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ON RELIABILITY ANALYSIS OF CONSECUTIVE k-OUT-OF-n SYSTEMS WITH ARBITRARILY DEPENDENT COMPONENTS

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Abstract. In this paper, we consider the linear and circular consecutive k-out-of-n systems consisting of arbitrarily dependent components. Under the condition that at least $n-r+1$ components ($r \leqslant n$) of the system are working at time t, we study the reliability properties of the residual lifetime of such systems. Also, we present some stochastic ordering properties of residual lifetime of consecutive k-out-of-n systems. In the following, we investigate the inactivity time of the component with lifetime $T_{r:n}$ at the system level for the consecutive k-out-of-n systems under the condition that the system is not working at time $t > 0$, and obtain some stochastic properties of this conditional random variable.

Keywords: residual lifetime; inactivity time; stochastic order; dependence; reliability

MSC 2010: 62N05, 60E15, 60E05, 60K10

1. Introduction

Consecutive systems, especially consecutive k-out-of-n systems, are among the most prominent coherent structures which are of great importance in various engineering industries. Linear and circular k-out-of-n are two types of consecutive k-out-of-n systems which are the most commonly used. Linear consecutive k-out-of-n:G (denoted Lin/Con/k/n:G) system is a system composed of n units in which the units are interconnected in a line and the system starts operating when k or more consecutive units are operating. Correspondingly, a linear consecutive k-out-of-n:F (denoted Lin/Con/k/n:F) system stops operating if and only if at least k consecutive units stop operating. Circular consecutive k-out-of-n:G (denoted Cir/Con/k/n:G) system is a type of system consisting of n units in which the units are interconnected in a circle and it starts operating if at least k consecutive units operate. In the same manner, a circular consecutive k-out-of-n:F (denoted Cir/Con/k/n:F) system stops operating if and only if at least k consecutive units stop operating. Numerous authors have conducted studies on these types of systems, among them we can refer to Chen and Hwang [8], Shanthikumar [34], Aki and Hirano [1], Boland and Samaniego [6], Eryılmaz [9], Navarro and Eryılmaz [21], Eryılmaz [10] and Sfakianakis and Papastavridis [32]; to have a comprehensive source on consecutive systems refer to Chang et al. [7], and Kuo and Zuo [17].

During the last several decades a strong interest has been shown in conducting studies on conditional reliability properties of coherent systems, especially the residual lifetime and the inactivity time of systems such as consecutive systems. Most results in the articles cited above have presumed a situation in which the components of the system are independent, or independent and identical. Bairamov et al. in [4] conducted a study on the mean residual lifetime of parallel systems under the condition that all independent and identical components were operating at time t. Asadi and Bayramoglu in [3] considered the mean residual lifetime of k-out-of-n systems including n independent and identical components for their study and presented some comparison results. Kochar and Xu in [15] generalized the results gained by Asadi and Bayramoglu in [3] for a situation in which the components are independent but non-identical. In order to gain more information on the studies conducted in this field, we refer the reader to Khaledi and Shaked [14], Navarro et al. [20], Li and Zhang [18], Li and Zhao [19], Zhao et al. [38]. Asadi in [2] considered the inactivity time of the components of a parallel system and presented the concept of mean inactivity time of the components. The extension of the results of [2] for the case when the components are independent and non-identical is obtained by Sadegh in [26]. Tavangar and Asadi in [35] expanded the results of [2] for an $(n-k+1)$-out-of-n system and defined the mean inactivity time of the components at the system level. In this regard, we can refer to Khaledi and Shaked [14], Zhao et al. [38], and Salehi and Asadi [28].

Moreover, some attempts have been made to explore the conditional reliability properties of consecutive k-out-of-n systems. Eryılmaz in [11] studied the residual lifetime of linear consecutive k-out-of-n systems with independent and identical components. Salehi et al. in [31] considered the residual lifetime of linear and circular consecutive k-out-of-n systems with independent and identical components, and then presented some results of stochastic comparisons for such systems. A study on the distribution of the residual lifetime of linear and circular k-out-of-n systems with independent and non-identical components was conducted by Salehi et al. in [30] according to permanent concepts. Recently, Salehi et al. in [29] have considered a new version of the conditional reliability of linear and circular k-out-of-n systems with independent and identical components, and have presented some significant results. However, in reality we are facing a situation in which there are dependencies between components of the system. We can have a more realistic assumption in comparison to the assumptions in the previous studies. The study of such systems with dependent components have been taken into consideration by researchers and engineers lately, see, e.g., Navarro et al. [23], Navarro and Spizzichino [24], Jia et al. [13], Zhang [37], Eryılmaz [12], Navarro and Rubio [22], Sadegh [27], Rezapour et al. [25], Tavangar and Asadi [36], Koutras et al. [16], and Bairamov et al. [5].

The variable of the residual lifetime of a system has been studied by numerous researchers and authors recently. This variable might be of prominent importance for engineers and system designers in order to conduct some maintenance procedures. Suppose that T is the lifetime of a coherent system which includes n components with lifetimes $T_1, T_2, \ldots, T_n$. Denote by $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ the ordered lifetimes of the components. The conditional variable of the residual lifetime of this system, which is shown by $\{T - t \mid T_{r:n} > t\}$, is in fact the residual lifetime of the system under the condition that at least $n-r+1$ components are operating at time t. Another important concept in reliability theory of systems is the inactivity time that has been also considered by many researchers. The inactivity time of the components at the system level is defined as $\{t - T_{r:n} \mid T \leqslant t\}$. This conditional random variable shows, in fact, the time elapsed from the failure of $T_{r:n}$ given that the system has failed at or before $t > 0$. In this paper we study the residual lifetime and inactivity time of linear and circular consecutive k-out-of-n systems when the lifetimes of components are arbitrarily dependent.

The present paper is organized as follows. In Section 2 some concepts and tools, which are required to get the main results of the paper, are presented. Moreover, in this section the form of the lifetime reliability function of linear and circular consecutive k-out-of-n systems with arbitrary components is also presented. The beginning of Section 3 is dedicated to the presentation of two important lemmas for obtaining an explicit formula of the reliability function of the residual lifetime and inactivity time for linear and circular consecutive k-out-of-n systems. Subsections 3.1 and 3.2 are dedicated to stochastic ordering properties of the residual lifetime and inactivity time, respectively, of linear and circular k-out-of-n systems consisting of arbitrarily dependent components with illustrative examples.

