

Sebastian M. Cioabă; Xiaofeng Gu

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CONNECTIVITY, TOUGHNESS, SPANNING TREES OF BOUNDED  
DEGREE, AND THE SPECTRUM OF REGULAR GRAPHS

SEBASTIAN M. CIOABĂ, Newark, XIAOFENG GU, Carrollton

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*Dedicated to the memory of Professor Miroslav Fiedler*

*Abstract.* The eigenvalues of graphs are related to many of its combinatorial properties. In his fundamental work, Fiedler showed the close connections between the Laplacian eigenvalues and eigenvectors of a graph and its vertex-connectivity and edge-connectivity.

We present some new results describing the connections between the spectrum of a regular graph and other combinatorial parameters such as its generalized connectivity, toughness, and the existence of spanning trees with bounded degree.

*Keywords:* spectral graph theory; eigenvalue; connectivity; toughness; spanning  $k$ -tree

*MSC 2010:* 05C50, 05C40, 05C42, 05E99, 05C05, 15A18

## 1. INTRODUCTION

The spectrum of a graph is related to many important combinatorial parameters. In his fundamental and ground-breaking works, Fiedler [17], [16] determined close connections between the Laplacian eigenvalues and eigenvectors of a graph and combinatorial parameters such as its vertex-connectivity or edge-connectivity. Fiedler's work has stimulated tremendous progress and growth in spectral graph theory since then.

In this paper, we study the connections between the spectrum of a regular graph and other combinatorial parameters such as generalized connectivity, toughness and the existence of spanning trees with bounded degree.

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Throughout this paper, we consider only finite, undirected and simple graphs. Given a graph  $G = (V, E)$  of order  $n$ , we denote by  $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$  the eigenvalues of its adjacency matrix. When the graph  $G$  is clear from the context, we use  $\lambda_i$  to denote  $\lambda_i(G)$ . We also use the notation  $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$ . If  $G$  is  $d$ -regular, then  $\lambda_1 = d$  and the multiplicity of  $d$  equals the number of components of  $G$ . We use  $\kappa(G)$ ,  $\kappa'(G)$  and  $c(G)$  to denote the vertex-connectivity, the edge-connectivity and the number of components of a graph  $G$ , respectively. For any undefined graph theoretic notions, see Bondy and Murty [3] or Brouwer and Haemers [6].

One of the well-known results of Fiedler in [17] implies that the vertex-connectivity of a  $d$ -regular graph is at least  $d - \lambda_2$ . This result was improved in certain ranges by Krivelevich and Sudakov in [24] who showed that the vertex-connectivity of a  $d$ -regular graph is at least  $d - 36\lambda^2/d$ . Given an integer  $l \geq 2$ , Chartrand, Kapoor, Lesniak and Lick in [8] defined the  $l$ -connectivity  $\kappa_l(G)$  of a graph  $G$  to be the minimum number of vertices of  $G$  whose removal produces a disconnected graph with at least  $l$  components or a graph with fewer than  $l$  vertices. Thus,  $\kappa_l(G) = 0$  if and only if  $c(G) \geq l$  or  $|V(G)| \leq l - 1$ . Note that  $\kappa_2(G) = \kappa(G)$ . For  $k \geq 1$ , a graph  $G$  is called  $(k, l)$ -connected if  $\kappa_l \geq k$ . See [8], [14], [23], [32] for more about  $l$ -connectivity and  $(k, l)$ -connected graphs. In particular, a structural characterization of  $(2, l)$ -connected graphs is presented in [23], as a generalization of the standard characterization of 2-connected graphs (see [3], Chapter 5).

Our results relating the generalized connectivity to the spectrum of a regular graph are below.

**Theorem 1.1.** *Let  $l, k$  be integers such as  $l \geq k \geq 2$ . For any connected  $d$ -regular graph  $G$  with  $|V(G)| \geq k + l - 1$ ,  $d \geq 3$  and edge connectivity  $\kappa'$ , if  $\kappa' = d$ , or if  $\kappa' < d$  and*

$$\lambda_{\lceil (l-k+1)d/(d-\kappa') \rceil}(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $\kappa_l(G) \geq k$ .

**Corollary 1.2.** *Let  $l \geq 2$ . For any connected  $d$ -regular graph  $G$  with  $|V(G)| \geq l + 1$  and  $d \geq 3$ , if*

$$\lambda_l(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $\kappa_l(G) \geq 2$ .

**Corollary 1.3.** For any connected  $d$ -regular graph  $G$  with  $d \geq 3$ , if

$$\lambda_2(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $\kappa(G) \geq 2$ .

Corollary 1.3 is a slight improvement of previous results of Krivelevich and Sudakov [24], Theorem 4.1, and Fiedler [17], Theorem 4.1.

