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QUOTIENT OF SPECTRAL RADIUS, (SIGNLESS) LAPLACIAN SPECTRAL RADIUS AND CLIQUE NUMBER OF GRAPHS

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Dedicated to the memory of Professor Miroslav Fiedler

Abstract. In this paper, the upper and lower bounds for the quotient of spectral radius (Laplacian spectral radius, signless Laplacian spectral radius) and the clique number together with the corresponding extremal graphs in the class of connected graphs with n vertices and clique number ω ($2 \leqslant \omega \leqslant n$) are determined. As a consequence of our results, two conjectures given in Aouchiche (2006) and Hansen (2010) are proved.

Keywords: spectral radius; (signless) Laplacian spectral radius; clique number

MSC 2010: 05C50, 05C75

1. Introduction

Throughout this paper, G is a simple undirected graph with n vertices. The maximum vertex degree of G is denoted by $\Delta(G)$. Let $A(G)$ and $D(G)$ be the adjacency matrix and the vertex degree matrix of G, respectively. Then, the Laplacian matrix of G is $L(G) = D(G) - A(G)$, and the signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. Denote by $\lambda(G)$, $\mu(G)$ and $q(G)$, respectively, the spectral radius, Laplacian spectral radius and signless Laplacian spectral radius of G. That means $\lambda(G)$, $\mu(G)$ and $q(G)$ are equal to the maximum eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively.

As usual, K_n, P_n and $K_{p,n-p}$ denote, respectively, the complete graph, the path and a complete bipartite graph on n vertices. Denote by $Ki_{n,\omega}$ the graph obtained

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by joining one vertex of K_ω to one end vertex of a path $P_{n-\omega}$ by a bridge. From the definition, $Ki_{n,2} \cong P_n$. Let $T_{n,\omega}$ define the complete ω-partite graph with almost equal parts, which is famous as Turán graph (see [4], [8]). Let $\Gamma(n,\omega)$ be the class of connected graphs with n vertices and clique number ω. When $\omega = 1$, $\Gamma(n,\omega)$ is trivial, and hence we always suppose that $2 \leqslant \omega \leqslant n$ in the following.

As early as in 1985, Brualdi and Hoffman in [2] investigated the maximum spectral radius of the adjacency matrix of a (not necessarily connected) graph in the set of all graphs with a given number of vertices and edges. Their work was followed by other authors, in the connected graph case as well as in the general case. Recently, the research on extremal graphs with maximum or minimum spectral radii (Laplacian spectral radii, signless Laplacian spectral radii) in $\Gamma(n,\omega)$ received much attention (see [3], [4], [5], [8], [10], [13], [14], [15]). Furthermore, some researchers were also concerned with the quotient of spectral radius (signless Laplacian spectral radius) and clique number of graphs. For instance, the following two conjectures were proposed not long ago.

Conjecture 1.1 ([1]). *If $G \in \Gamma(n,\omega)$ with $n \geqslant 3$, then $\lambda(G)/\omega$ is minimum for $Ki_{n,3}$.*

Conjecture 1.2 ([6]). *If $G \in \Gamma(n,\omega)$ with $n \geqslant 4$, then $q(G)/\omega \leqslant n/2$.*

Conjecture 1.2 was proved to be true for $n \geqslant 10$ by He et al. in [8], while Conjecture 1.1 is still open. Motivated by Conjectures 1.1 and 1.2, we consider the lower and upper bounds for the quotient of spectral radius (Laplacian spectral radius, signless Laplacian spectral radius) and the clique number of connected graphs, and we obtain better bounds for Conjectures 1.1 and 1.2.

Theorem 1.1. *If $G \in \Gamma(n,\omega)$ with $n \geqslant 5$ and $\omega \geqslant 2$, then*

$$\frac{\lambda(Ki_{n,3})}{3} + \frac{(\omega-3)^2}{\omega^3} \leqslant \frac{\lambda(G)}{\omega} \leqslant \frac{\lambda(T_{n,2})}{2} - \frac{(w-2)^2}{\omega^2},$$

where the left equality holds if and only if $G \cong Ki_{n,3}$, and the right equality holds if and only if $G \cong T_{n,2}$.