Notation.

n	the number of components
T_i	the lifetime of the ith component, $1 \leqslant i \leqslant n$
$(T_i)_t$	the residual lifetime of T_i at time t, i.e., $\{T_i - t \mid T_i > t\}$
$T_{r:n}$	rth smallest among $T_1, T_2, \ldots, T_n$, $1 \leqslant r \leqslant n$
$T_{k\vert n:\mathrm{G}}$	the lifetime of the Lin/Con/k/n:G system
$T_{k\vert n:\mathrm{F}}$	the lifetime of the Lin/Con/k/n:F system
$T^C_{k\vert n:\mathrm{G}}$	the lifetime of the Cir/Con/k/n:G system

$T^C_{k\|n:\mathrm{F}}$	the lifetime of the Cir/Con/k/n:F system
$T_{[i:m]}$	$\min\{T_i,\ldots,T_m\}$ for $i \leqslant m$
$T^{[i:m]}$	$\max\{T_i,\ldots,T_m\}$ for $i \leqslant m$
$F_{r:n}$ ($\overline{F}_{r:n}$)	the distribution (reliability) function of $T_{r:n}$
$F(t,\ldots,t)$ ($\overline{F}(t,\ldots,t)$)	the joint distribution (joint reliability) function of $(T_1,T_2,\ldots,T_n)$
$M_{1:k}(t)$	the mean residual lifetime of a series system with k components

2. Preliminaries

Let $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ be a random vector and assume that $\mathbf{T}$ has an arbitrary joint distribution function $F(t_1, t_2, \ldots, t_n)$ with corresponding joint reliability (or survival) function $\overline{F}(t_1, t_2, \ldots, t_n)$. Also, let $T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ denote the order statistics corresponding to T_i's. The reliability function of rth order statistic, i.e. $T_{r:n}$, denoted $\overline{F}_{r:n}(t)$, is equal to

$$\begin{aligned}\overline{F}_{r:n}(t) &= P(T_{r:n} > t)\\ &= P(\text{at least } n-r+1 \text{ of the } T_i\text{'s are greater than } t)\\ &= \sum_{j=n-r+1}^{n} \sum_{\pi\in C_j} P\{B_j^{(t,\pi)}\},\end{aligned} \tag{2.1}$$

where $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$, $B_j^{(t,\pi)}$ is the event

$$\{T_{\pi_1} > t, T_{\pi_2} > t, \ldots, T_{\pi_j} > t, T_{\pi_{j+1}} \leqslant t, \ldots, T_{\pi_n} \leqslant t\}, \tag{2.2}$$

and C_j is the set of all permutations $\{\pi_1, \pi_2, \ldots, \pi_n\}$ of $\{1, 2, \ldots, n\}$ for which $1 \leqslant \pi_1 < \ldots < \pi_j \leqslant n$ and $1 \leqslant \pi_{j+1} < \ldots < \pi_n \leqslant n$.

Remark 2.1. If $T_1, T_2, \ldots, T_n$ are exchangeable lifetimes of the components, the representation (2.1) is reduced to

$$\overline{F}_{r:n}(t) = \sum_{j=n-r+1}^{n} \binom{n}{j} P\{B_j^{(t)}\},$$

where

$$B_j^{(t)} = \{T_1 > t, T_2 > t, \ldots, T_j > t, T_{j+1} \leqslant t, \ldots, T_n \leqslant t\}.$$

If $T_1, T_2, \ldots, T_n$ are the indicators of arbitrary components' lifetimes which are interconnected in a Lin/Con/k/n:G structure, supposing that the lifetime of a Lin/Con/k/n:G system is shown by $T_{k|n:G}$, Eryılmaz in [10] indicated that for $2k \geqslant n$, the reliability function of the system equals

$$(2.3)\qquad P(T_{k|n:\mathrm{G}} > t) = \sum_{i=k}^{n}[P(T_{[i-k+1:i]} > t) - P(T_{[i-k+1:i+1]} > t)],$$

where $T_{[i:m]} = \min\{T_i, T_{i+1}, \ldots, T_m\}$ for $1 \leqslant i < m \leqslant n$. Since a Lin/Con/$k$/$n$:F system is the dual of a Lin/Con/k/n:G system, we have

$$(2.4)\qquad P(T_{k|n:\mathrm{F}} > t) = 1 - \sum_{i=k}^{n}[P(T^{[i-k+1:i]} \leqslant t) - P(T^{[i-k+1:i+1]} \leqslant t)],$$

where $T^{[i:m]} = \max\{T_i, T_{i+1}, \ldots, T_m\}$ for $1 \leqslant i < m \leqslant n$. For convenience, in equations (2.3) and (2.4), $P(T_{[n-k+1:n+1]} > t) = P(T^{[n-k+1:n+1]} \leqslant t) = 0$. It is worth mentioning that these representations do not impose any assumption on the lifetime of the system components.

If we denote the lifetime of the Cir/Con/k/n:F system by $T^{C}_{k|n:\mathrm{F}}$, Sfakianakis and Papastavridis in [32] obtained the survival function of the Cir/Con/k/n:F system consisting of arbitrarily dependent components for $2k+1 \geqslant n$ as follows:

$$(2.5)\quad P(T^{C}_{k|n:\mathrm{F}} > t) = 1 - P(T^{[1:n]} \leqslant t) - \sum_{i=1}^{n}[P(T^{[i+1:i+k]} \leqslant t) - P(T^{[i:i+k]} \leqslant t)],$$

where $T^{[i:i+k]} = \max\{T_i, T_{i+1}, \ldots, T_{i+k}\}$ for $1 \leqslant i \leqslant n$ and for convenience we take $T_{i+n} = T_i$, $i = 1, 2, \ldots, k$. Denoting by $T^{C}_{k|n:\mathrm{G}}$ the lifetime of the Cir/Con/k/n:G system, one can verify that the survival function of a Cir/Con/k/n:G system is equal to

$$(2.6)\qquad P(T^{C}_{k|n:\mathrm{G}} > t) = P(T_{[1:n]} > t) + \sum_{i=1}^{n}[P(T_{[i+1:i+k]} > t) - P(T_{[i:i+k]} > t)],$$

where $T_{[i:i+k]} = \min\{T_i, T_{i+1}, \ldots, T_{i+k}\}$ for $1 \leqslant i \leqslant n$.

In the following, we present the concept of usual stochastic order, and for more details we refer the reader to Shaked and Shanthikumar [33].

Definition 2.1. Let X and Y be random variables with survival functions $\overline{F}$ and $\overline{G}$, respectively. X is said to be smaller than Y in the usual stochastic order (denoted by $X \leqslant_{\mathrm{st}} Y$) if $\overline{F}(t) \leqslant \overline{G}(t)$ for all t.

Theorem 2.1. *Let $\{X_1, X_2, \ldots\}$ and $\{Y_1, Y_2, \ldots\}$ be two sequences of (not necessarily independent) random variables such that $(X_1, X_2, \ldots, X_k) \leqslant_{\mathrm{st}} (Y_1, Y_2, \ldots, Y_k)$, $k \geqslant 1$. Then $X_{i:m} \leqslant_{\mathrm{st}} Y_{j:n}$, whenever $i \leqslant j$ and $m - i \geqslant n - j$.*

Lemma 2.1. *Let $\{X_1, X_2, \ldots\}$ be a sequence of (not necessarily independent) random variables. Then $X_{i:m} \leqslant_{\rm st} X_{j:n}$, whenever $i \leqslant j$ and $m - i \geqslant n - j$.*

3. Main results

This part begins with two lemmas which are required for the extraction of the main results.

