Czechoslovak Mathematical Journal

Shaun Fallat; Abolghasem Soltani

Line graphs: their maximum nullities and zero forcing numbers

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 3, 743–755

Persistent URL: `http://dml.cz/dmlcz/145869`

Czechoslovak Mathematical Journal, 66 (141) (2016), 743–755

LINE GRAPHS: THEIR MAXIMUM NULLITIES AND ZERO FORCING NUMBERS

Shaun Fallat, Regina, Abolghasem Soltani, Tehran

(Received October 15, 2015)

Cordially dedicated to the mathematical legacy of Professor Miroslav Fiedler

Abstract. The maximum nullity over a collection of matrices associated with a graph has been attracting the attention of numerous researchers for at least three decades. Along these lines various zero forcing parameters have been devised and utilized for bounding the maximum nullity. The maximum nullity and zero forcing number, and their positive counterparts, for general families of line graphs associated with graphs possessing a variety of specific properties are analysed. Building upon earlier work, where connections to the minimum rank of line graphs were established, we verify analogous equations in the positive semidefinite cases and coincidences with the corresponding zero forcing numbers. Working beyond the case of trees, we study the zero forcing number of line graphs associated with certain families of unicyclic graphs.

Keywords: maximum nullity; zero forcing number; positive zero forcing number; line graphs; matrix; tree; positive semidefinite matrix; unicyclic graph

MSC 2010: 05C50, 15A18, 15A03

1. Introduction

Throughout this paper all graphs are assumed to be finite, simple, and undirected. We use the notation $G = (V, E)$ to denote the graph with nonempty vertex set $V = V(G)$ and edge set $E = E(G)$. An edge of G with end points u and v is denoted by uv. The order of the graph G is defined to be $|V(G)|$. A graph H is a subgraph of G (denoted $H \leqslant G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of a graph G is said to be *induced* if for any pair of vertices x and y of H, xy is an edge

The research has been supported by an NSERC Discovery Research Grant No. RGPIN-2014-06036.

of H if and only if xy is an edge of G. For a subset S of V, $G \setminus S$ is the subgraph induced on $V \setminus S$. More specifically, for any vertex v and any edge e in a graph G, we let $G - v$ and $G - e$ denote the graphs obtained from G by deleting the vertex v (and all edges incident with v), and by deleting the edge e, respectively. Similarly, if e is not an edge from G, then we let $G + e$ denote the graph obtained from G by adding the edge e.

A path on n vertices is a graph P_n with vertex set $V(P_n) = \{v_1, \dots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} \colon\ i = 1, 2, \dots, n-1\}$. A path in a graph G is said to be a *Hamilton path* if this path includes every vertex in G. A complete graph (or clique) is a graph $K_n = (\{v_1, \dots, v_n\}, E)$ such that $E = \{v_i v_j \colon\ i \neq j\}$. If G and H are two graphs, then the *Cartesian product of G and H*, denoted by $G \,\Box\, H$, is the graph with vertex set $V(G) \times V(H)$ with adjacency defined as: (u, v) is adjacent to (w, z) if and only if $u = w$ and $vz \in E(H)$ or $v = z$ and $uw \in E(G)$.

The *line graph* of a given graph G, denoted by $L(G)$, is obtained by associating a vertex with each edge of G and connecting two vertices with an edge if and only if the corresponding edges of G have a vertex in common.

To a given graph G with $V = \{1, \dots, n\}$, we associate a class of real, $n \times n$ symmetric matrices as follows. The set of symmetric matrices over $\mathbb{R}$ is denoted by $\mathcal{S}_n(\mathbb{R})$. For $A \in \mathcal{S}_n(\mathbb{R})$, the *graph of* A, denoted by $\mathcal{G}(A)$, is the graph with vertex set $\{1, \dots, n\}$ and with edge set $\{ij \colon\ a_{ij} \neq 0 \text{ and } i \neq j\}$. Further, we set $\mathcal{S}(G) = \{A \in \mathcal{S}_n(\mathbb{R}) \colon\ \mathcal{G}(A) \cong G\}$.

Given a graph G, the *minimum rank* of a graph G is defined to be

$$\mathrm{mr}(G) = \min\{\mathrm{rank}(A) \colon\ A \in \mathcal{S}(G)\},$$

while the *maximum nullity* of G is defined as

$$\mathrm{M}(G) = \max\{\mathrm{null}(A) \colon\ A \in \mathcal{S}(G)\}.$$

Both of these quantities have been studied extensively, see, for example [10], [11] and the numerous references listed within. Some basic properties include: $\mathrm{M}(G) + \mathrm{mr}(G) = |V(G)|$, $\mathrm{M}(P_n) = 1$, and $\mathrm{M}(K_n) = n - 1$.