The toughness  $t(G)$  of a connected graph  $G$  is defined as  $t(G) = \min\{|S| \times (c(G-S))^{-1}\}$ , where the minimum is taken over all proper subsets  $S \subset V(G)$  such that  $c(G-S) > 1$ . A graph  $G$  is  $t$ -tough if  $t(G) \geq t$ . This parameter was introduced by Chvátal [9] in 1973 and is closely related to many graph properties, including Hamiltonicity, pancyclicity and spanning trees, see [2]. By the definitions of toughness and generalized connectivity, for a noncomplete connected graph  $G$  we have  $t(G) = \min_{2 \leq l \leq \alpha} \{\kappa_l(G)/l\}$  where  $\alpha$  is the independence number of  $G$  (see also [14]).

The relationship between the toughness of a regular graph and the eigenvalues has been considered by many researchers, among which Alon [1] is the first.

**Theorem 1.4** (Alon [1]). For any connected  $d$ -regular graph  $G$ ,

$$t(G) > \frac{1}{3} \left( \frac{d^2}{d\lambda + \lambda^2} - 1 \right).$$

Around the same time, Brouwer [5] independently discovered a slightly better bound of  $t(G)$ .

**Theorem 1.5** (Brouwer [5]). For any connected  $d$ -regular graph  $G$ ,

$$t(G) > \frac{d}{\lambda} - 2.$$

Brouwer in [4] conjectured that the lower bound from the previous theorem can be improved to  $d/\lambda - 1$  for any connected  $d$ -regular graph  $G$ . For the special case of toughness 1, Liu and Chen in [27] improved Brouwer's previous result.

**Theorem 1.6** (Liu and Chen [27]). For any connected  $d$ -regular graph  $G$ , if

$$\lambda_2(G) < \begin{cases} d-1 + \frac{3}{d+1} & \text{if } d \text{ is even,} \\ d-1 + \frac{2}{d+1} & \text{if } d \text{ is odd,} \end{cases}$$

then  $t(G) \geq 1$ .

Recently, Cioabă and Wong in [12] further improved the above result.

**Theorem 1.7** (Cioabă and Wong [12]). *For any connected  $d$ -regular graph  $G$ , if*

$$\lambda_2(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $t(G) \geq 1$ .

Moreover, Cioabă and Wong in [12] showed that the previous result is the best possible by constructing  $d$ -regular graphs whose second largest eigenvalues equal the right-hand-side of the inequality from the previous theorem, but with toughness less than 1. An immediate corollary of the previous result is the following.

**Corollary 1.8** (Cioabă and Wong [12]). *For any bipartite connected  $d$ -regular graph  $G$ , if*

$$\lambda_2(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $t(G) = 1$ .

These authors also found the second largest eigenvalue condition for  $t(G) \geq \tau$ , where  $\tau \leq \kappa'/d$  is a positive number.

**Theorem 1.9** (Cioabă and Wong [12]). *Let  $G$  be a connected  $d$ -regular graph with edge connectivity  $\kappa'$  and  $d \geq 3$ . Suppose that  $\tau$  is a positive number such that  $\tau \leq \kappa'/d$ . If  $\lambda_2(G) < d - \tau d/(d+1)$ , then  $t(G) \geq \tau$ .*

In this paper, we continue investigating the relationship between toughness of a regular graph and its eigenvalues. The following theorems are the main results. As  $\lceil d/(d-\kappa') \rceil \geq 2$ , Theorem 1.10 is an improvement of Theorem 1.7. For bipartite regular graphs, Theorem 1.11 improves Corollary 1.8. We shall also mention that in Theorem 1.9 the eigenvalue condition is not needed, see Theorem 1.12. As an application of Theorem 1.12, Corollary 1.13 confirms a conjecture of Brouwer [4] when  $\kappa' < d$ .

**Theorem 1.10.** *Let  $G$  be a connected  $d$ -regular graph with  $d \geq 3$  and edge connectivity  $\kappa'$ . If  $\kappa' = d$ , or if  $\kappa' < d$  and*

$$\lambda_{\lceil d/(d-\kappa') \rceil}(G) < \begin{cases} \frac{d-2+\sqrt{d^2+12}}{2} & \text{if } d \text{ is even,} \\ \frac{d-2+\sqrt{d^2+8}}{2} & \text{if } d \text{ is odd,} \end{cases}$$

then  $t(G) \geq 1$ .