Lemma 3.1. *Let $T_1, T_2, \ldots, T_n$ be arbitrarily dependent lifetimes of n components with joint distribution $F(t_1, t_2, \ldots, t_n)$. Then, for $x, t > 0$, the joint survival function of $T_{[l:l+k-1]}$, $T_{r:n}$ is*

$$(3.1)\qquad \begin{aligned} \varphi^l_{k,r,n}(x,t) &= P(T_{[l:l+k-1]} > x + t, T_{r:n} > t) \\ &= \sum_{j=\max\{k,n-r+1\}}^{n} \sum_{\pi \in C_{j,l}} P(B_j^{(t,\pi)}) \sum_{m=0}^{j-k} \sum_{s \in C_{m(j),l}} p^l_{m,j}(x,t), \end{aligned}$$

where

$$\begin{aligned} p^l_{m,j}(x,t) = P(&T_l - t > x, \ldots, T_{l+k-1} - t > x, T_{s_1} - t \leqslant x, \ldots, \\ &T_{s_m} - t \leqslant x,\ T_{s_{m+1}} - t > x, \ldots, T_{s_{j-k}} - t > x \mid B_j^{(t,\pi)}), \end{aligned}$$

$T_{[l:l+k-1]}$ is the minimum of $T_l, T_{l+1}, \ldots, T_{l+k-1}$, $C_{j,l}$ is the set of all permutations $\{\pi_1, \pi_2, \ldots, \pi_{n-k}\}$ of $\{1, 2, \ldots, l-1, l+k, \ldots, n\}$, and the summation on $C_{m(j),l}$ extends over all permutations $s_1, s_2, \ldots, s_{j-k}$ of $\{\pi_1, \pi_2, \ldots, \pi_{j-k}\}$ for which $1 \leqslant s_1 < s_2 < \ldots < s_m \leqslant n$ and $1 \leqslant s_{m+1} < s_{m+2} < \ldots < s_{j-k} \leqslant n$.

P r o o f. For all $x, t > 0$,

$$\begin{aligned} &P(T_{[l:l+k-1]} > x + t, T_{r:n} > t) = P(T_l > x + t, \ldots, T_{l+k-1} > x + t, T_{r:n} > t) \\ &= \sum_{j=\max\{k,n-r+1\}}^{n} P(T_l > x + t, \ldots, T_{l+k-1} > x + t, \\ &\qquad \text{exactly } j - k \text{ of } T_1, \ldots, T_{l-1}, T_{l+k}, \ldots, T_n \text{ are greater than } t) \\ &= \sum_{j=\max\{k,n-r+1\}}^{n} \sum_{m=0}^{j-k} \sum_{\pi \in C_{j,l}} \sum_{s \in C_{m(j),l}} \\ &\qquad P(T_l > x + t, \ldots, T_{l+k-1} > x + t, t < T_{s_1} \leqslant x + t, \ldots, t < T_{s_m} \leqslant x + t, \\ &\qquad\qquad T_{s_{m+1}} > x + t, \ldots, T_{s_{j-k}} > x + t, T_{\pi_{j+1}} \leqslant t, \ldots, T_{\pi_n} \leqslant t) \end{aligned}$$

$$= \sum_{j=\max\{k,n-r+1\}}^{n} \sum_{m=0}^{j-k} \sum_{\pi\in C_{j,l}} \sum_{s\in C_{m(j),l}} P(B_j^{(t,\pi)}) P(T_l - t > x, \dots, T_{l+k-1} - t > x,$$
$$T_{s_1} - t \leqslant x, \dots, T_{s_m} - t \leqslant x, T_{s_{m+1}} - t > x, \dots, T_{s_{j-k}} - t > x \mid B_j^{(t,\pi)}).$$

Therefore, the proof is complete. □

Lemma 3.2. *Let $T_1, T_2, \dots, T_n$ be arbitrarily dependent lifetimes of n components with joint distribution $F(t_1, t_2, \dots, t_n)$. Then, for $x, t > 0$, the joint distribution function of $T^{[l:l+k-1]}$, $T_{r:n}$ is*

$$\begin{aligned} \phi^l_{k,r,n}(x,t) &= P(T^{[l:l+k-1]} \leqslant x+t,\ T_{r:n} \leqslant t) \\ &= \sum_{j=r}^{n} \sum_{m=\max\{0,k-j\}}^{\min\{k,n-j\}} \sum_{\pi\in C_{j,l}} \sum_{s\in C_{m(j),l}} q^l_{m,j}(x,t), \end{aligned} \tag{3.2}$$

where

$$q^l_{m,j}(x,t) = P(t < T_{s_1} \leqslant x+t, \dots, t < T_{s_m} \leqslant x+t, T_{s_{m+1}} \leqslant t, \dots, T_{s_k} \leqslant t,$$
$$T_{\pi_{k+1}} \leqslant t, \dots, T_{\pi_{j+m}} \leqslant t, T_{\pi_{j+m+1}} > t, \dots, T_{\pi_n} > t),$$

$T^{[l:l+k-1]}$ is the maximum of $T_l, T_{l+1}, \dots, T_{l+k-1}$, $C_{j,l}$ is the set of all permutations $\{\pi_{k+1}, \pi_{k+2}, \dots, \pi_n\}$ of $\{1, 2, \dots, l-1, l+k, \dots, n\}$, and the summation on $C_{m(j),l}$ extends over all permutations $s_1, s_2, \dots, s_k$ of $\{l, l+1, \dots, l+k-1\}$ for which $l \leqslant s_1 < s_2 < \dots < s_m \leqslant l+k-1$ and $l \leqslant s_{m+1} < s_{m+2} < \dots < s_{j-k} \leqslant l+k-1$.

Proof. For all $x, t > 0$,

$$\begin{aligned} &P(T^{[l:l+k-1]} \leqslant x+t,\ T_{r:n} \leqslant t) = P(T_l \leqslant x+t, \dots, T_{l+k-1} \leqslant x+t, T_{r:n} \leqslant t) \\ &= \sum_{j=r}^{n} P(T_l \leqslant x+t, \dots, T_{l+k-1} \leqslant x+t,\ \text{exactly } j \text{ of } n\ T\text{'s are less than } t). \end{aligned}$$

Hence, after some simplification, the proof is complete. □

Remark 3.1. If $T_1, T_2, \dots, T_n$ are the exchangeable random variables of the components' lifetimes, then the expressions (3.1) and (3.2) reduce to the result of Eryılmaz, see [12].

3.1. The residual lifetime. Let $T_1, T_2, \dots, T_n$ be arbitrary random variables representing the lifetimes of the components of a consecutive k-out-of-n system. Assume that $\mathbf{T} = (T_1, T_2, \dots, T_n)$ has an arbitrary joint survival function $\overline{F}(t_1, t_2, \dots, t_n)$.