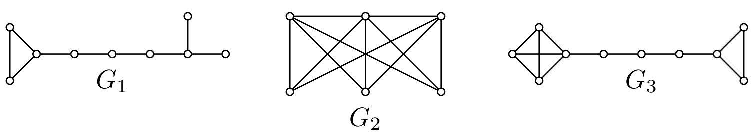

Figure 1. The graphs G_1, G_2 and G_3.

In the following, let G_1, G_2 and G_3 be the graphs as shown in Figure 1.

Remark 1.1. From Table 1, one can easily see that the lower bound is tight for $Ki_{6,4}$ and G_1, and the upper bound is tight for $K_{6,6}-e$ and G_2 in Theorem 1.1.

G	$\dfrac{\lambda(Ki_{n,3})}{3}+\dfrac{(\omega-3)^2}{\omega^3}$	$\dfrac{\lambda(G)}{\omega}$	$\dfrac{\lambda(T_{n,2})}{2}-\dfrac{(w-2)^2}{\omega^2}$
$Ki_{6,4}$	0.7584	0.7741	1.2500
G_1	0.7452	0.7463	2.1250
$K_{6,6}-e$	0.8703	2.9271	3.0000
G_2	0.7428	1.2555	1.3889

Table 1.

Theorem 1.2. *If $G\in\Gamma(n,\omega)$ with $n\geqslant 6$ and $\omega\geqslant 2$, then*

$$\frac{q(Ki_{n,3})}{3}+\frac{(\omega-2)(\omega-3)}{\omega^3}\leqslant\frac{q(G)}{\omega}\leqslant\frac{n}{2}-\Big(1-\frac{2}{\omega}\Big),$$

where the left equality holds if and only if $G\cong Ki_{n,3}$, and the right equality holds if and only if either $G\cong K_{p,n-p}$ for some positive integer p or G is isomorphic to a complete 3-partite graph with two vertices in each part.

Remark 1.2. From Table 2, one can easily see that our lower (for $Ki_{6,4}$ and G_1) and upper (for $K_{6,6}-e$ and G_2) bounds are tight in Theorem 1.2.

G	$\dfrac{q(Ki_{n,3})}{3}+\dfrac{(\omega-2)(\omega-3)}{\omega^3}$	$\dfrac{q(G)}{\omega}$	$\dfrac{n}{2}-\Big(1-\dfrac{2}{\omega}\Big)$
$Ki_{6,4}$	1.5831	1.5981	2.5000
G_1	1.5530	1.5538	4.1667
$K_{6,6}-e$	1.5530	5.8708	6.0000
G_2	1.5518	2.5863	2.6667

Table 2.

It is well-known (see [12]) that for any graph with n vertices, $\mu(G)\leqslant n$ with equality holding if and only if $\overline{G}$, namely, the complement graph of G, is disconnected. Thus, we can conclude that

$$\frac{\mu(G)}{\omega}\leqslant\frac{n}{\omega}$$

with equality holding if and only if $\overline{G}$ is disconnected. It is interesting to consider the upper bound for $\mu(G)/\omega$ when $\overline{G}$ is connected. For the lower bound of $\mu(G)/\omega$, we have

Theorem 1.3. *If $G\in\Gamma(n,\omega)$ with $2\leqslant\omega\leqslant n-1$, then*

$$\frac{\mu(G)}{\omega}\geqslant 1+\frac{1}{\omega}+\frac{n-\omega-1}{(n-\omega)\omega^3}\tag{1.1}$$

with equality holding if and only if $\omega=n-1$.

Remark 1.3. From Table 3, one can easily see that the lower bound is tight for $Ki_{6,4}$ and G_3 in Theorem 1.3.

G	$\dfrac{\mu(G)}{\omega}$	$1+\dfrac{1}{\omega}+\dfrac{n-\omega-1}{(n-\omega)\omega^3}$
$Ki_{6,4}$	1.2715	1.2578
G_3	1.2746	1.2630

Table 3.

2. Some useful preliminaries

The following lemma suggests that $Ki_{n,\omega}$ uniquely achieves the minimum spectral radius and the signless Laplacian spectral radius among $\Gamma(n,\omega)$ when $\omega \geqslant 2$.

Lemma 2.1 ([8], [14], [15]). *If $G \in \Gamma(n,\omega)$ with $\omega \geqslant 2$, then $\lambda(G) \geqslant \lambda(Ki_{n,\omega})$, and $q(G) \geqslant q(Ki_{n,\omega})$ with each equality holding if and only if $G \cong Ki_{n,\omega}$.*

The following lemma determines the extremal graph with maximum spectral radius and signless Laplacian spectral radius among $\Gamma(n,\omega)$ when $\omega \geqslant 2$.