**Theorem 1.11.** *For any bipartite connected  $d$ -regular graph  $G$  with  $\kappa' < d$ , if  $\lambda_{\lceil d/(d-\kappa') \rceil}(G) < d - (d-1)/2d$ , then  $t(G) = 1$ .*

**Theorem 1.12.** *Let  $G$  be a connected  $d$ -regular graph with edge connectivity  $\kappa'$ . Then  $t(G) \geq \kappa'/d$ .*

**Corollary 1.13.** *For any connected  $d$ -regular graph  $G$  with  $d \geq 3$  and edge connectivity  $\kappa' < d$ ,  $t(G) > d/\lambda_2 - 1 \geq d/\lambda - 1$ .*

Recently, there has been a lot of activity concerning connections between the eigenvalues of a graph and the maximum number of edge-disjoint spanning trees that can be packed in the graph [13], [21], [19], [22], [26], [29], [28], [35]. Another interesting problem would be to see how the eigenvalues of a graph influence the types of spanning trees contained in it. For an integer  $k \geq 2$ , a  $k$ -tree is a tree with the maximum degree at most  $k$ . This topic is related to connected factors. A  $[1, k]$ -factor is a spanning subgraph in which each vertex has the degree at least one and at most  $k$ . By definition, a graph  $G$  has a spanning  $k$ -tree if and only if  $G$  has a connected  $[1, k]$ -factor. For more about degree bounded trees, we refer the readers to survey [33]. For spectral conditions of  $k$ -factors in regular graphs, see [11], [20], [31], [30]. In his PhD Dissertation, Wong [35] proved the following sufficient spectral condition for the existence of spanning  $k$ -trees in regular graphs for  $k \geq 3$ .

**Theorem 1.14** (Wong [35]). *Let  $k \geq 3$  and let  $G$  be a connected  $d$ -regular graph. If  $\lambda_4 < d - d/((k-2)(d+1))$ , then  $G$  has a spanning  $k$ -tree.*

In this paper, we improve this result.

**Theorem 1.15.** *Let  $k \geq 3$  and let  $G$  be a connected  $d$ -regular graph with edge connectivity  $\kappa'$ . Let  $l = d - (k-2)\kappa'$ . Each of the following statements holds.*

- (i) *If  $l \leq 0$ , then  $G$  has a spanning  $k$ -tree.*
- (ii) *If  $l > 0$  and  $\lambda_{\lceil 3d/l \rceil} < d - d/((k-2)(d+1))$ , then  $G$  has a spanning  $k$ -tree.*

Note that eigenvalue conditions for the existence of spanning 2-trees (Hamiltonian paths) and Hamiltonian cycles have been obtained by Krivelevich and Sudakov in [25] and Butler and Chung in [7].

## 2. PRELIMINARIES

In this section, we present some eigenvalue interlacing results to be used in our arguments. For a real and symmetric matrix  $M$  of order  $n$  and a natural number  $1 \leq i \leq n$ , we denote by  $\lambda_i(M)$  the  $i$ -th largest eigenvalue of  $M$ . The following interlacing theorem can be found in many textbooks, for example [6], page 35, or [18], page 193, and is usually referred to as Cauchy eigenvalue interlacing.

**Theorem 2.1.** *Let  $A$  be a real symmetric  $n \times n$  matrix and  $B$  a principal  $m \times m$  submatrix of  $A$ . Then  $\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{n-m+i}(A)$  for  $1 \leq i \leq m$ .*

**Corollary 2.2.** *Let  $S_1, S_2, \dots, S_p$  be disjoint subsets of  $V(G)$  with  $e(S_i, S_j) = 0$  for  $i \neq j$ . For  $1 \leq i \leq p$  let  $G[S_i]$  denote the subgraph of  $G$  induced by  $S_i$ . Then*

$$\lambda_p(G) \geq \lambda_p\left(G\left[\bigcup_{i=1}^p S_i\right]\right) \geq \min_{1 \leq i \leq p} \{\lambda_1(G[S_i])\}.$$

Let  $d \geq 3$  be an integer, and let  $\mathcal{X}(d)$  denote the family of all connected irregular graphs with maximum degree  $d$ , order  $n \geq d + 1$  and size  $m$  with  $2m \geq dn - d + 1$  that have at least two vertices of degree  $d$  if  $d$  is odd, and at least three vertices of degree  $d$  if  $d$  is even. If  $t \geq 2$  is an even integer, let  $M_t$  denote the disjoint union of  $t/2$  edges. If  $G$  and  $H$  are two vertex disjoint graphs, the join  $G \vee H$  of  $G$  and  $H$  is the graph obtained by taking the union of  $G$  and  $H$  and adding all the edges between the vertex set of  $G$  and the vertex set of  $H$ . The complement of  $G$  is denoted by  $\overline{G}$ . For  $d \geq 3$ , define  $X_d$  as  $\overline{M}_{d-1} \vee K_2$  if  $d$  is odd and  $\overline{M}_{d-2} \vee K_3$  if  $d$  is even.