In the following, we represent the reliability function of the conditional random variables $\{T_{k|n:\mathrm{G}} - t \mid T_{r:n} > t\}$ ($\{T^C_{k|n:\mathrm{G}} - t \mid T_{r:n} > t\}$) and $\{T_{k|n:\mathrm{F}} - t \mid T_{r:n} > t\}$ ($\{T^C_{k|n:\mathrm{F}} - t \mid T_{r:n} > t\}$), for linear and circular consecutive k-out-of-n systems.

Proposition 3.1. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n arbitrarily dependent components of a Lin/Con/k/n system. Then for $2k \geqslant n$ and $x, t > 0$,*

$$P(T_{k|n:\mathrm{G}} - t > x \mid T_{r:n} > t) = \frac{1}{\overline{F}_{r:n}(t)}\left[\sum_{l=1}^{n-k+1} \varphi^l_{k,r,n}(x,t) - \sum_{l=1}^{n-k} \varphi^l_{k+1,r,n}(x,t)\right], \tag{3.3}$$

and

$$\begin{aligned} P(T_{k|n:\mathrm{F}} - t > x \mid T_{r:n} > t) \\ = 1 - \frac{1}{\overline{F}_{r:n}(t)}\left[\sum_{l=1}^{n-k+1} \psi^l_{k,r,n}(x,t) - \sum_{l=1}^{n-k} \psi^l_{k+1,r,n}(x,t)\right], \end{aligned} \tag{3.4}$$

where

$$\psi^l_{k,r,n}(x,t) = P(T^{[l:l+k-1]} \leqslant x+t) - \phi^l_{k,r,n}(x,t), \quad l = 1,2,\ldots,n-k+1, \tag{3.5}$$

$\varphi^l_{k,r,n}(x,t)$ and $\phi^l_{k,r,n}(x,t)$ are as defined in (3.1) and (3.2), respectively. Note that

$$\begin{aligned} P(T^{[l:l+k-1]} \leqslant t) &= P(\max\{T_l, \ldots, T_{l+k-1}\} \leqslant t) \\ &= P(T_l \leqslant t, T_{l+1} \leqslant t, \ldots, T_{l+k-1} \leqslant t). \end{aligned}$$

P r o o f. From (2.3), (2.4) and also using Lemmas 3.1, 3.2 and common arguments similar to those used in the proof of Theorem 3.3 in Salehi et al. [30], the proof can be obtained. □

In the following we obtain some results on the residual lifetime of circular consecutive systems.

Proposition 3.2. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n arbitrarily dependent components of a* Cir/Con/k/n *system with joint survival function $\overline{F}(t_1, t_2, \ldots, t_n)$. Then for $2k+1 \geqslant n$ and $x, t > 0$,*

$$\begin{aligned} P(T^C_{k|n:\mathrm{G}} - t > x \mid T_{r:n} > t) \\ = \frac{1}{\overline{F}_{r:n}(t)}\left[\sum_{l=1}^{n} \varphi^l_{k,r,n}(x,t) - \sum_{l=1}^{n} \varphi^l_{k+1,r,n}(x,t) + \overline{F}_{1:n}(t+x)\right], \end{aligned}$$

and

$$P(T^C_{k|n:\mathrm{F}} - t > x \mid T_{r:n} > t)$$
$$= 1 - \frac{1}{\overline{F}_{r:n}(t)}\biggl[\sum_{l=1}^{n} \psi^l_{k,r,n}(x,t) - \sum_{l=1}^{n} \psi^l_{k+1,r,n}(x,t) + \psi^1_{n,r,n}(x,t)\biggr],$$

where $\varphi^l_{k,r,n}(x,t)$ *and* $\psi^l_{k,r,n}(x,t)$ *are defined in* (3.1) *and* (3.5), *respectively.*

Proof. Using (2.5), (2.6), Lemmas 3.1, 3.2, and the same steps as used to prove Proposition 3.1, the survival function of the residual lifetime of the Cir/Con/k/n:G system and Cir/Con/k/n:F system for $2k+1 \geqslant n$, can be obtained. □

Theorem 3.1. *Let* $T_1, T_2, \ldots, T_n$ *be the arbitrary random variables showing the lifetimes of the components of a* Lin/Con/k/n:G *system with a joint survival function* $\overline{F}(t_1, t_2, \ldots, t_n)$. *Then for* $k \geqslant \max\{n-r+1, [n/2]\}$ *and for all* $t > 0$,

$$\{T_{k|n:\mathrm{G}} - t \mid T_{r+1:n} > t\} \leqslant_{\mathrm{st}} \{T_{k|n:\mathrm{G}} - t \mid T_{r:n} > t\}.$$

Proof. To prove the required result, we have to show that for $t > 0$,

$$P(T_{k|n:\mathrm{G}} - t > x \mid T_{r+1:n} > t) \leqslant P(T_{k|n:\mathrm{G}} - t > x \mid T_{r:n} > t) \quad \forall\, x > 0.$$

Using (3.3), for $2k \geqslant n$, we have

$$P(T_{k|n:\mathrm{G}} - t > x \mid T_{r:n} > t) - P(T_{k|n:\mathrm{G}} - t > x \mid T_{r+1:n} > t) = \frac{\Delta(k) - \Delta(k+1)}{\overline{F}_{r:n}(t)\overline{F}_{r+1:n}(t)},$$

where

$$\Delta(k) = \sum_{l=1}^{n-k+1} [\overline{F}_{r+1:n}(t)\varphi^l_{k,r,n}(x,t) - \overline{F}_{r:n}(t)\varphi^l_{k,r+1,n}(x,t)], \quad 2k \geqslant n.$$

If $k \geqslant n-r+1$, using (3.1), $\varphi^l_{k,r,n}(x,t) = \varphi^l_{k,r+1,n}(x,t)$ and $\varphi^l_{k+1,r,n}(x,t) = \varphi^l_{k+1,r+1,n}(x,t)$, for $l = 1, 2, \ldots, n-k+1$. Therefore,

$$\begin{aligned}(3.6)\quad \Delta(k) - \Delta(k+1) &= [\overline{F}_{r+1:n}(t) - \overline{F}_{r:n}(t)] \sum_{l=1}^{n-k+1} \varphi^l_{k,r,n}(x,t) \\ &\quad - [\overline{F}_{r+1:n}(t) - \overline{F}_{r:n}(t)] \sum_{l=1}^{n-k} \varphi^l_{k+1,r,n}(x,t) \\ &= [\overline{F}_{r+1:n}(t) - \overline{F}_{r:n}(t)] \biggl[\sum_{l=1}^{n-k+1} \varphi^l_{k,r,n}(x,t) - \sum_{l=1}^{n-k} \varphi^l_{k+1,r,n}(x,t)\biggr].\end{aligned}$$

From Lemma 2.1, the first bracket in expression (3.6) is non-negative. Since $\varphi^l_{k,r,n}(x,t) \geqslant \varphi^l_{k+1,r,n}(x,t)$, when $k \geqslant n-r+1$, the second bracket in expression (3.6) is also non-negative. Hence, for $k \geqslant n-r+1$, $\Delta(k)-\Delta(k+1) \geqslant 0$, and it means that $\{T_{k|n:\mathrm{G}} - t \mid T_{r+1:n} > t\} \leqslant_{\mathrm{st}} \{T_{k|n:\mathrm{G}} - t \mid T_{r:n} > t\}$. Therefore, the proof is complete. $\square$

The following example gives an application of this theorem.