Lemma 2.2. *Suppose that $G \in \Gamma(n,\omega)$.*

(i) ([13]). *If $2 \leqslant \omega \leqslant n$, then $\lambda(G) < \lambda(T_{n,\omega})$ unless $G \cong T_{n,\omega}$.*
(ii) ([3], [8]). *If $\omega = 2$, then $q(G) < q(K_{p,n-p})$ unless $G \cong K_{p,n-p}$.*
(iii) ([3], [8]). *If $3 \leqslant \omega \leqslant n$, then $q(G) < q(T_{n,\omega})$ unless $G \cong T_{n,\omega}$.*

Lemma 2.3. *For any $\omega \geqslant 2$, we have*

(i) *$\lambda(T_{n,\omega}) \leqslant n - n/\omega$, where the equality implies that n/ω is a positive integer.*
(ii) ([7], [8]). *$q(T_{n,\omega}) \leqslant 2n - 2n/\omega$, where the equality implies that either n/ω is a positive integer or $\omega = 2$.*

Proof. We only need to show (i). From the definition, $T_{n,\omega}$ is a Turán graph with k parts of size $d+1$ and $\omega - k$ parts of size d, where $d = \lfloor n/\omega \rfloor$, $n = k + \omega d$, and $0 \leqslant k < \omega$. It is proved that [4]

$$\lambda(T_{n,\omega}) = \frac{1}{2}\Big(n - \frac{2(n-k)}{\omega} - 1 + \sqrt{(n+1)^2 - 4k\Big(\frac{n-k}{\omega}+1\Big)}\Big). \tag{2.1}$$

Since

$$\Big(n - \frac{2k}{\omega} + 1\Big)^2 - \Big((n+1)^2 - 4k\Big(\frac{n-k}{\omega}+1\Big)\Big) = \frac{4k(\omega-1)(\omega-k)}{\omega^2} \geqslant 0,$$

by (2.1) we have $\lambda(T_{n,\omega}) \leqslant n - n/\omega$, where the equality implies that $k = 0$ and hence n/ω is a positive integer. $\square$

Lemma 2.4 ([14]). *For any integers $\omega \geqslant 3$ and $n \geqslant 4$,*

$$\lambda(Ki_{n,\omega}) > \omega - 1 + \frac{1}{\omega^2} + \frac{1}{\omega^3}.$$

Let G be a connected graph, and $uv \in E(G)$. The graph $G^*_{u,v}$ is obtained from G by subdividing the edge uv, i.e., adding a new vertex w and edges wu, wv in $G - uv$. An *internal path*, say $v_1v_2\dots v_{s+1}$, $s \geqslant 1$, is a path joining v_1 and v_{s+1} (which need not be distinct) so that v_1 and v_{s+1} have degrees greater than 2, while all other vertices $v_2, v_3, \dots, v_s$ are of degree 2. Let W_n be a tree on n vertices obtained from a path $P_{n-2} = v_1v_2\dots v_{n-2}$ by adding two new vertices v_{n-1} and v_n, and two new edges v_2v_{n-1} and $v_{n-3}v_n$.

Lemma 2.5. *Let uv be an edge of the connected graph G. If uv belongs to an internal path of G, then*

(i) ([9]) $\lambda(G^*_{u,v}) < \lambda(G)$ *if* $G \ncong W_n$;
(ii) ([11]) $q(G^*_{u,v}) < q(G)$.

3. Proofs of Theorems 1.1–1.3

A vertex with degree one is called a *pendent vertex*. When $n \geqslant 5$, let H_{n+1} be the graph of order $n+1$ obtained from $Ki_{n,3}$ by adding one new vertex and adding one new edge between this new vertex and the vertex with degree two of $Ki_{n,3}$, which is adjacent with the unique pendent vertex of $Ki_{n,3}$.

Proof of Theorem 1.1. We first prove the lower case. Taking Lemma 2.1 into consideration, we may suppose that $\omega \neq 3$ and divide the proof into two cases.