**Lemma 2.3** (Cioabă and Wong [12]). *Let  $d \geq 3$  be an integer and  $H \in \mathcal{X}(d)$ . Then*

$$\lambda_1(H) \geq \theta(d) = \begin{cases} \frac{1}{2}(d - 2 + \sqrt{d^2 + 12}) & \text{if } d \text{ is even,} \\ \frac{1}{2}(d - 2 + \sqrt{d^2 + 8}) & \text{if } d \text{ is odd.} \end{cases}$$

*Equality occurs if and only if  $G = X_d$ .*

**Theorem 2.4** (Cioabă [10]). *Let  $k$  and  $d$  be two integers such that  $d \geq k \geq 2$ . If  $G$  is a  $d$ -regular graph with  $\lambda_2(G) < d - 2(k - 1)/(d + 1)$ , then  $\kappa'(G) \geq k$ .*

**Corollary 2.5.** *Let  $G$  be a  $d$ -regular graph with  $d \geq 2$  and edge connectivity  $\kappa' < d$ . Then  $\lambda_2(G) \geq d - 2\kappa'/(d + 1)$ .*

**Proof.** Let  $k = \kappa' + 1$  in the contrapositive of Theorem 2.4. □

### 3. SPECTRUM AND GENERALIZED CONNECTIVITY OF REGULAR GRAPHS

In this section, we prove Theorem 1.1. Corollaries 1.2 and 1.3 obviously follow from Theorem 1.1.

**Proof of Theorem 1.1.** We prove it by contradiction, i.e., we assume that  $\kappa_l(G) < k$ . By definition, there exists a subset  $S \subset V(G)$  with  $|S| \leq k - 1$  such that  $c(G - S) \geq l$ . Let  $s = |S|$ ,  $c = c(G - S)$  and let  $H_1, H_2, \dots, H_c$  be the components of  $G - S$ . For  $1 \leq i \leq c$  let  $n_i = |V(H_i)|$  and let  $t_i$  be the number of edges between  $H_i$  and  $S$ . Then  $t_i \geq \kappa'$  for  $1 \leq i \leq c$ . Since  $G$  is  $d$ -regular,  $\sum_{i=1}^c t_i \leq ds \leq d(k - 1)$ .

As  $d(k - 1) \geq \sum_{i=1}^c t_i \geq c\kappa' \geq l\kappa'$ , we have  $ld - d(k - 1) \leq ld - l\kappa'$ . If  $\kappa' = d$ , then the previous inequality is impossible, a contradiction. Thus, we may assume that  $\kappa' < d$ , and hence  $l \geq (l - k + 1)d/(d - \kappa')$ . We claim that there are at least  $\lceil (l - k + 1)d/(d - \kappa') \rceil$  indices  $i$  such that  $t_i < d$ . Otherwise, there would be at most  $\lceil (l - k + 1)d/(d - \kappa') \rceil - 1$  indices  $i$  such that  $t_i < d$ . In other words, there would be at least  $c - \lceil (l - k + 1)d/(d - \kappa') \rceil + 1$  indices  $i$  with  $t_i \geq d$ . Thus,

$$\begin{aligned} \sum_{i=1}^c t_i &\geq \left( c - \left\lceil \frac{(l - k + 1)d}{d - \kappa'} \right\rceil + 1 \right) d + \left( \left\lceil \frac{(l - k + 1)d}{d - \kappa'} \right\rceil - 1 \right) \kappa' \\ &= cd - \left( \left\lceil \frac{(l - k + 1)d}{d - \kappa'} \right\rceil - 1 \right) (d - \kappa') \\ &> cd - \frac{(l - k + 1)d}{d - \kappa'} (d - \kappa') \\ &= cd - (l - k + 1)d = (c - l)d + (k - 1)d \geq ds, \end{aligned}$$

contrary to  $\sum_{i=1}^c t_i \leq ds$ . Hence, there are at least  $\lceil (l - k + 1)d/(d - \kappa') \rceil$  indices  $i$  such that  $t_i < d$ . Without loss of generality, we may assume these indices are  $1, 2, \dots, \lceil (l - k + 1)d/(d - \kappa') \rceil$ .

For  $1 \leq i \leq \lceil (l - k + 1)d/(d - \kappa') \rceil$ ,  $n_i \geq d + 1$ . Otherwise, if  $n_i \leq d$ , then  $dn_i = t_i + 2|E(H_i)| \leq t_i + n_i(n_i - 1) \leq t_i + d(n_i - 1)$ , which implies  $t_i \geq d$ , contrary to  $t_i < d$ .