Also, we let $\mathcal{S}_+(G)$ denote the subset of $\mathcal{S}(G)$ consisting of all real positive semidefinite matrices. Further, we let

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}(A) \colon\ A \in \mathcal{S}_+(G)\},$$

and

$$\mathrm{M}_+(G) = \max\{\mathrm{null}(A) \colon\ A \in \mathcal{S}_+(G)\}.$$

The parameter $\mathrm{mr}_+(G)$ is called the *(real) minimum positive semidefinite rank* of G, while $\mathrm{M}_+(G)$ is called the *maximum positive semidefinite nullity* of G (see also [11]). As with the case of standard minimum rank, it is clear that for any graph G, $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |V(G)|$.

Finally, we define some additional concepts and terminology. A *clique covering* of a graph G is a family of cliques in G that cover (or contain) all of the edges of G. The *clique cover number* of a graph G, denoted by $\mathrm{cc}(G)$, is the minimum number of cliques over all clique coverings of G. For every graph G we have $\mathrm{mr}(G) \leqslant \mathrm{mr}_+(G) \leqslant \mathrm{cc}(G)$ (see [5], [10]). Furthermore, if G is chordal (that is, a graph with no induced cycles on four or more vertices), then $\mathrm{mr}_+(G) = \mathrm{cc}(G)$, whereas $\mathrm{mr}(G)$ is often strictly less than $\mathrm{cc}(G)$ for chordal graphs (see also [5]).

2. Zero forcing sets and zero forcing numbers

Let G be a graph with every vertex initially coloured either black or white. If u is a black vertex of G and u has exactly one white neighbour, say v, then we change the colour of v to black; this rule is known as the *colour change rule*. In this case we say "u forces v" which is denoted by $u \to v$. The procedure of colouring a graph using the colour change rule is called the *zero forcing process* (abbreviated to the forcing process). Given an initial colouring of G, the *derived set* is the set of all black vertices resulting from repeatedly applying the colour change rule until no more changes are possible. A *zero forcing set*, Z, is a subset of vertices of G such that if initially the vertices in Z are coloured black and the remaining vertices are coloured white, then the derived set of G is $V(G)$. The *zero forcing number* of a graph G, denoted by $\mathrm{Z}(G)$, is the smallest size of a zero forcing set of G. A zero forcing process is called *optimal* if the initial set of black vertices forms a minimum zero forcing set. In [1], it was verified that for any connected graph G is $\mathrm{Z}(G) \geqslant \mathrm{M}(G)$.

Recently, a variant of the zero forcing number, called positive semidefinite zero forcing number or the positive zero forcing number, was introduced in [3], and a collection of its properties were discussed in [7] and [8]. The *positive zero forcing number* is also based on a colour change rule that is similar to the zero forcing colour change rule. To begin, let G be a graph and B a set of vertices; we will initially colour the vertices of B black and all other vertices white. Let $W_1, \ldots, W_k$ be the set of vertices of the connected components of $G \setminus B$. If u is a vertex in B and w is the only white neighbour of u in the graph induced by $V(W_i) \cup B$, then u can force the colour of w to black. This is the *positive colour change rule*. Definitions and terminology for the positive zero forcing process, such as colouring and positive zero forcing number, etc., are analogous to those for the zero forcing number, except we use the positive semidefinite colour change rule.

The size of the smallest positive zero forcing set of a graph G is denoted by $\mathrm{Z}_+(G)$. Moreover, for all graphs G, since a zero forcing set is also a positive zero forcing set, we have that $\mathrm{Z}_+(G) \leqslant \mathrm{Z}(G)$. Also, in [3] it was observed that $\mathrm{M}_+(G) \leqslant \mathrm{Z}_+(G)$ for any graph G.

3. On the zero forcing numbers of line graphs

We recall some basic results on the minimum rank of line graphs, which can be found in [1]. These known results represent the starting point for our research and will be relied upon throughout our work. We state them here for easier referencing later.