Case 1: $n = 5$. When $\omega = 2$, by Lemma 2.1 we get

$$\frac{\lambda(Ki_{5,3})}{3} < 0.739 < \cos\frac{\pi}{6} = \frac{\lambda(P_5)}{2} \leqslant \frac{\lambda(T)}{2}. \tag{3.1}$$

When $4 \leqslant \omega \leqslant 5$, by Lemma 2.1, Lemma 2.4 and the upper bound in (3.1), it is easy to check that

$$\frac{\lambda(G)}{\omega} \geqslant \frac{\lambda(Ki_{n,\omega})}{\omega} > 1 - \frac{1}{\omega} + \frac{1}{\omega^3} > 0.739 + \frac{(\omega-3)^2}{\omega^3} > \frac{\lambda(Ki_{5,3})}{3} + \frac{(\omega-3)^2}{\omega^3}.$$

Case 2: $n \geqslant 6$. By Lemmas 2.1 and 2.5, we have

$$\frac{\lambda(Ki_{n,3})}{3} < \frac{\lambda(H_{n+1})}{3} < \frac{\lambda(H_n)}{3} < \dots < \frac{\lambda(H_7)}{3} < 0.752. \tag{3.2}$$

When $\omega = 2$, since G is connected, by Lemma 2.1 and the upper bound in (3.2),

$$\frac{\lambda(T)}{2} \geqslant \frac{\lambda(P_n)}{2} = \cos\frac{\pi}{n+1} > 0.9 > 0.752 + 0.125 > \frac{\lambda(Ki_{n,3})}{3} + \frac{1}{8}.$$

When $\omega = 4$, by Lemma 2.1, Lemma 2.4 and the upper bound in (3.2),

$$\frac{\lambda(G)}{4} \geqslant \frac{\lambda(Ki_{n,4})}{4} > 1 - \frac{1}{4} + \frac{1}{64} + \frac{1}{256} > 0.769 > 0.752 + \frac{1}{64} > \frac{\lambda(Ki_{n,3})}{3} + \frac{1}{64}.$$

When $\omega \geqslant 5$, since

$$1 - \frac{1}{\omega} + \frac{1}{\omega^3} - 0.752 - \frac{(\omega-3)^2}{\omega^3} = \frac{\omega^2(31\omega - 250) + 250(3\omega - 4)}{125\omega^3} > 0,$$

by Lemma 2.1, Lemma 2.4 and the upper bound in (3.2),

$$\frac{\lambda(G)}{\omega} \geqslant \frac{\lambda(Ki_{n,\omega})}{\omega} > 1 - \frac{1}{\omega} + \frac{1}{\omega^3} > 0.752 + \frac{(\omega-3)^2}{\omega^3} > \frac{\lambda(Ki_{n,3})}{3} + \frac{(\omega-3)^2}{\omega^3}.$$

Thus, the lower bound of Theorem 1.1 holds. Now, we turn to prove the upper bound of Theorem 1.1. When $\omega = 2$, the result follows from Lemma 2.2. When $\omega \geqslant 3$, by Lemmas 2.2 and 2.3, it suffices to show that

$$\frac{n}{\omega} - \frac{n}{\omega^2} < \frac{\sqrt{n^2-1}}{4} - \frac{(\omega-2)^2}{\omega^2} \quad (\text{as } \lambda(T_{n,2}) \geqslant 0.5\sqrt{n^2-1} \text{ by (2.1)}),$$

which is equivalent to

$$\frac{n}{\omega} - \frac{n}{\omega^2} + \frac{(\omega-2)^2}{\omega^2} < \frac{\sqrt{n^2-1}}{4}. \tag{3.3}$$

Note that $f_1(x) = n/x - n/x^2 + (x-2)^2/x^2$ is a decreasing function on $x \geqslant 3$. Thus,

$$f_1(\omega) \leqslant f_1(3) = \frac{n}{3} - \frac{n}{9} + \frac{1}{9} = \frac{2n+1}{9}.$$

If $n \geqslant 6$, since $f_1(\omega) \leqslant f_1(3) = (2n+1)/9 < (8n-1)/32 < \sqrt{n^2-1}/4$, (3.3) holds.