Since  $dn_i = t_i + 2|E(H_i)|$  for  $1 \leq i \leq \lceil (l - k + 1)d/(d - \kappa') \rceil$ , if  $d$  is even, hence  $t_i$  is also even, and thus  $t_i \leq d - 2$ . If  $d$  is odd, then  $t_i \leq d - 1$ . As  $n_i \geq d + 1$ , each  $H_i$  contains at least three vertices of degree  $d$  if  $d$  is even, and at least two vertices of degree  $d$  if  $d$  is odd. Thus,  $H_i \in \mathcal{X}_d$  for  $1 \leq i \leq \lceil (l - k + 1)d/(d - \kappa') \rceil$ . By Corollary 2.2 and Lemma 2.3,  $\lambda_{\lceil (l - k + 1)d/(d - \kappa') \rceil}(G) \geq \min_{1 \leq i \leq \lceil (l - k + 1)d/(d - \kappa') \rceil} \{\lambda_1(H_i)\} \geq \theta(d)$ , contrary to the assumption. This completes the proof.  $\square$



4. SPECTRUM AND TOUGHNESS OF REGULAR GRAPHS

In this section, we prove Theorems 1.10, 1.11, 1.12 and Corollary 1.13.

**Proof of Theorem 1.10.** We prove it by contradiction, i.e., we assume that  $t(G) < 1$ . By definition, there exists a subset  $S \subset V(G)$  such that  $|S|/(c(G - S)) < 1$ . Let  $s = |S|$ ,  $c = c(G - S)$  and let  $H_1, H_2, \dots, H_c$  be the components of  $G - S$ . For  $1 \leq i \leq c$  let  $n_i = |V(H_i)|$  and let  $t_i$  be the number of edges between  $H_i$  and  $S$ . Then  $s < c$  and  $t_i \geq \kappa'$  for  $1 \leq i \leq c$ . Since  $G$  is  $d$ -regular,  $\sum_{i=1}^c t_i \leq ds$ .

As  $c\kappa' \leq \sum_{i=1}^c t_i \leq ds \leq d(c - 1)$ , we have  $c(d - \kappa') \geq d$ . If  $\kappa' = d$ , then we get a contradiction. Thus, we may assume that  $\kappa' < d$ , and so  $c \geq d/(d - \kappa')$ . We claim that there are at least  $\lceil d/(d - \kappa') \rceil$  indices  $i$  such that  $t_i < d$ . Otherwise, there would be at most  $\lceil d/(d - \kappa') \rceil - 1$  indices  $i$  such that  $t_i < d$ . In other words, there would be at least  $c - \lceil d/(d - \kappa') \rceil + 1$  indices  $i$  with  $t_i \geq d$ . Thus,

$$\begin{aligned} \sum_{i=1}^c t_i &\geq \left(c - \left\lceil \frac{d}{d - \kappa'} \right\rceil + 1\right)d + \left(\left\lceil \frac{d}{d - \kappa'} \right\rceil - 1\right)\kappa' \\ &= cd - \left(\left\lceil \frac{d}{d - \kappa'} \right\rceil - 1\right)(d - \kappa') \\ &> cd - \frac{d}{d - \kappa'}(d - \kappa') = cd - d \geq ds, \end{aligned}$$

contrary to  $\sum_{i=1}^c t_i \leq ds$ . Thus, there are at least  $\lceil d/(d - \kappa') \rceil$  indices  $i$  such that  $t_i < d$ .

Without loss of generality, we may assume these indices are  $1, 2, \dots, \lceil d/(d - \kappa') \rceil$ .

For  $1 \leq i \leq \lceil d/(d - \kappa') \rceil$  we have  $n_i \geq d + 1$ . Otherwise, if  $n_i \leq d$ , then  $dn_i = t_i + 2|E(H_i)| \leq t_i + n_i(n_i - 1) \leq t_i + d(n_i - 1)$ , which implies  $t_i \geq d$ , contrary to  $t_i < d$ .

Since  $dn_i = t_i + 2|E(H_i)|$  for  $1 \leq i \leq \lceil d/(d - \kappa') \rceil$ , so if  $d$  is even, then  $t_i$  is also even, and thus  $t_i \leq d - 2$ . If  $d$  is odd, then  $t_i \leq d - 1$ . As  $n_i \geq d + 1$ , each  $H_i$  contains at least three vertices of degree  $d$  if  $d$  is even, and at least two vertices of degree  $d$  if  $d$  is odd. Thus,  $H_i \in \mathcal{X}_d$  for  $1 \leq i \leq \lceil d/(d - \kappa') \rceil$ . By Corollary 2.2 and Lemma 2.3,  $\lambda_{\lceil d/(d - \kappa') \rceil}(G) \geq \min_{1 \leq i \leq \lceil d/(d - \kappa') \rceil} \{\lambda_1(H_i)\} \geq \theta(d)$ , contrary to the assumption. This completes the proof.  $\square$

**Lemma 4.1.** For any bipartite regular graph  $G$ ,  $t(G) \leq 1$ .

**Proof.** Let  $S$  be the set of vertices of one part of the bipartition. Then this equation  $c(G - S) = |S|$  holds. Thus,  $t(G) \leq |S|/(c(G - S)) = 1$ .  $\square$