Theorem 3.1 ([1], Corollary 3.19). *If G has $n \geqslant 2$ vertices, then $\mathrm{mr}(L(G)) \leqslant n-2$ or equivalently $\mathrm{M}(L(G)) \geqslant |E(G)| - n + 2$.*

Corollary 3.2 ([1], Corollary 3.20). *If G has $n \geqslant 2$ vertices and contains a Hamilton path, then $\mathrm{mr}(L(G)) = n-2$.*

A connected graph is called *non-separable* if it does not have a cut-vertex. A *block* of a graph is a maximal non-separable subgraph. A graph is *block-clique* if every block is clique. A graph is the line graph of a tree if and only if it is block-clique and no vertex is contained in more than 2 blocks, see also [1].

The minimum rank and zero forcing number of such a block-clique graph are determined in the next fact and can be found in [1].

Proposition 3.3 ([1], Proposition 3.23). *Let G be a block-clique graph of order at least 2 such that no vertex is contained in more than 2 blocks. Then $\mathrm{mr}(G) = \mathrm{cc}(G)$ and $\mathrm{M}(G) = \mathrm{Z}(G)$.*

As noted in [1], Corollary 3.24, it follows that if T is a tree with l pendant vertices, then $\mathrm{Z}(L(T)) = l - 1 = \mathrm{M}(L(T))$.

Moving beyond trees, it was demonstrated in [1] that if G contains a Hamilton path, then $\mathrm{M}(L(G)) = |E(G)| - |V(G)| + 2$. A similar conclusion was also shown if G contains the complete bipartite graph $K_{k,n-k}$. These facts were then extended to include the zero forcing number as follows:

Theorem 3.4 ([1], Proposition 4.9). *Suppose G is a graph that contains a Hamilton path or $K_{k,n-k}$. Then*

$$\mathrm{Z}(L(G)) = \mathrm{M}(L(G)) = |E(G)| - |V(G)| + 2.$$

Using a very basic, yet useful, idea concerning edge addition, we can re-prove these facts differently, which may be of interest to the readers.

Lemma 3.5. *Suppose G is a connected graph and contains a Hamilton path. If $\mathrm{Z}(L(G)) = |E(G)| - |V(G)| + 2$, then for any edge e with $e \notin E(G)$ we have*

$$\mathrm{Z}(L(G+e)) = |E(G)| - |V(G)| + 3.$$

Proof. Define $G_1 \cong G + e$. Then $L(G_1) \cong L(G) + v_e$, where the vertex v_e is adjacent to all the edges of G that are incident with e. Suppose B is a minimum zero forcing set for $L(G)$.

Claim: $B \cup \{v_e\}$ is a zero forcing set for $L(G_1)$.

Proof of claim. Since $L(G)$ and $L(G_1)$ only differ by a single vertex v_e, choosing v_e initially black and using the zero forcing set for $L(G)$ will result in a zero forcing set for $L(G_1)$ in which v_e need not force any vertices. Hence $\mathrm{Z}(L(G_1)) \leqslant \mathrm{Z}(L(G))+1$. □

On the other hand, using Theorem 3.1, we have

$$\mathrm{Z}(L(G_1)) \geqslant |E(G_1)| - |V(G_1)| + 2 = |E(G)| + 1 - |V(G)| + 2 = \mathrm{Z}(L(G)) + 1.$$

Thus $\mathrm{Z}(L(G_1)) = \mathrm{Z}(L(G)) + 1 = |E(G)| - |V(G)| + 3$. □

Now we are ready to provide another proof of Theorem 3.4.

Proof of Theorem 3.4. First suppose G is a graph that contains a Hamilton path. Define a sequence of connected spanning subgraphs of G as:

$$G_0 \cong P_n, \quad G_1 \cong G_0 + e_1, \quad \ldots, \quad G_k \cong G_{k-1} + e_k \cong G.$$

Since G contains a Hamilton path, G contains P_n as a spanning subgraph. Hence, G can be obtained from P_n by adding a sequence of k edges as outlined above. It is easy to show that $\mathrm{Z}(L(G_0)) = \mathrm{M}(L(G_0)) = |E(G_0)| - |V(G_0)| + 2$. By Lemma 3.5, the same type of equality holds for G_1. Furthermore, by a sequential application of Lemma 3.5, we obtain the desired result.