If $n = 5$, since $f_1(\omega) \leqslant f_1(3) = 11/9 < \sqrt{24}/4$, (3.3) also holds. $\square$

Remark 3.1. When $n \geqslant 13$ and $\omega \geqslant 2$, since $(2n+3)/9 < (16n-1)/64 < \sqrt{n^2-1}/4$, it can be proved similarly to the proof of Theorem 1.1 that

$$\frac{\lambda(G)}{\omega} \leqslant \frac{\lambda(T_{n,2})}{2} - \left(1 - \frac{2}{\omega}\right),$$

where the equality holds if and only if $G \cong T_{n,2}$.

P r o o f of Theorem 1.2. We first prove the lower bound. By Lemma 2.5, we get

$$(3.4)\qquad \begin{aligned} &\frac{q(Ki_{6,3})}{3} < 1.552, \quad \text{and} \\ &\frac{q(Ki_{n,3})}{3} < \frac{q(H_{n+1})}{3} < \ldots < \frac{q(H_8)}{3} < 1.557 \quad \text{for } n \geqslant 7. \end{aligned}$$

When $\omega = 2$, by Lemma 2.1 and the upper bounds in (3.4), we have

$$\frac{q(G)}{2} \geqslant \frac{q(P_n)}{2} = 1 + \cos\frac{\pi}{n} \geqslant 1 + \cos\frac{\pi}{6} > 1.866 > \frac{q(Ki_{n,3})}{3}.$$

When $\omega = 3$, by Lemma 2.1 it follows that

$$\frac{q(G)}{\omega} \geqslant \frac{q(Ki_{n,3})}{3}$$

with equality holding if and only if $G \cong Ki_{n,3}$.

Otherwise, $\omega \geqslant 4$. For $\omega = n$, $G \cong K_n$ and hence by the upper bounds in (3.4) we have

$$\frac{q(G)}{\omega} = 2 - \frac{2}{n} > 1.557 + \frac{1}{n} > \frac{q(Ki_{n,3})}{3} + \frac{(n-2)(n-3)}{n^3} \quad \text{for } n \geqslant 7,$$

and

$$\frac{q(G)}{\omega} = 2 - \frac{2}{6} > 1.552 + \frac{1}{18} > \frac{q(Ki_{6,3})}{3} + \frac{1}{18} \quad \text{for } n = 6.$$

Now we have to prove our result for $4 \leqslant \omega \leqslant n-1$. By Lemma 2.1, we have

$$\frac{q(G)}{\omega} \geqslant \frac{q(Ki_{n,\omega})}{\omega} \geqslant \frac{q(Ki_{\omega+1,\omega})}{\omega} = \frac{2\omega - 1 + \sqrt{4\omega^2 - 12w + 17}}{2\omega}.$$

Now it suffices to show that

$$\frac{2\omega - 1 + \sqrt{4\omega^2 - 12w + 17}}{2\omega} > \frac{q(Ki_{n,3})}{3} + \frac{(\omega-2)(\omega-3)}{\omega^3},$$

that is, by the upper bounds in (3.4), it suffices to show that

$$(3.5)\qquad \frac{2\omega - 1 + \sqrt{4\omega^2 - 12w + 17}}{2\omega} > 1.557 + \frac{(\omega-2)(\omega-3)}{\omega^3}.$$

For $4 \leqslant \omega \leqslant 7$, we can check directly that the result in (3.5) holds. For $\omega \geqslant 8$,

$$\frac{2\omega - 1 + \sqrt{4\omega^2 - 12w + 17}}{2\omega} > 1.769 > 1.557 + \frac{1}{\omega} > 1.557 + \frac{(\omega-2)(\omega-3)}{\omega^3}$$

as $f_2(x) = 1 - 1/2x + \sqrt{1 - 3/x + 17/(4x^2)}$ is an increasing function on $x \geqslant 8$.

Therefore (3.5) holds and hence this completes the proof of the lower bound.

Now, we turn to proving the upper bound of Theorem 1.2.

By Lemma 2.2, we can get our required result for $\omega = 2$. When $\omega \geqslant 3$, combining Lemmas 2.2 and 2.3, it suffices to show that

$$\frac{q(G)}{\omega} \leqslant \frac{q(T_{n,\omega})}{\omega} \leqslant \frac{2n}{\omega} - \frac{2n}{\omega^2} \leqslant \frac{n}{2} - \Big(1 - \frac{2}{\omega}\Big),$$

that is,

$$\frac{n}{2} - 1 - \frac{2n}{\omega} + \frac{2n}{\omega^2} + \frac{2}{\omega} \geqslant 0. \tag{3.6}$$

Note that $f_3(x) = n/2 - 1 - 2n/x + 2n/x^2 + 2/x$ is an increasing function on $x \geqslant 3$. Thus, $f_3(\omega) \geqslant f_3(3) = n/2 - 1 - 2n/3 + 2n/9 + 2/3 = (n-6)/18 \geqslant 0$, and hence (3.6) holds.