Example 3.1. Let T_1, T_2, T_3, and T_4 denote the arbitrarily dependent lifetimes of four components which are connected in a Lin/Con/3/4:G system and assume that T_i's have a Farlie-Gumbel-Morgenstern multivariate exponential distribution with parameters $\boldsymbol{\theta} = (\theta_{12}, \theta_{13}, \ldots, \theta_{1234})$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, the joint survival function

$$\begin{aligned}\overline{F}(t_1,t_2,t_3,t_4) = 1 + \sum_{p=1}^{4}(-1)^p \sum_{1\leqslant i_1<i_2<\ldots<i_p\leqslant 4} &(1-\mathrm{e}^{-\lambda_{i_1}t_{i_1}})\ldots(1-\mathrm{e}^{-\lambda_{i_p}t_{i_p}}) \\ &\times\biggl(1+\sum_{d=2}^{p}\sum_{j_1<j_2<\ldots<j_d}\theta_{j_1j_2\ldots j_d}\mathrm{e}^{-\lambda_{j_1}t_{j_1}}\ldots\mathrm{e}^{-\lambda_{j_d}t_{j_d}}\biggr),\end{aligned}$$

where $j_1, j_2, \ldots, j_d \in \{i_1, i_2, \ldots, i_p\}$, $i_1, i_2, \ldots, i_p \in \{1,2,3,4\}$, and for convenience we define $\sum\limits_{d=b}^{a} = 0$ when $a < b$. Hence, using Theorem 3.1, it can be shown that

$$\begin{aligned}P(T_{3|4:G} - t > x \mid T_{2:4} > t) &- P(T_{3|4:G} - t > x \mid T_{3:4} > t) \\ &= \frac{\overline{F}_{3:4}(t) - \overline{F}_{2:4}(t)}{\overline{F}_{2:4}(t)\overline{F}_{3:4}(t)}[P(T^{[1:3]} > x+t) + P(T^{[2:4]} > x+t) \\ &\quad - \overline{F}(x+t,x+t,x+t,x+t)] \geqslant 0.\end{aligned}$$

The graphs of the survival functions of $\{T_{3|4:G} - t \mid T_{r:n} > t\}$, $r = 2,3$, for $\theta_{12} = \theta_{13} = \ldots = \theta_{1234} = 0.15$ and $\boldsymbol{\lambda} = (1, 1.5, 0.5, 2)$, at a fixed point $t = 1$, are plotted in Figure 1. Also, from (2.1), we can extract the value of $\overline{F}_{3:4}(t)$ and $\overline{F}_{2:4}(t)$ at the fixed point $t = 1$.

Theorem 3.2. *Let $T_1, T_2, \ldots, T_n$ be the arbitrary random variables showing the lifetimes of the components of a* Cir/Con/k/n:G *system with a joint survival function $\overline{F}(t_1, t_2, \ldots, t_n)$. For $k \geqslant \max\{n-r+1, [\frac{1}{2}(n-1)]\}$ and for all $t > 0$,*

$$\{T^C_{k|n:\mathrm{G}} - t \mid T_{r+1:n} > t\} \leqslant_{\mathrm{st}} \{T^C_{k|n:\mathrm{G}} - t \mid T_{r:n} > t\}.$$

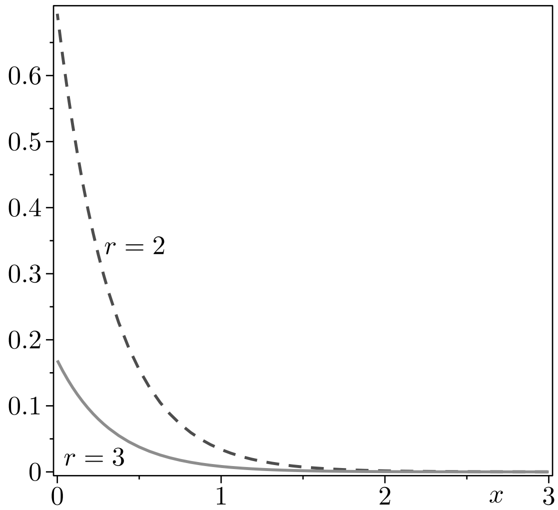

Figure 1. The curves of survival functions of $\{T_{3|4:G} - t \mid T_{r:n} > t\}$, $r = 2, 3$ at $t = 1$, for Example 3.1.

P r o o f. From Proposition 3.2, for $x, t > 0$ and $2k + 1 \geqslant n$,

$$
\begin{aligned}
(3.7)\quad & P(T^C_{k|n:\mathrm{G}} - t > x \mid T_{r:n} > t) - P(T^C_{k|n:\mathrm{G}} - t > x \mid T_{r+1:n} > t) \\
&= \frac{\overline{F}_{1:n}(t+x)}{\overline{F}_{r:n}(t)} - \frac{\overline{F}_{1:n}(t+x)}{\overline{F}_{r+1:n}(t)} \\
&\quad + \sum_{l=1}^{n} \left[\left(\frac{\varphi^l_{k,r,n}(x,t)}{\overline{F}_{r:n}(t)} - \frac{\varphi^l_{k,r+1,n}(x,t)}{\overline{F}_{r+1:n}(t)} \right) - \left(\frac{\varphi^l_{k+1,r,n}(x,t)}{\overline{F}_{r:n}(t)} - \frac{\varphi^l_{k+1,r+1,n}(x,t)}{\overline{F}_{r+1:n}(t)} \right) \right] \\
&= \frac{1}{\overline{F}_{r:n}(t)\overline{F}_{r+1:n}(t)} \left(\overline{F}_{1:n}(t+x)[\overline{F}_{r+1:n}(t) - \overline{F}_{r:n}(t)] + \sum_{l=1}^{n} [\Delta^*_l(k) - \Delta^*_l(k+1)] \right),
\end{aligned}
$$

where

$$\Delta^*_l(k) = \overline{F}_{r+1:n}(t)\varphi^l_{k,r,n}(x,t) - \overline{F}_{r:n}(t)\varphi^l_{k,r+1,n}(x,t), \quad 2k + 1 \geqslant n.$$