Now assume that G contains the bipartite graph $K_{k,n-k}$ with $1 < k < n-1$. The proof in this case is similar to the proof in the previous case, in which we start with a base graph and derive a sequence of graphs ending with G. Observe that if we choose $G_0 \cong K_{k,n-k}$, then using the fact that $L(K_{k,n-k})$ is isomorphic to $K_k \,\square\, K_{n-k}$, we have $\mathrm{M}(K_s \,\square\, K_t) = \mathrm{Z}(K_s \,\square\, K_t) = st - s - t + 2$ for any positive integers s and t (see also [1], Corollary 3.11). For G_0 we have $\mathrm{M}(L(G_0)) = \mathrm{Z}(L(G_0)) = k(n-k) - n + 2 = |E(G_0)| - n + 2$. Now define an increasing sequence of connected spanning subgraphs of G as:

$$G_0 \cong K_{k,n-k}, \quad G_1 \cong G_0 + e_1, \quad \ldots, \quad G_k \cong G_{k-1} + e_k \cong G.$$

Beginning with G_0 and applying, in succession, Lemma 3.5, we obtain the desired result. □

Similarly to the vertex colour change rule we could define an *edge colour change rule*. Initially, we colour each edge in $E(G)$ either black or white. A black edge f may change the colour of a white edge g to black if and only if g is the only white edge that shares a vertex with the edge f. An edge set $F \subset E(G)$ is called an *edge forcing set for* G if when the edges in F are initially coloured black, and after successive application of the edge colour change rule, the result is that all edges are black. Further, the *edge zero forcing number* $\mathrm{Z}_e(G)$ is the size of the smallest edge forcing set in G. The reader may refer to [9] for additional details and facts about the edge zero forcing number. For our purpose, we observe that $\mathrm{Z}_e(G) = \mathrm{Z}(L(G))$, which follows easily from the definitions of both Z and Z_e.

4. Line graphs: extensions to the positive semidefinite case

The objective of this section is to extend most of the results presented in [1] on the minimum rank of line graphs to the positive semidefinite setting.

The next lemma is easy to establish and can be found in [14].

Lemma 4.1. *Let T be a tree of order n with l pendant vertices. Then $\mathrm{mr}_+(L(T)) = n - l$ and $\mathrm{Z}_+(L(T)) = l - 1 = \mathrm{M}_+(L(T))$.*

The next result, which is simply the positive semidefinite analog of Theorem 3.18 in [1], follows by simply confirming that the matrix

$$M = \begin{bmatrix} I_{n-1} - \frac{1}{n-1} J_{n-1} & D \\ D^{\mathrm{T}} & DD^{\mathrm{T}} \end{bmatrix}$$

is a positive semidefinite matrix (see the proof of [1], Theorem 3.18, where M is defined and used).

Theorem 4.2. *For $n \geqslant 2$ we have $\mathrm{mr}_+(L(K_n)) = n - 2$.*

Corollary 4.3. *For all graphs G of order n we have $\mathrm{mr}_+(L(G)) \leqslant n - 2$, and hence $\mathrm{M}_+(L(G)) \geqslant |E(G)| - |V(G)| + 2$.*

As in [1], we can extend the inequality in Corollary 4.3 to an equality when G is assumed to possess certain additional properties.

Corollary 4.4. *For any connected graph G with a Hamilton path we have*

$$\mathrm{Z}_+(L(G)) = \mathrm{M}_+(L(G)) = |E(G)| - |V(G)| + 2.$$

Proof. For any such graph G, $\mathrm{Z}_+(L(G)) \leqslant \mathrm{Z}(L(G)) = |E(G)| - |V(G)| + 2$. On the other hand, by Corollary 4.3, $|E(G)| - |V(G)| + 2 \leqslant \mathrm{M}_+(L(G))$. Now as $\mathrm{M}_+(L(G)) \leqslant \mathrm{Z}_+(L(G))$, we obtain the result. □

Suppose G of order n contains $K_{k,n-k}$. Then $L(K_{k,n-k})$ is an induced subgraph of $L(G)$. Hence,

$$\mathrm{mr}_+(K_k \,\square\, K_{n-k}) = \mathrm{mr}_+(L(K_{k,n-k})) \leqslant \mathrm{mr}_+(L(G)).$$

Now using [1], Corollary 3.11, we have $\mathrm{mr}_+(K_k \,\square\, K_{n-k}) \geqslant \mathrm{mr}(K_k \,\square\, K_{n-k}) = n-2$, which implies $\mathrm{mr}_+(L(G)) = n - 2$. Consequently, we have the following result for such graphs.