Furthermore, if the equality holds in the upper bound of Theorem 1.2, then the equality of (3.6) holds, which implies that $n = 6$, $\omega = 3$ and G is isomorphic to a complete 3-partite graph with two vertices in each part by Lemma 2.3 (as $f_3(\omega) = f_3(3) = 0$). Conversely, if G is isomorphic to a complete 3-partite graph with two vertices in each part, then it can be directly checked that the equality holds in the upper bound of Theorem 1.2. $\square$

To prove Theorem 1.3, we need to introduce some more notation: Let D_1 be a graph of order $\omega + 2$ obtained from a complete graph K_ω by attaching two pendent vertices at two different vertices in the clique, and let D_2 be a graph of order $\omega + 2$ obtained from a complete graph K_ω by attaching two pendent vertices at one vertex in the clique.

Proof of Theorem 1.3. If $\omega = n - 1$, then $\mu(G) = n$ as G is connected. Therefore, the equality holds in (1.1). Otherwise, $\omega \leqslant n - 2$. Then $Ki_{\omega+2,\omega} \subseteq G$ or $D_1 \subseteq G$ or $D_2 \subseteq G$. It is easy to see that $\mu(D_2) = \omega + 2$, and we have to determine the values of $\mu(Ki_{\omega+2,\omega})$ and $\mu(D_1)$.

One can easily see that $\mu(Ki_{\omega+2,\omega})$ satisfies the system of equations

$$\begin{cases} (\mu - \omega)x_1 = -(\omega - 1)\,x_2 - x_3, \\ (\mu - 1)x_2 = -x_1, \\ (\mu - 2)x_3 = -x_1 - x_4, \\ (\mu - 1)x_4 = -x_3. \end{cases}$$

Thus, $\mu(Ki_{\omega+2,\omega})$ satisfies $f_4(x) = 0$, where

$$f_4(x) = x^3 - (\omega + 4)x^2 + (3\omega + 4)x - (\omega + 2).$$

Note that $f_4(x) \to \infty$ as $x \to \infty$ and $f_4(\omega + 1 + 1/w^2) = -(\omega - 1)(\omega^4 + 2\omega^3 + \omega + 1)/\omega^6 < 0$. Therefore, we have

$$\mu(Ki_{\omega+2,\omega}) > \omega + 1 + \frac{1}{w^2}.$$

One can easily see that $\mu(D_1)$ satisfies the following system of equations:

$$\begin{cases} (\mu - \omega)x_1 = -(\omega - 2)x_2 - x_3 - x_4, \\ (\mu - 2)x_2 = -x_1 - x_3, \\ (\mu - \omega)x_3 = -(\omega - 2)x_2 - x_1 - x_5, \\ (\mu - 1)x_4 = -x_1, \\ (\mu - 1)x_5 = -x_3. \end{cases}$$

Thus, $\mu(D_1)$ satisfies $f_5(x) = 0$, where

$$f_5(x) = x^2 - (\omega + 2)x + \omega.$$

Since $f_5(x) \to \infty$ as $x \to \infty$ and $f_5(\omega + 1 + 1/w^2) = -(\omega^4 - \omega^3 - 1)/\omega^4 < 0$, we have

$$\mu(D_2) > \mu(D_1) > \omega + 1 + \frac{1}{\omega^2}.$$

Recall that $\omega \leqslant n - 2$. Thus,

$$\frac{\mu(G)}{\omega} > 1 + \frac{1}{\omega} + \frac{1}{\omega^3} > 1 + \frac{1}{\omega} + \frac{n - \omega - 1}{(n - \omega)\omega^3}.$$

This completes the proof. $\square$

Remark 3.2. From the proof of Theorem 1.3, one can easily see that

$$\frac{\mu(G)}{\omega} > 1 + \frac{1}{\omega} + \frac{1}{\omega^3}$$

for $2 \leqslant \omega \leqslant n - 2$.

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