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The Group of Invertible Elements of the Algebra of Quaternions

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Abstract

We have, that all two-dimensional subspaces of the algebra of quaternions, containing a unit, are 2-dimensional subalgebras isomorphic to the algebra $\mathbb{C}$ of complex numbers. It was proved in the papers of N. E. Belova. In the present article we consider a 2-dimensional subalgebra $\mathbb{R}(i)$ of complex numbers with basis $\{1, i\}$ and we construct the principal locally trivial bundle which is isomorphic to the Hopf fibration.

Key words: Group of invertible elements, algebra of quaternions, principal locally trivial bundle, 2-dimensional subalgebras, structural group, unit, Hopf fibration.

2010 Mathematics Subject Classification: 53B20, 53B30, 53C21

1 Introduction

The Hopf fibration describes a 3-sphere (a hypersphere in four-dimensional space) in terms of circles and an ordinary sphere in the mathematical field of topology. Discovered by Heinz Hopf in 1931, it is an influential early example of a fiber bundle. Technically, Hopf found a many-to-one continuous function (or "map") from the 3-sphere onto the 2-sphere such that each distinct point of the 2-sphere comes from a distinct circle of the 3-sphere (Hopf 1931). Thus the 3-sphere is composed of fibers, where each fiber is a circle—one for each point of the 2-sphere.

The Hopf fibration, like any fiber bundle, has the important property that it is locally a product space. However it is not a trivial fiber bundle, i.e., S^3

is not globally a product of S^2 and S^1 although locally it is indistinguishable from it.

This has many implications: for example the existence of this bundle shows that the higher homotopy groups of spheres are not trivial in general. It also provides a basic example of a principal bundle, by identifying the fiber with the circle group.

There are numerous generalizations of the Hopf fibration. The unit sphere in complex coordinate space C^{n+1} fibers naturally over the complex projective space CP^n with circles as fibers, and there are also real, quaternionic, and octonionic versions of these fibrations. In particular, the Hopf fibration belongs to a family of four fiber bundles in which the total space, base space, and fiber space are all spheres.

The Hopf fibration is important in twistor theory.

Mathematicians from Kazan developed geometric problems on manifolds over algebras, see V. V. Vishnevski, A. P. Shirokov, V. V. Shurigin [22], B. N. Shapukov [21]. These problems were initiated by A. P. Norden [17].

Some problems were studied onto algebras of quaternions [12] and anti-quaternions [13]. Problems of differential geometry associated with quaternions and antiquaternions have been studied in e.g. [1, 2, 3, 9, 10, 17, 11, 14, 15].

We have, that all two-dimensional subspaces of the algebra of quaternions, containing a unit, are 2-dimensional subalgebras isomorphic to the algebra $\mathbb{C}$ of complex numbers. It was proved in the papers of N. E. Belova [4, 5].

In the present article we consider a 2-dimensional subalgebra $\mathbb{R}(i)$ of complex numbers with basis $\{1, i\}$ and we construct the principal locally trivial bundle which is isomorphic to the Hopf fibration.

2 Expression of quaternions with the basis $\{1, i, j, k\}$

The Hopf fibration $\pi\colon S^{2n+1} \to \mathbb{C}P^n$ [6], [8] is given as a submersion of the sphere

$$z_1\bar{z}_1 + \ldots + z_{n+1}\bar{z}_{n+1} = 1$$

onto the complex projective space

$$\pi(z_1, \ldots, z_{n+1}) = (z_1 : \cdots : z_{n+1})$$

and the action of the structural group $SO(2, \mathbb{R})$ has the form

$$(z_1, \ldots, z_{n+1}) \to (e^{i\varphi}z_1, \ldots, e^{i\varphi}z_{n+1}). \tag{1}$$

The orbits of this action are great circles of the sphere. Particularly, if $n = 1$, the formula (1) gives the principal bundle of the 3-dimensional sphere above the complex projective line.