Theorem 4.5. *Suppose G is a graph of order n that contains $K_{k,n-k}$. Then $\mathrm{mr}_+(L(G)) = n - 2$ and*

$$\mathrm{Z}_+(L(G)) = \mathrm{M}_+(L(G)) = |E(G)| - |V(G)| + 2.$$

We note in passing that the quantity $|E(G)| - |V(G)| + 1$ is a well known number in graph theory and other areas of mathematics, often referred to as the *cyclomatic number* of a connected graph G and is denoted by $\mathrm{cy}(G)$ (see [16]). The cyclomatic number is also called the circuit rank, and is the minimum number of edges that are to be deleted so as to remove all of the cycles in a connected graph. In our work, we have established that any connected graph G on n vertices which contains a Hamilton path or $K_{k,n-k}$ satisfies

$$\mathrm{Z}_+(L(G)) = \mathrm{M}_+(L(G)) = \mathrm{Z}(L(G)) = \mathrm{M}(L(G)) = \mathrm{cy}(G) + 1.$$

Furthermore, for any connected graph G we have

$$\mathrm{Z}(L(G)) \geqslant \mathrm{M}(L(G)) \geqslant \mathrm{cy}(G) + 1 \quad \text{and} \quad \mathrm{Z}_+(L(G)) \geqslant \mathrm{M}_+(L(G)) \geqslant \mathrm{cy}(G) + 1.$$

5. Zero forcing number of line graphs of unicyclic graphs

In this section we study the zero forcing number of line graphs associated with unicyclic graphs, satisfying a mild necessary condition.

Recall that a *unicyclic graph* U is a connected graph containing exactly one cycle. Obviously, $|V(U)| = |E(U)|$, and eliminating an edge on the cycle produces a tree. In order to determine the zero forcing number of a unicyclic graph U, the following concepts are employed. A *path covering* of a graph is a family of induced disjoint paths in the graph that cover (or include) all vertices of the graph. The minimum number of such paths that cover the vertices of a graph G is called the *path cover number* of G, and is denoted by $\mathcal{P}(G)$. In [15], it is shown that for any unicyclic graph U, $\mathrm{Z}(U) = \mathcal{P}(U)$. For the remainder of this section, we assume that all unicyclic graphs are connected.

Deleting an edge from a graph may increase or decrease the zero forcing number, but by at most one in either direction. Accordingly, we have the following results, which can be found in [6] and in [13] and [2] from different viewpoints.

Proposition 5.1. *Let G be a connected graph. If $e = uv$ is an edge of G, then* $\mathrm{Z}(G) - 1 \leqslant \mathrm{Z}(G - e) \leqslant \mathrm{Z}(G) + 1$.

Proposition 5.2. *Let $e = uv$ be an edge in a graph G. If there is a minimum zero forcing process for G in which neither $u \to v$ nor $v \to u$, then* $\mathrm{Z}(G) - 1 \leqslant \mathrm{Z}(G - e) \leqslant \mathrm{Z}(G)$. *Otherwise* $\mathrm{Z}(G) \leqslant \mathrm{Z}(G - e) \leqslant \mathrm{Z}(G) + 1$.

Using the above facts, we establish a key result concerning the zero forcing number of the line graph of certain unicyclic graphs.

Theorem 5.3. *Suppose U is a unicyclic graph of girth k with s pendant vertices ($s = 0$ is allowed) and at least one vertex of degree two on its cycle. Then* $s \leqslant \mathrm{Z}(L(U)) \leqslant s + 2$.

Proof. Observe that $L(U)$ contains an induced cycle of length k, call it C_k. Further, note that $L(U)$ is constructed from a cycle of length k, and by identifying edges along this cycle with a collection of block-clique graphs, depending on the nature of the adjacency between the vertices along the cycle of U and the trees or branches attached to these vertices. (Recall that the line graph of a tree is a certain block-clique graph.) Let y be a vertex on the unique cycle C_k of U having degree two, and suppose x and z are the neighbours of y on the cycle. Further, let $e_1 = xy$ and $e_2 = yz$. Then in $L(U)$ let e be the edge incident with vertices e_1 and e_2. It follows $L(U) - e$ is connected and is a block-clique graph as y is of degree two in U. To see this, in U replace the vertex y by two new pendant vertices y' and y'', and replace the edges e_1 and e_2 by $e_1' = xy'$ and $e_2' = zy''$. Call this new graph T. Observe that T is a tree on $|V(U)| + 1$ vertices with exactly $s + 2$ pendant vertices. Thus, $L(T)$ is a connected block-clique graph in which each vertex belongs to at most two cliques, and $L(U) - e \cong L(T)$.