**Proof of Theorem 1.11.** We prove it by contradiction, i.e., we assume that  $t(G) \neq 1$ . By Lemma 4.1,  $t(G) < 1$ . By definition, there exists a subset  $S \subset V(G)$  such that  $|S|/(c(G-S)) < 1$ . An argument similar to that in the proof of Theorem 1.10 shows that there are at least  $\lceil d/(d-\kappa') \rceil$  components  $H_i$  of  $G-S$  such that  $t_i < d$ , where  $t_i$  is the number of edges between  $H_i$  and  $S$  for  $1, 2, \dots, \lceil d/(d-\kappa') \rceil$ . Let  $n_i = |V(H_i)|$  and let  $m_i = |E(H_i)|$  for  $1, 2, \dots, \lceil d/(d-\kappa') \rceil$ . Then  $2m_i = dn_i - t_i \geq dn_i - d + 1$ . As each  $H_i$  is also bipartite,  $m_i \leq n_i^2/4$ . Thus,  $n_i^2/2 \geq 2m_i \geq dn_i - d + 1$ , which implies that  $n_i^2 - 2dn_i + 2d - 2 \geq 0$ . Hence,  $n_i \geq 2d$ . By Corollary 2.2,

$$\begin{aligned} \lambda_{\lceil d/(d-\kappa') \rceil}(G) &\geq \min_{1 \leq i \leq \lceil d/(d-\kappa') \rceil} \{\lambda_1(H_i)\} \geq \min_{1 \leq i \leq \lceil d/(d-\kappa') \rceil} \left\{ \frac{2m_i}{n_i} \right\} \\ &\geq \frac{dn_i - d + 1}{n_i} \geq d - \frac{d-1}{2d}, \end{aligned}$$

contrary to the assumption. This completes the proof.  $\square$

**Proof of Theorem 1.12.** Suppose that  $S$  is a vertex-cut of  $G$ . Let  $s = |S|$ ,  $c = c(G-S)$  and let  $H_1, H_2, \dots, H_c$  be the components of  $G-S$ . For  $1 \leq i \leq c$  let  $n_i = |V(H_i)|$  and let  $t_i$  be the number of edges between  $H_i$  and  $S$ . Then  $t_i \geq \kappa'$  for  $1 \leq i \leq c$ . As  $G$  is  $d$ -regular,  $\sum_{i=1}^c t_i \leq ds$ . Thus,  $c\kappa' \leq \sum_{i=1}^c t_i \leq ds$ , which implies that  $s/c \geq \kappa'/d$ . Hence,  $t(G) \geq \kappa'/d$ .  $\square$

**Proof of Corollary 1.13.** By Corollary 2.5,  $\lambda_2 \geq d - 2\kappa'/(d+1)$ , which implies that  $2\kappa'/(d - 2\kappa'/(d+1)) \geq d/\lambda_2 - 1$ . If  $d \geq 4$ , then  $\lambda_2 \geq d - 2\kappa'/(d+1) > 2$ . If  $d = 3$ , then  $\kappa' \leq 2$ , and thus  $\lambda_2 \geq d - 2\kappa'/(d+1) \geq 2$ . By Theorem 1.12,

$$t(G) \geq \frac{\kappa'}{d} > \frac{\kappa'/d}{(\lambda_2/2)(1+1/d)} = \frac{2\kappa'}{\lambda_2(d+1)} \geq \frac{d}{\lambda_2} - 1,$$

which completes the proof.  $\square$

## 5. SPECTRUM AND SPANNING $k$ -TREES IN REGULAR GRAPHS

In this section, we prove Theorem 1.15. We will use the following sufficient condition of the existence of a spanning  $k$ -tree obtained by Win [34], which was also proved by Ellingham and Zha [15] with a new proof later.

**Theorem 5.1** (Ellingham and Zha [15], Win [34]). *Let  $k \geq 2$  and let  $G$  be a connected graph. If for any  $S \subseteq V(G)$ ,  $c(G-S) \leq (k-2)|S| + 2$ , then  $G$  has a spanning  $k$ -tree.*

Now we are ready to prove Theorem 1.15.

**Proof of Theorem 1.15.** We prove it by contradiction, i.e., we assume that  $G$  does not have spanning  $k$ -trees for  $k \geq 3$ . By Theorem 5.1, there exists a subset  $S \subseteq V(G)$  such that

$$(5.1) \quad c(G - S) \geq (k - 2)|S| + 3.$$

Let  $s = |S|$ ,  $c = c(G - S)$  and let  $H_1, H_2, \dots, H_c$  be the components of  $G - S$ . For  $1 \leq i \leq c$  let  $n_i = |V(H_i)|$  and let  $t_i$  be the number of edges between  $H_i$  and  $S$ . Then  $t_i \geq \kappa'$  for  $1 \leq i \leq c$ . Since  $G$  is  $d$ -regular,  $c\kappa' \leq \sum_{i=1}^c t_i \leq ds$ . By (5.1),  $s \leq (c - 3)/(k - 2)$ . Thus,  $c\kappa' \leq d(c - 3)/(k - 2)$ , which implies that

$$(5.2) \quad c(d - (k - 2)\kappa') \geq 3d.$$

Thus,  $l = d - (k - 2)\kappa' > 0$ , contrary to (i). This proves (i). Now, we continue to prove (ii).