If $k \geqslant n - r + 1$, using (3.1) in Lemma 3.1, $\varphi^l_{k,r,n}(x,t) = \varphi^l_{k,r+1,n}(x,t)$ and $\varphi^l_{k+1,r,n}(x,t) = \varphi^l_{k+1,r+1,n}(x,t)$. Therefore,

$$(3.8)\qquad \Delta^*_l(k) - \Delta^*_l(k+1) = [\overline{F}_{r+1:n}(t) - \overline{F}_{r:n}(t)][\varphi^l_{k,r,n}(x,t) - \varphi^l_{k+1,r,n}(x,t)].$$

From Theorem 2.1, the first bracket in expression (3.8) is non-negative. Since $\varphi^l_{k,r,n}(x,t) \geqslant \varphi^l_{k+1,r,n}(x,t)$, when $k \geqslant n - r + 1$, the second bracket in expression (3.8) is also non-negative. Hence, for $k \geqslant n - r + 1$, $\Delta^*_l(k) - \Delta^*_l(k+1) \geqslant 0$, and then expression (3.7) is non-negative. Therefore, the proof is complete. □

Here we assume that the lifetimes of components are exchangeable. In the special case when $r = 1$, the residual lifetime of the consecutive k-out-of-n system can be defined as

$$\{T_{k|n:\mathrm{G}} - t \mid T_{1:n} > t\}.$$

It means that the residual lifetime of the system depends on the condition that all exchangeable components of the system are alive at time t. For $x, t > 0$, one can show that

$$\begin{aligned}(3.9)\qquad & P(T_{k|n:\mathrm{G}} - t > x \mid T_{1:n} > t) \\ &\quad = (n-k+1)\overline{F}_t(\underbrace{x,\dots,x}_{k}) - (n-k)\overline{F}_t(\underbrace{x,\dots,x}_{k+1}), \quad 2k \geqslant n,\end{aligned}$$

$$\begin{aligned}(3.10)\qquad & P(T^C_{k|n:\mathrm{G}} - t > x \mid T_{1:n} > t) \\ &\quad = \overline{F}_t(\underbrace{x,\dots,x}_{n}) + n\Big[\overline{F}_t(\underbrace{x,\dots,x}_{k}) - \overline{F}_t(\underbrace{x,\dots,x}_{k+1})\Big], \quad 2k+1 \geqslant n,\end{aligned}$$

where $\overline{F}_t(x,\dots,x)$ is the joint survival function of $((T_1)_t, (T_2)_t, \dots, (T_n)_t)$, with $(T_i)_t = \{T_i - t \mid T_i > t\}$, $i = 1, 2, \dots, n$. In other words, $\overline{F}_t(x,\dots,x)$ is the survival function of the series system consisting of exchangeable components with lifetimes $(T_i)_t$.

Using (3.9) and (3.10), the mean residual lifetime functions of the linear and circular consecutive k-out-of-n:G systems, under the condition that all components are operating at time t, are equal to

$$\begin{aligned}E(T_{k|n:\mathrm{G}} - t \mid T_{1:n} > t) &= (n-k+1)M_{1:k}(t) - (n-k)M_{1:k+1}(t), \quad 2k \geqslant n,\\ E(T^C_{k|n:\mathrm{G}} - t \mid T_{1:n} > t) &= M_{1:k}(t) + n[M_{1:k}(t) - M_{1:k+1}(t)], \quad 2k+1 \geqslant n,\end{aligned}$$

where $M_{1:k}(t)$ denotes the mean residual lifetime of a series system with k exchangeable components, i.e.,

$$M_{1:k}(t) = E(T_{1:k} - t \mid T_{1:k} > t) = \int_0^\infty \overline{F}_t(\underbrace{x,\dots,x}_{k})\,\mathrm{d}x.$$

Now we can prove the following theorems.

Theorem 3.3. *Let $T_1, T_2, \dots, T_{n+1}$ denote the exchangeable lifetimes of the components of a* Lin/Con/k/n *system. Assume that $T_1, T_2, \dots, T_{n+1}$ have the joint survival function $\overline{F}$. Then for $2k \geqslant n+1$ and $t > 0$,*

$$\{T_{k|n:\mathrm{G}} - t \mid T_{1:n} > t\} \leqslant_{\mathrm{st}} \{T_{k|n+1:G} - t \mid T_{1:n+1} > t\}.$$

P r o o f. To prove the statement we have to show that for $2k \geqslant n+1$, and $t > 0$,

$$P(T_{k|n:\mathrm{G}} - t > x \mid T_{1:n} > t) \leqslant P(T_{k|n+1:G} - t > x \mid T_{1:n+1} > t) \quad \forall\, x > 0.$$

From (3.9), for the Lin/Con/k/n:G system,

$$\begin{aligned} &P(T_{k|n+1:G} - t > x \mid T_{1:n+1} > t) - P(T_{k|n:\mathrm{G}} - t > x \mid T_{1:n} > t) \\ &\quad = (n-k+2)\overline{F}_t(\underbrace{x,\ldots,x}_{k}) - (n-k+1)\overline{F}_t(\underbrace{x,\ldots,x}_{k+1}) \\ &\qquad - \Big[(n-k+1)\overline{F}_t(\underbrace{x,\ldots,x}_{k}) - (n-k)\overline{F}_t(\underbrace{x,\ldots,x}_{k+1})\Big] \\ &\quad = \overline{F}_t(\underbrace{x,\ldots,x}_{k}) - \overline{F}_t(\underbrace{x,\ldots,x}_{k+1}) \geqslant 0. \end{aligned}$$

The last inequality follows from Lemma 2.1, and hence the proof is complete. $\square$

E x a m p l e 3.2. Let the exchangeable lifetimes of components $T_1, T_2, \ldots, T_n$ have Marshal and Olkin's multivariate exponential distribution with the joint survival function

$$\overline{F}(t_1, t_2, \ldots, t_n) = \exp\Big\{-\lambda \sum_{i=1}^{n} t_i - \lambda^* \max\{t_1, t_2, \ldots, t_n\}\Big\}, \quad \lambda > 0,\ \lambda^* \geqslant 0.$$

It is easily shown that

$$\overline{F}(\underbrace{t,\ldots,t}_{i}, \underbrace{0,\ldots,0}_{n-i}) = \exp\{-(i\lambda + \lambda^*)t\},$$

and then

$$\overline{F}_t(\underbrace{x,\ldots,x}_{k}) = \frac{\overline{F}(\overbrace{x+t,\ldots,x+t}^{k}, \overbrace{0,\ldots,0}^{n-k})}{\overline{F}(\underbrace{t,\ldots,t}_{k}, \underbrace{0,\ldots,0}_{n-k})} = \exp\{-(k\lambda + \lambda^*)x\}.$$

To show the result of Theorem 3.3, using (3.9) and (3.10) we plot the graphs of the survival functions of $\{T_{4|n:\mathrm{G}} - t \mid T_{1:n} > t\}$, $n = 6, 7$, in Figure 2 for $\lambda = 0.5$ and $\lambda^* = 0.1$. It is obvious that $\{T_{4|6:\mathrm{G}} - t \mid T_{1:6} > t\} \leqslant_{\mathrm{st}} \{T_{4|7:\mathrm{G}} - t \mid T_{1:7} > t\}$.