Assume the associative unital 4-dimensional algebra $\mathbb{A}$ of type VIa. It is the algebra of quaternions [19, 20] with the basis $1, i, j, k$ and the multiplication table

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Any quaternion can be expressed as $\mathbf{x} = x^0 + x^1 i + x^2 j + x^3 k$, conjugation is given by $\mathbf{x} \mapsto \bar{\mathbf{x}} = x^0 - x^1 i - x^2 j - x^3 k$, $\overline{\mathbf{xy}} = \bar{\mathbf{y}}\bar{\mathbf{x}}$ holds, the number $\mathbf{x}\bar{\mathbf{x}} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ is real, and $\mathbf{x} \mapsto |\mathbf{x}| = \sqrt{\mathbf{x}\bar{\mathbf{x}}}$ defines a module corresponding to the scalar product $\mathbf{xy} = \frac{1}{2}(\mathbf{x}\bar{\mathbf{y}} + \mathbf{y}\bar{\mathbf{x}})$. $|1| = |i| = |j| = |k| = 1$ and the basis elements are called units. For any $\mathbf{x}$ with $|\mathbf{x}| \neq 0$ there exists the inverse element $\mathbf{x}^{-1} = \frac{\bar{\mathbf{x}}}{|\mathbf{x}|^2}$. The set of all invertible elements from the algebra $\mathbb{A}$ of quaternions $\tilde{\mathbb{A}} = \{\mathbf{x} \mid \mathbf{x} \neq 0\}$ is a Lie group [21].

3 Quaternions on 2-dimensional subalgebra $\mathbb{R}(i)$

We have, that all two-dimensional subspaces of the algebra of quaternions, containing a unit, are 2-dimensional subalgebras isomorphic to the algebra $\mathbb{C}$ of complex numbers. It was proved in the papers of N. E. Belova [4, 5]. Consider a 2-dimensional subalgebra $\mathbb{R}(i)$ of complex numbers with basis $\{1, i\}$. The set of its invertible elements

$$\tilde{\mathbb{R}(i)} = \{\lambda = a + bi \mid \lambda \neq 0\},\ a, b \in \mathbb{R}$$

is a Lie subgroup of the group $\tilde{\mathbb{A}}$, a 2-plane with exception of one point.

Let us write a quaternion as

$$\mathbf{x} = x^0 + x^1 i + (x^2 + x^3 i)j = z_1 + z_2 j,\ z_1, z_2 \in \mathbb{R}(i).$$

Then $\bar{\mathbf{x}} = \bar{z}_1 - z_2 j$ and $\mathbf{x}\bar{\mathbf{x}} = z_1\bar{z}_1 + z_2\bar{z}_2$.

Consider the factorset of the right cosets $\tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$. The quaternions $\mathbf{x}, \mathbf{y} \in \tilde{\mathbb{A}}$ belong to the same right coset by $\tilde{\mathbb{R}}(i)$ if and only if $\mathbf{xy}^{-1} \in \tilde{\mathbb{R}}(i)$ holds. But

$$\mathbf{xy}^{-1} = \frac{\mathbf{x}\bar{\mathbf{y}}}{|\mathbf{y}|^2},$$

where

$$\mathbf{x}\bar{\mathbf{y}} = (z_1 + z_2 j)(\bar{w}_1 - w_2 j) = (z_1\bar{w}_1 + z_2\bar{w}_2) + (z_2 w_1 - z_1 w_2)j\,.$$

Therefore, this quaternion is nonzero complex number, if the conditions

$$z_2 w_1 - z_1 w_2 = 0, \quad z_1\bar{w}_1 + z_2\bar{w}_2 \neq 0$$

are satisfied.