Now by Proposition 3.3, we have $\mathrm{Z}(L(T)) = \mathrm{M}(L(T)) = l - 1$, where l is the number of pendant vertices in T, and hence $\mathrm{Z}(L(U) - e) = l - 1$. So by Proposition 5.1, $l - 2 \leqslant \mathrm{Z}(L(U)) \leqslant l$. We have already established that $l = s + 2$. So in terms of the number of pendants in U, we have $s \leqslant \mathrm{Z}(L(U)) \leqslant s + 2$, which completes the proof. $\square$

Consider the two examples of specific unicyclic graphs given in Figure 1 with $k \geqslant 3$. Observe that $\mathrm{Z}(L(U_1)) = s + 1$ and $\mathrm{Z}(L(U_2)) = s + r$ (the number of pendant vertices in U_2). In fact the upper bound $s + 2$ in Theorem 5.3 is never attained when U contains a vertex of degree two on its unique cycle, except for the case when U has no pendant vertices. To prove this, we need the following lemma.

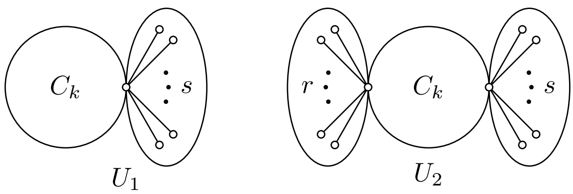

Figure 1. Two unicyclic graphs U_1 and U_2.

Following the derived notation in the above proof, if U is a unicyclic graph with a vertex y of degree two on its unique cycle, then we define the edge e from $L(U)$ as $e = uv$, where $u = xy$ and $v = yz$ are the two edges in U incident with y.

Lemma 5.4. *Suppose U is a unicyclic graph with at least one vertex of degree two on its unique cycle. Let $e = uv$ be the edge defined above based upon this vertex of degree two. Then there is a minimum zero forcing set for $L(U)$ in which either $u \to v$ or $v \to u$.*

Proof. If there is no clique adjacent to either of the vertices incident with e, then we have the desired result. Otherwise, there exists at least one clique K_m with size $m \geqslant 3$, such that $e \cap K_m = u$ or v.

Without loss of generality, suppose that $e \cap K_m = u$. However, $L(T_i')$ is a block-clique graph of order at least 3 and from the proof presented in [1] of Proposition 3.3, since u is not a cut-vertex of $L(T_i')$ (see Figure 2), we can choose u and all remaining non cut-vertices except one as minimum zero forcing set for $L(T_i')$. Hence u forces v, and the proof is complete. $\square$

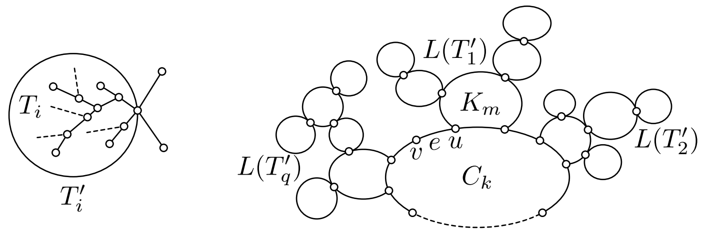

Figure 2. A branch T'_i and the line graph of U.

Corollary 5.5. *If U is a unicyclic graph with at least one vertex of degree two on the unique cycle and with $s > 0$ pendants, then $s \leqslant \mathrm{Z}(L(U)) \leqslant s + 1$.*

P r o o f. According to Proposition 5.2 and Lemma 5.4, the desired result follows. $\square$

An important problem is to characterize the families of unicyclic graphs U with s pendants, such that $\mathrm{Z}(L(U)) = s$ or $\mathrm{Z}(L(U)) = s + 1$. In the next example we introduce a family of unicyclic graphs U with $\mathrm{Z}(U) = \mathrm{Z}(L(U)) = s$.