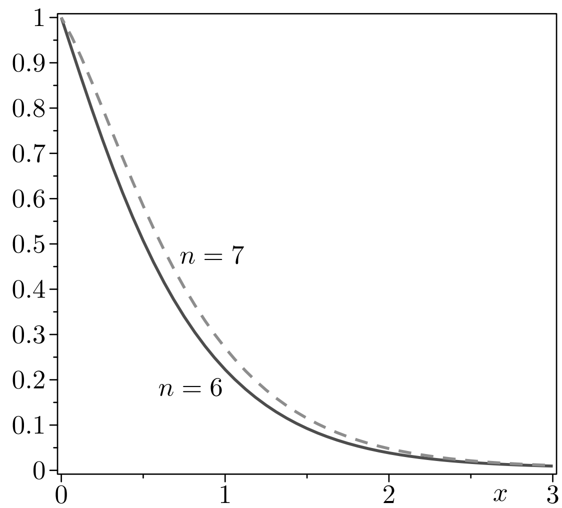

Figure 2. The curves of survival functions of $\{T_{4|n:\mathrm{G}} - t \mid T_{1:n} > t\}$, $n = 6, 7$, for Example 3.2.

R e m a r k 3.2. The result of Theorem 3.3 says that for a fixed value k, the mean residual lifetime function of a Lin/Con/k/n:G system under the condition that all components are operating at time t, increases in n, $n = k, k+1, \ldots, 2k-1$, i.e., for all $t > 0$,

$$E(T_{k|n:\mathrm{G}} - t \mid T_{1:n} > t) \leqslant E(T_{k|n+1:G} - t \mid T_{1:n+1} > t).$$

In the next result, we show that for any fixed k, the residual lifetime of a Cir/Con/k/n:G system under the condition that all components are working at time t, (i.e., $\{T^C_{k|n:\mathrm{G}} - t \mid T_{1:n} > t\}$) is stochastically increasing in n, when the components of the system are independent and identical.

Theorem 3.4. *Let $T_1, T_2, \ldots, T_{n+1}$ denote the lifetimes of the components of a* Cir/Con/k/n:G *system. Assume that T_i's are independent and have identical distribution function F. Then for $2k \geqslant n$ and $t > 0$,*

$$\{T^C_{k|n:\mathrm{G}} - t \mid T_{1:n} > t\} \leqslant_{\mathrm{st}} \{T^C_{k|n+1:G} - t \mid T_{1:n+1} > t\}.$$

P r o o f. From (2.6), we have for $2k \geqslant n$,

$$P(T^C_{k|n+1:\mathrm{G}} - t > x \mid T_{1:n+1} > t) - P(T^C_{k|n:\mathrm{G}} - t > x \mid T_{1:n} > t) = \eta(k) - \eta(n),$$

where

$$\eta(k) = F_t(x)(\overline{F}_t(x))^k,$$

with $\overline{F}_t(x)=\overline{F}(x+t)/\overline{F}(t)$, and $F_t(x)=1-\overline{F}_t(x)$. Therefore,

$$\eta(k)-\eta(n)=F_t(x)(\overline{F}_t(x))^k[1-(\overline{F}_t(x))^{n-k}]\geqslant 0.$$

Hence, the proof is complete. □

3.2. The inactivity time. In this section, we study the inactivity time of the failed components of the linear and circular consecutive k-out-of-n:G (F) systems consisting of arbitray components. First, we obtain the reliability function of the inactivity time for such systems, which is given in the following theorems.

Theorem 3.5. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n arbitrarily dependent components of a* Lin/Con/k/n *system. Then for $2k\geqslant n$ and $x,t>0$,*

$$\begin{aligned}(3.11)\qquad P(t-T_{r:n}>x\mid T_{k|n:\mathrm{G}}\leqslant t)&=\frac{1}{P(T_{k|n:\mathrm{G}}\leqslant t)}\\ &\times\Bigg[F_{r:n}(t-x)-\Bigg(\sum_{l=1}^{n-k+1}[P(T_{[l:l+k-1]}>t)-\varphi^l_{k,r,n}(x,t-x)]\\ &-\sum_{l=1}^{n-k}[P(T_{[l:l+k]}>t)-\varphi^l_{k+1,r,n}(x,t-x)]\Bigg)\Bigg],\end{aligned}$$

and

$$\begin{aligned}(3.12)\qquad &P(t-T_{r:n}>x\mid T_{k|n:\mathrm{F}}\leqslant t)\\ &=\frac{1}{P(T_{k|n:\mathrm{F}}\leqslant t)}\Bigg[\sum_{l=1}^{n-k+1}\phi^l_{k,r,n}(x,t-x)-\sum_{l=1}^{n-k}\phi^l_{k+1,r,n}(x,t-x)\Bigg],\end{aligned}$$

where $\varphi^l_{k,r,n}(x,t)$ and $\phi^l_{k,r,n}(x,t)$ are defined in (3.1) *and* (3.2), *respectively. Note that*

$$\begin{aligned}P(T_{[l:l+k-1]}>t)&=P(\min\{T_l,\ldots,T_{l+k-1}\}>t)\\ &=P(T_l>t,T_{l+1}>t,\ldots,T_{l+k-1}>t).\end{aligned}$$

Proof. For $0<x<t$,

$$P(t-T_{r:n}>x\mid T_{k|n:\mathrm{G}}\leqslant t)=\frac{1}{P(T_{k|n:\mathrm{G}}\leqslant t)}P(T_{r:n}<t-x,T_{k|n:\mathrm{G}}\leqslant t).$$

Using (2.3) and Lemma 3.1, we have

$$
\begin{aligned}
&P(T_{r:n} < t-x, T_{k|n:\mathrm{G}} \leqslant t) \\
&\quad = P(T_{r:n} < t-x) - P(T_{r:n} < t-x, T_{k|n:\mathrm{G}} > t) \\
&\quad = F_{r:n}(t-x) - \Bigg[\sum_{l=1}^{n-k+1} (P(T_{[l:l+k-1]} > t) - \varphi^{l}_{k,r,n}(x, t-x)) \\
&\qquad\qquad - \sum_{l=1}^{n-k} (P(T_{[l:l+k]} > t) - \varphi^{l}_{k+1,r,n}(x, t-x)) \Bigg].
\end{aligned}
$$