The first of them means that $z_1 : z_2 = w_1 : w_2$ and then the second condition gives $z_1\bar{z}_1 + z_2\bar{z}_2 \neq 0$ and similarly $w_1\bar{w}_1 + w_2\bar{w}_2 \neq 0$ holds. Consequently, firstly, the canonical projection $\pi\colon \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ has the form

$$\pi(\mathbf{x}) = (z_1 : z_2). \tag{2}$$

Secondly, the factorset $\tilde{\mathbb{A}}/\tilde{\mathbb{R}}(i)$ is a complex projective line $P(i)$ which is covered by two maps:

$$U_1 = \{[z_1 : z_2] \mid z_2 \neq 0\} \quad \text{with the coordinate } z = \frac{z_1}{z_2},$$

$$U_2 = \{[z_1 : z_2] \mid z_1 \neq 0\} \quad \text{with the coordinate } \tilde{z} = \frac{z_2}{z_1}.$$

4 Principal locally trivial bundle above the complex projective line with the typical fiber

So, we have the principal bundle $E = (\tilde{\mathbb{A}}, \pi, P(i))$ of right cosets $\tilde{\mathbb{R}}(i)\mathbf{x}$. Here the structural group $\tilde{\mathbb{R}}(i)$ acts on the left $\tilde{\mathbb{R}}(i)\mathbf{x} \to \tilde{\mathbb{R}}(i)(\tilde{\mathbb{R}}(i)\mathbf{x}) = \tilde{\mathbb{R}}(i)\mathbf{x}$. The mappings of trivialization and their inverse have the form

$$\varphi_1\colon \pi^{-1}(U_1) \longrightarrow U_1 \times \tilde{\mathbb{R}}(i), \quad \varphi_1(z_1 + z_2 j) = \left(\frac{z_1}{z_2}, z_2\right),$$

$$\varphi_1^{-1}(z,\ \lambda) = \lambda(z + j),$$

$$\varphi_2\colon \pi^{-1}(U_2) \longrightarrow U_2 \times \tilde{\mathbb{R}}(i), \quad \varphi_2(z_1 + z_2 j) = \left(\frac{z_2}{z_1}, z_1\right),$$

$$\varphi_2^{-1}(\tilde{z},\ \lambda) = \lambda(1 + \tilde{z} j).$$

These formulas follow from the equalities

$$\mathbf{x} = z_1 + z_2 j = z_2\left(\frac{z_1}{z_2} + j\right) = z_2(z + j), \quad \text{when } z_2 \neq 0,$$

$$\mathbf{x} = z_1 + z_2 j = z_1\left(1 + \frac{z_2}{z_1} j\right) = z_1(1 + \tilde{z} j), \quad \text{when } z_1 \neq 0.$$

The gluing function is

$$\psi_{12}(z,\ \lambda) = \varphi_2 \circ \varphi_1^{-1} = \left(\frac{1}{z}, \lambda z\right).$$

Therefore, the bundle is locally trivial. Hence, the following theorem is valid.

Theorem 1 *The bundle $(\tilde{\mathbb{A}}, \pi, P(i))$, defined by the formula (2), is the principal locally trivial bundle above the complex projective line with the typical fiber, diffeomorphic to the two-plane without one point and with the structural group $\tilde{\mathbb{R}}(i)$.*

It is known [7, 19], the complex projective line can be regarded as a conformal plane C^2 and using stereographic mapping it can be identified with the two-dimensional sphere S^2. Thus, the examined bundle is isomorphic to the Hopf fibration [18].

Let us find the preimage of an arbitrary point of the base under the mapping π. Let $d = u + iv \in P(i)$ belongs to the neighborhood U_1. We get 2-planes $L_2\colon z_1 - dz_2 = 0$, which are defined by two real equations

$$\begin{aligned} x^0 - ux^2 + vx^3 &= 0, \\ x^1 - vx^2 - ux^3 &= 0. \end{aligned}$$

Similarly, if $\tilde{d} = \tilde{u} + i\tilde{v} \in U_2$ holds, then we obtain 2-planes $\tilde{L_2}\colon z_2 - \tilde{d}z_1 = 0$ with equations

$$\begin{aligned} x^2 - \tilde{u}x^0 + \tilde{v}x^1 &= 0, \\ x^3 - \tilde{v}x^0 - \tilde{u}x^1 &= 0, \end{aligned}$$

where

$$\tilde{u} = \frac{u}{u^2 + v^2}, \qquad \tilde{v} = \frac{-v}{u^2 + v^2}.$$

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