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A TREATMENT OF A DETERMINANT INEQUALITY OF FIEDLER AND MARKHAM

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To the memory of Miroslav Fiedler (1926–2015)

Abstract. Fiedler and Markham (1994) proved

$$\Big(\frac{\det \widehat{H}}{k}\Big)^k \geqslant \det H,$$

where $H = (H_{ij})_{i,j=1}^n$ is a positive semidefinite matrix partitioned into $n \times n$ blocks with each block $k \times k$ and $\widehat{H} = (\operatorname{tr} H_{ij})_{i,j=1}^n$. We revisit this inequality mainly using some terminology from quantum information theory. Analogous results are included. For example, under the same condition, we prove

$$\det(I_n + \widehat{H}) \geqslant \det(I_{nk} + kH)^{1/k}.$$

Keywords: determinant inequality; partial trace

MSC 2010: 15A45

1. Introduction

We are interested in complex matrices partitioned into $n \times n$ blocks

$$H = \begin{pmatrix} H_{11} & \dots & H_{1n} \\ \vdots & \ddots & \vdots \\ H_{n1} & \dots & H_{nn} \end{pmatrix}$$

with each block $k \times k$. To save room, we usually put $H = (H_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$. A map (not necessarily linear) $\varphi\colon\ \mathbb{M}_k \to \mathbb{M}_m$ is completely positive, see [1], page 92, if for each $n \geqslant 1$, $(\varphi(H_{ij}))_{i,j=1}^n$ is (Hermitian) positive semidefinite whenever $(H_{ij})_{i,j=1}^n$ is positive semidefinite.

It is known that the trace map and determinant map are completely positive. Let $E_r(H_{ij})$, $1 \leqslant r \leqslant k$, denote the r-th elementary symmetric function of the eigenvalues of the matrix H_{ij}. As early as last 60s, de Pillis [3] showed that $\varphi = E_r$ is a completely positive map. In [4], Fiedler and Markham gave an almost "visualized" proof of this fact. As a byproduct of their argument (plus a clever use of Oppenheim's inequality), Fiedler and Markham derived the following determinant inequality, which is the starting point of this paper.

Proposition 1.1 ([4], Corollary 1). *Let $H = (H_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ be positive semidefinite. Then*

$$\left(\frac{\det \widehat{H}}{k}\right)^k \geqslant \det H, \tag{1.1}$$

where $\widehat{H} = (\operatorname{tr} H_{ij})_{i,j=1}^n$. Moreover, the constant k in (1.1) is optimal.

Now we introduce the definition of partial traces, a notion from quantum information theory. For any $H \in \mathbb{M}_n(\mathbb{M}_k)$, we may write $H = \sum\limits_{i=1}^{p} A_i \otimes B_i$ for some $A_i \in \mathbb{M}_n$, $B_i \in \mathbb{M}_k$ and some positive integer p. Here "$\otimes$" stands for the tensor product. The partial traces $\operatorname{tr}_1 H \in \mathbb{M}_n$ and $\operatorname{tr}_2 H \in \mathbb{M}_m$ are defined (see, e.g., [10], page 12), respectively, as

$$\operatorname{tr}_1 H = \sum_{i=1}^{p} (\operatorname{tr} A_i) B_i, \quad \operatorname{tr}_2 H = \sum_{i=1}^{p} (\operatorname{tr} B_i) A_i.$$

In other words, tr_1 "traces out" the first factor and tr_2 "traces out" the second factor. The actual forms of the partial traces are seen in [10], page 12:

$$\operatorname{tr}_1 H = \sum_{i=1}^{n} H_{ii}, \quad \operatorname{tr}_2 H = (\operatorname{tr} H_{ij})_{i,j=1}^n.$$

It is easy to see that partial trace maps are completely positive. With what has been just defined, (1.1) can be written as

$$\left(\frac{\det(\operatorname{tr}_2 H)}{k}\right)^k \geqslant \det H. \tag{1.2}$$

A natural question is whether a result analogous to (1.2) is true for $\operatorname{tr}_1 H$. The answer turns out to be yes. In the next section, we consider some related results, moreover, we give a new proof of (1.2) using an identity connecting $\operatorname{tr}_2 H$ and H.

2. Main results

Proposition 2.1. *Let $H \in \mathbb{M}_n(\mathbb{M}_k)$ be positive semidefinite. Then*

$$\Big(\frac{\det(\operatorname{tr}_1 H)}{n}\Big)^n \geqslant \det H. \tag{2.1}$$

Moreover, the constant n in (2.1) is optimal.