By (5.2),  $c \geq \lceil 3d/l \rceil$ . We claim that there are at least  $\lceil 3d/l \rceil$  indices  $i$  such that  $t_i < d/(k - 2)$ . Otherwise, there would be at most  $\lceil 3d/l \rceil - 1$  indices  $i$  such that  $t_i < d/(k - 2)$ . In other words, there would be at least  $c - \lceil 3d/l \rceil + 1$  indices  $i$  with  $t_i \geq d/(k - 2)$ . Thus,

$$\begin{aligned} ds &\geq \sum_{i=1}^c t_i \geq \left(c - \left\lceil \frac{3d}{l} \right\rceil + 1\right) \frac{d}{k - 2} + \left(\left\lceil \frac{3d}{l} \right\rceil - 1\right) \kappa' \\ &= \frac{cd}{k - 2} - \left(\left\lceil \frac{3d}{l} \right\rceil - 1\right) \left(\frac{d}{k - 2} - \kappa'\right) \\ &> \frac{cd}{k - 2} - \frac{3d}{l} \left(\frac{d}{k - 2} - \kappa'\right) \\ &= \frac{cd}{k - 2} - \frac{3d}{k - 2} = d \frac{c - 3}{k - 2} \geq ds, \end{aligned}$$

a contradiction. This proves that there are at least  $\lceil 3d/l \rceil$  indices  $i$  such that  $t_i < d/(k - 2)$ . Without loss of generality, we may assume these indices are  $1, 2, \dots, \lceil 3d/l \rceil$ .

For  $1 \leq i \leq \lceil 3d/l \rceil$ , since  $t_i < d/(k - 2)$ , it is not hard to get  $n_i \geq d + 1$  by counting the total degree of  $H_i$ . By Corollary 2.2,  $\lambda_{\lceil 3d/l \rceil}(G) \geq \min_{1 \leq i \leq \lceil 3d/l \rceil} \{\lambda_1(H_i)\} \geq d - d/((k - 2)(d + 1))$ , contrary to the assumption. This completes the proof.  $\square$

## 6. FINAL REMARKS

In this paper, we established some new connections between the spectrum of a regular graph and its generalized connectivity, toughness or the existence of spanning  $k$ -trees. Some of our results are the best possible. For example, the constructions from [12], Section 3, show that the upper bound from Theorem 1.10 is the best possible. Also, Corollary 1.3 is the best possible when  $d = 4$ . To see this, construct a 4-regular graph by taking two disjoint copies of  $X_4$  and adding a new vertex adjacent to 4 vertices (2 in each copy of  $X_4$ ) of degree 3. The resulting graph is 4-regular, has vertex-connectivity 1 and its second largest eigenvalue equals the upper bound from Corollary 1.3.

It would be interesting to improve and generalize our results to general graphs and eigenvalues of Laplacian matrix, signless Laplacian or normalized Laplacian.

### References

- [1] *N. Alon*: Tough Ramsey graphs without short cycles. *J. Algebr. Comb.* *4* (1995), 189–195.
- [2] *D. Bauer, H. J. Broersma, E. Schmeichel*: Toughness of graphs—a survey. *Graphs Comb.* *22* (2006), 1–35.
- [3] *J. A. Bondy, U. S. R. Murty*: *Graph Theory*. Graduate Texts in Mathematics 244, Springer, Berlin, 2008.
- [4] *A. E. Brouwer*: Spectrum and connectivity of graphs. *CWI Quarterly* *9* (1996), 37–40.
- [5] *A. E. Brouwer*: Toughness and spectrum of a graph. *Linear Algebra Appl.* *226/228* (1995), 267–271.
- [6] *A. E. Brouwer, W. H. Haemers*: *Spectra of Graphs*. Universitext, Springer, Berlin, 2012.
- [7] *S. Butler, F. Chung*: Small spectral gap in the combinatorial Laplacian implies Hamiltonian. *Ann. Comb.* *13* (2010), 403–412.
- [8] *G. Chartrand, S. F. Kapoor, L. Lesniak, D. R. Lick*: Generalized connectivity in graphs. *Bull. Bombay Math. Colloq.* *2* (1984), 1–6.
- [9] *V. Chvátal*: Tough graphs and Hamiltonian circuits. *Discrete Math.* *5* (1973), 215–228.
- [10] *S. M. Cioabă*: Eigenvalues and edge-connectivity of regular graphs. *Linear Algebra Appl.* *432* (2010), 458–470.
- [11] *S. M. Cioabă, D. A. Gregory, W. H. Haemers*: Matchings in regular graphs from eigenvalues. *J. Comb. Theory, Ser. B* *99* (2009), 287–297.
- [12] *S. M. Cioabă, W. Wong*: The spectrum and toughness of regular graphs. *Discrete Appl. Math.* *176* (2014), 43–52.
- [13] *S. M. Cioabă, W. Wong*: Edge-disjoint spanning trees and eigenvalues of regular graphs. *Linear Algebra Appl.* *437* (2012), 630–647.
- [14] *D. P. Day, O. R. Oellermann, H. C. Swart*: Bounds on the size of graphs of given order and  $l$ -connectivity. *Discrete Math.* *197/198* (1999), 217–223.
- [15] *M. N. Ellingham, X. Zha*: Toughness, trees, and walks. *J. Graph Theory* *33* (2000), 125–137.
- [16] *M. Fiedler*: A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory. *Czech. Math. J.* *25* (1975), 619–633.
- [17] *M. Fiedler*: Algebraic connectivity of graphs. *Czech. Math. J.* *23* (1973), 298–305.
- [18] *C. Godsil, G. Royle*: *Algebraic Graph Theory*. Graduate Texts in Mathematics 207, Springer, New York, 2001.