Example 5.6. We construct unicyclic graph U of girth $k \geqslant 4$ and s pendant vertices. Denote the ordered vertices of C_k by $\{u_1, \ldots, u_k\}$. Add the star graphs $S_{n_1}, S_{n_2}, \ldots, S_{n_t}$, where $1 \leqslant t \leqslant \lfloor \frac{1}{2}k \rfloor$, to the vertices $u_1, u_3, \ldots, u_{2t-1}$ and choose $n_1, n_2, \ldots, n_t$ such that $s = n_1 + n_2 + \ldots + n_t - t$. By a process similar to the one outlined in the proof of Theorem 5.3, we may conclude that $\mathrm{Z}(U) = \mathrm{Z}(L(U)) = s$. This equality holds when we replace each branch of S_{n_i} by different path of arbitrary length, denoted by PS_{n_i} for all $1 \leqslant i \leqslant t$. Also, all unicyclic subgraphs U' of U satisfy $\mathrm{Z}(U') = \mathrm{Z}(L(U'))$.

Note that in Figure 3, we have the special case of unicyclic graph constructed by the method in the above example for which $\mathrm{Z}(U) = \mathrm{Z}(L(U)) = 9$.

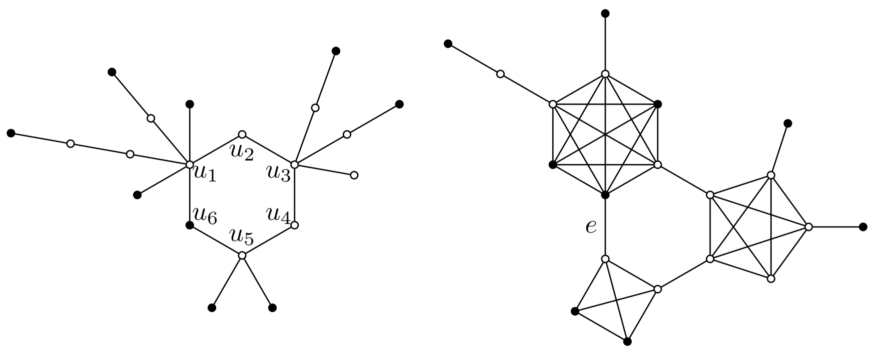

Figure 3. The unicyclic graph U and its line graph.

We actually suspect that all unicyclic graphs U with at least one pendant vertex satisfy

$$s \leqslant \mathrm{Z}(L(U)) \leqslant s+1.$$

The restriction that there is at least one vertex of degree two on the unique cycle was required for our proof technique, although we are not convinced this restriction is necessary. Note though, we do need U to have at least one pendant vertex as the zero forcing number of any cycle is two, whereas the number of pendant vertices is zero.

Thus far we have concentrated on the zero forcing number of unicyclic graphs, but the maximum nullity of unicyclic graphs has also been determined (see [4], Corollary 5.3). In this case the maximum nullity can be expressed in terms of the 'trimmed graph' associated with a given unicyclic graph. In fact, for any unicyclic graph U, the maximum nullity is either $\mathcal{P}(G)$ or $\mathcal{P}(G)-1$, depending on the trimmed form of U, see [4], Corollary 5.3.

Suppose U is a connected unicyclic graph with at least one vertex of degree two on its unique cycle. For such a graph it is not difficult to determine that the trimmed form of U is never a k-sun with k-odd (see [4] or a definition of an n-sun), as every vertex of a k-sun has degree three. Therefore it follows that $\mathrm{M}(U)=\mathcal{P}(U)=\mathrm{Z}(U)$ for such a unicyclic graph U.

Looking at other related parameters, it is well known (see, for example, [10]) that for any graph G and any edge e of G, each of the following inequalities holds: $\mathrm{Z}_+(G)-1 \leqslant \mathrm{Z}_+(G-e) \leqslant \mathrm{Z}_+(G)+1$ (see [7]); $\mathrm{M}_+(G)-1 \leqslant \mathrm{M}_+(G-e) \leqslant \mathrm{M}_+(G)+1$ (see [7]); $\mathrm{M}(G)-1 \leqslant \mathrm{M}(G-e) \leqslant \mathrm{M}(G)+1$ (see [12]).

Combining all of the above inequalities with the proof established for Theorem 5.3 and Lemma 4.1, we have the following consequences.