P r o o f. As H is positive semidefinite, it is clear that

$$\sum_{i=1}^n \det H_{ii} \leqslant \det\Big(\sum_{i=1}^n H_{ii}\Big).$$

By Fischer's inequality, see [6], page 506, and the Arithmetic mean-Geometric mean inequality,

$$\begin{aligned}\det H &\leqslant \prod_{i=1}^n \det H_{ii} \leqslant \Big(\frac{1}{n}\sum_{i=1}^n \det H_{ii}\Big)^n \\ &\leqslant \Big(\frac{1}{n}\det\Big(\sum_{i=1}^n H_{ii}\Big)\Big)^n = \Big(\frac{\det(\operatorname{tr}_1(H))}{n}\Big)^n.\end{aligned}$$

This proves the inequality. To see the inequality is sharp, we simply take $H = I_n$, the $n \times n$ identity matrix, with $k = 1$. $\square$

We remark that both (1.2) or (2.1) can be tighter than the other. For example, take $n = 2$ and

$$H = \begin{pmatrix} 2I_k & I_k \\ I_k & I_k \end{pmatrix}.$$

Then $\det H = 1$ and

$$\Big(\frac{\det(\operatorname{tr}_2 H)}{k}\Big)^k = k^k, \quad \Big(\frac{\det(\operatorname{tr}_1 H)}{n}\Big)^n = \Big(\frac{3^k}{2}\Big)^2.$$

Clearly, when $k = 2$,

$$\Big(\frac{\det(\operatorname{tr}_2 H)}{k}\Big)^k = k^k < \Big(\frac{\det(\operatorname{tr}_1 H)}{n}\Big)^n = \Big(\frac{3^k}{2}\Big)^2, \tag{2.2}$$

but the inequality (2.2) reverses for large k.

Inequality (2.1) seems easier to prove than its counterpart (1.2). As promised, we shall give a new proof of (1.2). Our proof relies on the following identity connecting $\operatorname{tr}_2 H$ and H, which can be found in [8]; see also [11], equation (14).

Lemma 2.2. *Let X and Y be generalized Pauli matrices on $\mathbb{C}^k$; these operators act as $Xe_j = e_{j+1}$, $Ye_j = \mathrm{e}^{2\pi \mathrm{i} j/k} e_j$, where i is the imaginary unit, e_j is the j-th column of I_k and $e_{k+1} = e_1$. Then*

$$\frac{1}{k}\sum_{l,\,j=1}^{k}(I_n \otimes X^l Y^j)H(I_n \otimes X^l Y^j)^* = (\mathrm{tr}_2 H)\otimes I_k. \tag{2.3}$$

With relevant changes, a similar identity can be posed for $I_n \otimes (\mathrm{tr}_1 H)$.

P r o o f of (1.2). First of all, note that X, Y in (2.3) are unitary. As the determinant functional is log-concave over the cone of positive semidefinite matrices, see [6], page 488, we have

$$\begin{aligned}\det\biggl(\frac{1}{k^2}\sum_{l,\,j=1}^{k}(I_n \otimes X^l Y^j)H(I_n \otimes X^l Y^j)^*\biggr)&\\ &\geqslant \prod_{j=1}^{k^2}\bigl(\det(I_n \otimes X^l Y^j)H(I_n \otimes X^l Y^j)^*\bigr)^{1/k^2}\\ &= \prod_{j=1}^{k^2}(\det H)^{1/k^2} = \det H.\end{aligned}$$

Then by (2.3), $\det H$ is bounded above by

$$\det\Bigl(\frac{(\mathrm{tr}_2 H)\otimes I_k}{k}\Bigr) = \det\Bigl(\Bigl(\frac{\mathrm{tr}_2 H}{k}\Bigr)\otimes I_k\Bigr) = \Bigl(\frac{\det(\mathrm{tr}_2 H)}{k}\Bigr)^k.$$

Therefore, (1.2) follows. □

A matrix $H = (H_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is said to be positive partial transpose (i.e., PPT) if H is positive semidefinite and its partial transpose $(H_{ji})_{i,j=1}^n$ is also positive semidefinite. It is known, see [7], that if $H = (H_{ij})_{i,j=1}^n \in \mathbb{M}_n(\mathbb{M}_k)$ is PPT, then both

$$\mathrm{tr}_2 H \otimes I_k - H \quad \text{and} \quad I_n \otimes \mathrm{tr}_1 H - H$$

are positive semidefinite.