- [19] *X. Gu*: Spectral conditions for edge connectivity and packing spanning trees in multi-graphs. *Linear Algebra Appl.* *493* (2016), 82–90.
- [20] *X. Gu*: Regular factors and eigenvalues of regular graphs. *Eur. J. Comb.* *42* (2014), 15–25.
- [21] *X. Gu*: Connectivity and Spanning Trees of Graphs. PhD Dissertation, West Virginia University, 2013.
- [22] *X. Gu, H.-J. Lai, P. Li, S. Yao*: Edge-disjoint spanning trees, edge connectivity and eigenvalues in graphs. *J. Graph Theory* *81* (2016), 16–29.
- [23] *X. Gu, H.-J. Lai, S. Yao*: Characterization of minimally  $(2, l)$ -connected graphs. *Inf. Process. Lett.* *111* (2011), 1124–1129.
- [24] *M. Krivelevich, B. Sudakov*: Pseudo-random graphs. *More Sets, Graphs and Numbers* (E. Györi, ed.). Bolyai Soc. Math. Stud. 15, Springer, Berlin, 2006, pp. 199–262.
- [25] *M. Krivelevich, B. Sudakov*: Sparse pseudo-random graphs are Hamiltonian. *J. Graph Theory* *42* (2003), 17–33.
- [26] *G. Li, L. Shi*: Edge-disjoint spanning trees and eigenvalues of graphs. *Linear Algebra Appl.* *439* (2013), 2784–2789.
- [27] *B. Liu, S. Chen*: Algebraic conditions for  $t$ -tough graphs. *Czech. Math. J.* *60* (2010), 1079–1089.
- [28] *Q. Liu, Y. Hong, X. Gu, H.-J. Lai*: Note on edge-disjoint spanning trees and eigenvalues. *Linear Algebra Appl.* *458* (2014), 128–133.
- [29] *Q. Liu, Y. Hong, H.-J. Lai*: Edge-disjoint spanning trees and eigenvalues. *Linear Algebra Appl.* *444* (2014), 146–151.
- [30] *H. Lu*: Regular graphs, eigenvalues and regular factors. *J. Graph Theory* *69* (2012), 349–355.
- [31] *H. Lu*: Regular factors of regular graphs from eigenvalues. *Electron. J. Comb.* *17* (2010), Research paper 159, 12 pages.
- [32] *O. R. Oellermann*: On the  $l$ -connectivity of a graph. *Graphs Comb.* *3* (1987), 285–291.
- [33] *K. Ozeki, T. Yamashita*: Spanning trees: A survey. *Graphs Comb.* *27* (2011), 1–26.
- [34] *S. Win*: On a connection between the existence of  $k$ -trees and the toughness of a graph. *Graphs Comb.* *5* (1989), 201–205.
- [35] *W. Wong*: Spanning Trees, Toughness, and Eigenvalues of Regular Graphs. PhD Dissertation, University of Delaware, 2013, available at <http://pqdtopen.proquest.com/doc/1443835286.html?FMT=ABS>.

*Authors' addresses:* Sebastian M. Cioabă, Department of Mathematical Sciences, University of Delaware, 501 Ewing Hall, Newark, Delaware 19716, USA, e-mail: [cioaba@udel.edu](mailto:cioaba@udel.edu); Xiaofeng Gu, Department of Mathematics, University of West Georgia, 1601 Maple St., Carrollton 30118, Georgia, USA, e-mail: [xgu@westga.edu](mailto:xgu@westga.edu).