Corollary 5.7. *If U is a unicyclic graph with at least one vertex of degree two on the unique cycle and with $s>0$ pendants, then*

$$s \leqslant \mathrm{M}(L(U)) \leqslant \mathrm{Z}(L(U)) \leqslant s+1,$$

and

$$s \leqslant \mathrm{M}_+(L(U)) \leqslant \mathrm{Z}_+(L(U)) \leqslant s+1.$$

The various cases of equality have not been characterized. However, for the class of unicyclic graphs U defined in Example 5.6, we have

$$s=\mathrm{M}(U)=\mathrm{Z}(U)=\mathrm{Z}(L(U))=\mathrm{M}(L(U)).$$

The positive semidefinite cases of equality between $\mathrm{M}_+(L(U))$ and $\mathrm{Z}_+(L(U))$ with their counterparts $\mathrm{M}_+(U)$ and $\mathrm{Z}_+(U)$ seem less useful to consider, as it is known that both $\mathrm{M}_+(U)$ and $\mathrm{Z}_+(U)$ are equal to the tree cover number of U (see [10]), where the quantities $\mathrm{M}_+(L(U))$ and $\mathrm{Z}_+(L(U))$ are constrained by the number of pendant vertices in U. In general, the tree cover number, defined in a similar manner as the path cover number, is not typically related to the number of pendant vertices of U.

References

[1] AIM Minimum Rank – Special Graphs Work Group: Zero forcing sets and the minimum rank of graphs. Linear Algebra Appl. *428* (2008), 1628–1648.

[2] *F. Alinaghipour*: Zero Forcing Set for Graphs. PhD Dissertation, University of Regina, 2013.

[3] *F. Barioli, W. Barrett, S. Fallat, H. T. Hall, L. Hogben, B. Shader, P. van den Driessche, H. van der Holst*: Zero forcing parameters and minimum rank problems. Linear Algebra Appl. *433* (2010), 401–411.

[4] *F. Barioli, S. Fallat, L. Hogben*: On the difference between the maximum multiplicity and path cover number for tree-like graphs. Linear Algebra Appl. *409* (2005), 13–31.

[5] *M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, W. Wang*: On the minimum rank among positive semidefinite matrices with a given graph. SIAM J. Matrix Anal. Appl. *30* (2008), 731–740.

[6] *C. J. Edholm, L. Hogben, M. Huynh, J. LaGrange, D. D. Row*: Vertex and edge spread of zero forcing number, maximum nullity, and minimum rank of a graph. Linear Algebra Appl. *436* (2012), 4352–4372.

[7] *J. Ekstrand, C. Erickson, H. T. Hall, D. Hay, L. Hogben, R. Johnson, N. Kingsley, S. Osborne, T. Peters, J. Roat, A. Ross, D. D. Row, N. Warnberg, M. Young*: Positive semidefinite zero forcing. Linear Algebra Appl. *439* (2013), 1862–1874.

[8] *J. Ekstrand, C. Erickson, D. Hay, L. Hogben, J. Roat*: Note on positive semidefinite maximum nullity and positive semidefinite zero forcing number of partial 2-trees. Electron. J. Linear Algebra. (electronic only) *23* (2012), 79–87.

[9] *L. Eroh, C. X. Kang, E. Yi*: A Comparison between the Metric Dimension and Zero Forcing Number of Line Graphs. 2012, 14 pages, arXiv:1207.6127v1 [math.CO].

[10] *S. M. Fallat, L. Hogben*: Chapter 46: Minimum rank, maximum nullity, and zero forcing number of graphs. Handbook of Linear Algebra (L. Hogben, ed.). CRC Press, Boca Raton, 2014, pp. 46-1–46-36.

[11] *S. Fallat, L. Hogben*: The minimum rank of symmetric matrices described by a graph: a survey. Linear Algebra Appl. *426* (2007), 558–582.

[12] *P. M. Nylen*: Minimum-rank matrices with prescribed graph. Linear Algebra Appl. *248* (1996), 303–316.

[13] *K. Owens*: Properties of the Zero Forcing Number. Master's Thesis, Brigham Young University, 2009.

[14] *T. Peters*: Positive semidefinite maximum nullity and zero forcing number. Electron. J. Linear Algebra (electronic only) *23* (2012), 815–830.

[15] *D. D. Row*: A technique for computing the zero forcing number of a graph with a cut-vertex. Linear Algebra Appl. *436* (2012), 4423–4432.

[16] *X. Yu*: Cyclomatic numbers of connected induced subgraphs. Discrete Math. *105* (1992), 275–284.

Authors' addresses: Shaun Fallat, Department of Mathematics and Statistics, University of Regina, 3737 Wascana Parkway, S4S 0A4 Regina SK, Canada, e-mail: `shaun.fallat@uregina.ca`; Abolghasem Soltani, Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, Jalal AleAhmad Nasr No. 7, 14115-134, Tehran, Iran, e-mail: `a.soltani@modares.ac.ir`.