Making use of a majorization result of Hiroshima [5] (see also [9], Theorem 2.1) and mimicking the proof of [2], Corollary 2.3, we have

Proposition 2.3. *Let $H \in \mathbb{M}_n(\mathbb{M}_k)$ be PPT. Then*

$$\det(I_k + \mathrm{tr}_1 H) \leqslant \det(I_{nk} + H); \tag{2.4}$$

$$\det(I_n + \mathrm{tr}_2 H) \leqslant \det(I_{nk} + H). \tag{2.5}$$

In general, neither (2.4) nor (2.5) need to be true if the PPT assumption is dropped. This motivates us to consider possible extensions of (2.4) and (2.5) without the PPT assumption or at least some analogous results. We need a lemma before presenting our last result.

For a Hermitian matrix $X \in \mathbb{M}_n$, we denote by $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$ the vector of the eigenvalues of X such that $\lambda_1(X) \geqslant \ldots \geqslant \lambda_n(X)$. For two Hermitian matrices $X, Y \in \mathbb{M}_n$, we write $\lambda(X) \prec \lambda(Y)$ if $\sum\limits_{j=1}^{m} \lambda_j(X) \leqslant \sum\limits_{j=1}^{m} \lambda_j(Y)$ for all $m = 1, \ldots, n-1$ and $\sum\limits_{j=1}^{m} \lambda_j(X) = \sum\limits_{j=1}^{m} \lambda_j(Y)$. This is the so called majorization relation. It is clear that if X, Y are positive semidefinite and $\lambda(X) \prec \lambda(Y)$, then $\det(I_n + X) \geqslant \det(I_n + Y)$.

Lemma 2.4 ([6], page 250)**.** *Let X and Y be two Hermitian matrices of the same size. Then*

$$\lambda(X+Y) \prec \lambda(X) + \lambda(Y).$$

Theorem 2.5. *Let $H \in \mathbb{M}_n(\mathbb{M}_k)$ be positive semidefinite. Then*

$$\det(I_k + \operatorname{tr}_1 H) \geqslant \det(I_{nk} + nH)^{1/n}; \tag{2.6}$$

$$\det(I_n + \operatorname{tr}_2 H) \geqslant \det(I_{nk} + kH)^{1/k}. \tag{2.7}$$

Moreover, these inequalities are sharp.

Proof. We prove (2.7) only, as (2.6) can be proved in exactly the same way. By the identity (2.3) and Lemma 2.4, we have

$$\begin{aligned}\lambda(\operatorname{tr}_2 H \otimes I_k) &= \lambda\biggl(\frac{1}{k}\sum_{l,j=1}^{k}(I_n \otimes X^l Y^j) H (I_n \otimes X^l Y^j)^*\biggr)\\ &\prec \frac{1}{k}\sum_{l,j=1}^{k}\lambda((I_n \otimes X^l Y^j) H (I_n \otimes X^l Y^j)^*)\\ &= \frac{1}{k}\sum_{l,j=1}^{k}\lambda(H) = k\lambda(H).\end{aligned}$$

Therefore,

$$\det(I_{nk} + (\operatorname{tr}_2 H) \otimes I_k) \geqslant \det(I_{nk} + kH).$$

But now

$$\det(I_{nk} + (\operatorname{tr}_2 H) \otimes I_k) = \det((I_n + \operatorname{tr}_2 H) \otimes I_k) = \det(I_n + \operatorname{tr}_2 H)^k,$$

yielding the desired result (2.7). To see that (2.6) is sharp, we just take $n = 1$. If we take $k = 1$, then (2.7) becomes an equality. $\square$

Remark 2.6. If we apply (1.2), we could also get a lower bound for $\det(I_n + \operatorname{tr}_2 H)$. The argument goes as follows:

$$\begin{aligned}\det(I_n + \operatorname{tr}_2 H) &= \det\left(\operatorname{tr}_2\left(\frac{I_{nk}}{k} + H\right)\right) \\ &\geqslant k \det\left(\frac{I_{nk}}{k} + H\right)^{1/k} \quad \text{by (1.2)} \\ &= \frac{1}{k^{n-1}} \det(I_{nk} + kH)^{1/k}.\end{aligned}$$

However, this lower bound is weaker than (2.7).

To the author's best knowledge, determinant inequalities involving partial traces have not been extensively investigated in the literature. The author expects there would be a fruitful interplay between determinant and partial traces.

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