac{1}{2}\overline{N}(r,0;F)+\frac{1}{2}\overline{N}(r,\infty;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))\\
&\quad+\frac{1}{2}N_{k+1}(r,0;f^nP(f))+\frac{k+5}{2}\overline{N}(r,\infty;f)\\
&\quad+(k+2)\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant\Bigl(2k+\frac{3m}{2}+5-\Bigl(\frac{k}{2}+2\Bigr)\Theta(\infty,f)-\frac{1}{2}\Theta(\infty,f)+\varepsilon\Bigr)T(r,f)\\
&\quad+\Bigl(2k+m+4-\Bigl(\frac{k}{2}+2\Bigr)\Theta(\infty,g)-\frac{k}{2}\Theta(\infty,g)+\varepsilon\Bigr)T(r,g)\\
&\quad+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant\Bigl(4k+\frac{5m}{2}+9-\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\varepsilon\Bigr)T(r)+S(r).\end{aligned}$$

Similarly

$$(3.8)\quad (n+m)T(r,g)\leqslant\Bigl(4k+\frac{5m}{2}+9-\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))+2\varepsilon\Bigr)T(r)+S(r).$$

From (3.7) and (3.8) we obtain

$$\Bigl(n-4k-\frac{3m}{2}-9+\frac{k+5}{2}(\Theta(\infty,f)+\Theta(\infty,g))-2\varepsilon\Bigr)T(r)\leqslant S(r),$$

a contradiction since $n\geqslant\max\{2k+3m,4k+\frac{3}{2}m+9\}$, $\Theta(\infty,f)+\Theta(\infty,g)>4/n$ and $\varepsilon>0$ is arbitrary. Therefore $H=0$. That is,

$$\Bigl(\frac{F''}{F'}-\frac{2F'}{F-1}\Bigr)-\Bigl(\frac{G''}{G'}-\frac{2G'}{G-1}\Bigr)=0.$$

Integrating both sides of the above equality twice we get

$$(3.9)\qquad \frac{1}{F-1}=\frac{A}{G-1}+B,$$

where A ($\neq 0$) and B are constants. From (3.9) it is clear that F, G share the value 1 CM and so they share $(1,2)$. Hence we have $n \geqslant \max\{2k+3m, 3k+m+8\}$. Now we discuss the following three subcases separately.

Subcase 1. Let $B \neq 0$ and $A = B$. Then from (3.9) we get

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.10}$$

If $B = -1$, then from (3.10) we get $FG = 1$, a contradiction by Lemma 7.

If $B \neq -1$, from (3.10) we obtain $1/F = BG/((1+B)G-1)$ and so

$$\overline{N}\Big(r, \frac{1}{1+B}; G\Big) = \overline{N}(r,0;F).$$

Using the second fundamental theorem of Nevanlinna, we obtain

$$\begin{aligned} T(r,G) &\leqslant \overline{N}(r,0;G) + \overline{N}\Big(r, \frac{1}{1+B}; G\Big) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leqslant \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G). \end{aligned}$$

Using (2.1) and (2.2) we obtain from the above inequality

$$\begin{aligned} T(r,G) \leqslant\ & N_{k+1}(r,0;f^nP(f)) + k\overline{N}(r,\infty;f) + T(r,G) \\ & + N_{k+1}(r,0;g^nP(g)) - (n+m)T(r,g) + \overline{N}(r,\infty;g) + S(r,g). \end{aligned}$$

Hence

$$(n+m)T(r,g) \leqslant (2k+m+1)T(r,f) + (k+m+2)T(r,g) + S(r,g).$$

This gives

$$(n-3k-m-3)(T(r,f)+T(r,g)) \leqslant S(r,f) + S(r,g),$$

a contradiction as $n \geqslant \max\{2k+3m, 3k+m+8\}$.

Subcase 2. Let $B \neq 0$ and $A \neq B$. Then from (3.9) we get $F = ((B+1)G - (B-A+1))/(BG + (A-B))$ and so $\overline{N}(r,(B-A+1)/(B+1);G) = \overline{N}(r,0;F)$. Proceeding similarly to Subcase 1 we obtain a contradiction.

Subcase 3. Let $B = 0$ and $A \neq 0$. Then from (3.9) we get $F = (G+A-1)/A$ and $G = AF-(A-1)$. If $A \neq 1$ then $\overline{N}(r,(A-1)/A;F) = \overline{N}(r,0;G)$ and $\overline{N}(r,1-A;G) = \overline{N}(r,0;F)$. So by Lemma 9 we have $n \leqslant 3k+m+3$, a contradiction. Thus $A = 1$ and hence $F = G$. Then the result follows from Case 1.

Case 3. Let $l=0$ and $H \not\equiv 0$. Using Lemma 6 and (3.2) we obtain from (3.3)

$$
\begin{aligned}
(3.11)\quad (n+m)T(r,f) &\\
&\leqslant N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)+2\overline{N}(r,0;F)\\
&\quad+\overline{N}(r,0;G)+N_{k+2}(r,0;f^nP(f))+2\overline{N}(r,\infty;F)\\
&\quad+\overline{N}(r,\infty;G)+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant N_{k+2}(r,0;f^nP(f))+N_{k+2}(r,0;g^nP(g))+2N_{k+1}(r,0;f^nP(f))\\
&\quad+N_{k+1}(r,0;g^nP(g))+(2k+4)\overline{N}(r,\infty;f)\\
&\quad+(2k+3)\overline{N}(r,\infty;g)+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant \bigl(5k+3m+8-(2k+4)\Theta(\infty,f)-\varepsilon\bigr)T(r,f)\\
&\quad+\bigl(4k+2m+6-(2k+3)\Theta(\infty,g)-\varepsilon\bigr)T(r,g)\\
&\quad+O\{\log r\}+S(r,f)+S(r,g)\\
&\leqslant \bigl(9k+5m+14-(2k+3)(\Theta(\infty,f)+\Theta(\infty,g))\\
&\quad-\min\{\Theta(\infty,f),\Theta(\infty,g)\}+2\varepsilon\bigr)T(r)+S(r).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(3.12)\qquad (n+m)T(r,g)\leqslant (9k+5m+14&-(2k+3)(\Theta(\infty,f)+\Theta(\infty,g))\\
&-\min\{\Theta(\infty,f),\Theta(\infty,g)\}+2\varepsilon)T(r)+S(r).
\end{aligned}
$$

Combining (3.11) and (3.12) we obtain

$$
\begin{aligned}
\bigl(n-9k-4m-14&+(2k+3)(\Theta(\infty,f)+\Theta(\infty,g))\\
&+\min\{\Theta(\infty,f),\Theta(\infty,g)\}-2\varepsilon\bigr)T(r)\leqslant S(r),
\end{aligned}
$$

which contradicts the fact that

$$
n\geqslant \max\{2k+3m,9k+4m+14\},\quad \Theta(\infty,f)+\Theta(\infty,g)>4/n
$$

and $\varepsilon>0$ is arbitrary. Therefore $H=0$ and then proceeding in the same manner as in Case 2 the result follows.

This completes the proof of the theorem. □

Proof of Theorem 1. Proceeding along the lines of the proof of Theorem 2 and using the case $l_i \leqslant k$ of Lemma 7 and Lemma 8 we can easily deduce the conclusions of Theorem 1. We omit the details here. □

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