Hence, we get (3.11). For a Lin/Con/k/n:F system, from (2.4) and after some simplifications, it is easy to show that for $0 < x < t$,

$$
\begin{aligned}
&P(T_{r:n} < t-x, T_{k|n:\mathrm{F}} \leqslant t) \\
&= \sum_{i=k}^{n} [P(T^{[i-k+1:i]} \leqslant t, T_{r:n} < t-x) - P(T^{[i-k+1:i+1]} \leqslant t, T_{r:n} < t-x)] \\
&= \sum_{l=1}^{n-k+1} P(T^{[l:l+k-1]} \leqslant t, T_{r:n} < t-x) - \sum_{l=1}^{n-k} P(T^{[l:l+k]} \leqslant t, T_{r:n} < t-x),
\end{aligned}
$$

hence using Lemma 3.2, the proof is complete. □

Theorem 3.6. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n arbitrarily dependent components of a* Cir/Con/k/n *system with joint survival function $\overline{F}(t_1, t_2, \ldots, t_n)$. Then for $2k+1 \geqslant n$ and $x, t > 0$,*

$$
\begin{aligned}
&P(t - T_{r:n} > x | T^{C}_{k|n:\mathrm{G}} \leqslant t) \\
&\quad = \frac{1}{P(T^{C}_{k|n:\mathrm{G}} \leqslant t)} \Bigg[F_{r:n}(t-x) - \sum_{l=1}^{n} (P(T_{[l:l+k-1]} > t) - \varphi^{l}_{k,r,n}(x, t-x)) \\
&\qquad\qquad + \sum_{l=1}^{n} (P(T_{[l:l+k]} > t) - \varphi^{l}_{k+1,r,n}(x, t-x)) \Bigg],
\end{aligned}
$$

and

$$
\begin{aligned}
&P(t - T_{r:n} > x \mid T^{C}_{k|n:\mathrm{F}} \leqslant t) \\
&\qquad = \frac{1}{P(T^{C}_{k|n:\mathrm{F}} \leqslant t)} \Bigg[H_{r,n}(x,t) + \sum_{l=1}^{n} (\phi^{l}_{k,r,n}(x, t-x) - \phi^{l}_{k+1,r,n}(x, t-x)) \Bigg],
\end{aligned}
$$

where $\varphi^{l}_{k,r,n}(x,t)$ and $\phi^{l}_{n,r,n}(x,t)$ are defined in (3.1) *and* (3.2)*, respectively, and also*

$$
H_{r,n}(x,t) = P(T_{r:n} < t-x, T_{n:n} \leqslant t) = \sum_{\pi \in C_n} P(A^{(t,\pi)}_{n}) \overline{F}^{(t,\pi)}_{n-r+1:n}(x),
$$

where
$$A_i^{(t,\pi)} = (T_{\pi_1} \leqslant t, T_{\pi_2} \leqslant t, \ldots, T_{\pi_i} \leqslant t, T_{\pi_{i+1}} > t, \ldots, T_{\pi_n} > t),$$
C_i *is the set of all permutations* $\{\pi_1, \pi_2, \ldots, \pi_n\}$ *of* $\{1, 2, \ldots, n\}$ *for which* $1 \leqslant \pi_1 < \ldots < \pi_i \leqslant n$ *and* $1 \leqslant \pi_{i+1} < \ldots < \pi_n \leqslant n$, *and* $\overline{F}^{(t,\pi)}_{n-r+1:n}(x)$ *is the joint survival function of the vector* $(t - T_{\pi_1}, \ldots, t - T_{\pi_n} \mid A_n^{(t,\pi)})$ *(see the proof of Theorem* 1 *in Tavangar and Asadi* [36]*).*

P r o o f. The proof can be obtained from (2.5) and (2.6) and using Lemmas 3.1 and 3.2 in the same way as in the proof of Theorem 3.5. □

Theorem 3.7. *Let* $T_1, T_2, \ldots, T_n$ *be the arbitrarily dependent random variables showing the lifetimes of the components of a consecutive* k-out-of-n:G *system with a joint survival function* $\overline{F}(t_1, t_2, \ldots, t_n)$. *Then*

(a) *for* $k \geqslant \max\{n - r + 1, [n/2]\}$ *and for all* $t > 0$,
$$\{t - T_{r+1:n} \mid T_{k|n:\mathrm{G}} \leqslant t\} \leqslant_{\mathrm{st}} \{t - T_{r:n} \mid T_{k|n:\mathrm{G}} \leqslant t\};$$
(b) *for* $k \geqslant \max\{n - r + 1, [\frac{1}{2}(n - 1)]\}$ *and for all* $t > 0$,
$$\{t - T_{r+1:n} \mid T^C_{k|n:\mathrm{G}} \leqslant t\} \leqslant_{\mathrm{st}} \{t - T_{r:n} \mid T^C_{k|n:\mathrm{G}} \leqslant t\}.$$

P r o o f. We only present the proof of case (a), because the proof of case (b) is analogous to (a), using Theorem 3.6. From Theorem 3.5, for $2k \geqslant n$,

$$\begin{aligned} &P(t - T_{r:n} > x \mid T_{k|n:\mathrm{G}} \leqslant t) - P(t - T_{r+1:n} > x \mid T_{k|n:\mathrm{G}} \leqslant t) \\ &\quad = \frac{1}{P(T_{k|n:\mathrm{G}} \leqslant t)}(F_{r:n}(t - x) - F_{r+1:n}(t - x) + D(k) - D(k + 1)), \end{aligned} \tag{3.13}$$

where
$$D(k) = \sum_{l=1}^{n-k+1} (\varphi^l_{k,r,n}(x, t - x) - \varphi^l_{k,r+1,n}(x, t - x)).$$
When $k \geqslant n - r + 1$, from Lemma 3.1, $\varphi^l_{k,r+1,n}(x, t - x) = \varphi^l_{k,r,n}(x, t - x)$ for all $0 < x < t$, and then $D(k) = D(k + 1) = 0$. Hence, since for all $0 < x < t$, $F_{r+1:n}(t - x) \leqslant F_{r:n}(t - x)$, expression (3.13) is non-negative. Therefore, the proof is complete. □

4. Summary & conclusions

In this paper, we studied the conditional variables (the residual lifetime and inactivity time) of linear and circular consecutive k-out-of-n:G(F) systems with arbitrarily dependent components; here we used two important lemmas to extract an explicit form for the reliability function of these two well-known conditional variables under a set of conditions for k. In Subsection 3.1, we presented stochastic ordering properties of the residual lifetimes of linear and circular consecutive k-out-of-n:G systems. We showed that for a fixed k and n, the residual lifetime of such systems is stochastically decreasing with respect to r, and we then illustrated this feature with a numerical example. In the remainder of Subsection 3.1, under the assumption that all components are alive at time t, i.e., $r = 1$, we proved that for a fixed k, the residual lifetime of a Lin/Con/k/n:G system is stochastically increasing with respect to n (the number of components). This property was also proved for the residual lifetime of Cir/Con/k/n:G systems consisting of independent and identical components. In Subsection 3.2, we stochastically compare the inactivity times of linear and circular consecutive k-out-of-n:G systems, and showed that for a fixed k and n, the inactivity time of such systems is stochastically decreasing with respect to r, too. The significance of this study is the lack of any conditions on component lifetimes, and this means that the lifetimes of components can have dependent or arbitrary distributions.

A c k n o w l e d g e m e n t s. The author would like to thank the editor and anonymous referee for their valuable and constructive comments, which improved the presentation of the paper.

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