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Geometric Structures in Bundles of Associative Algebras

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Abstract

The article deals with bundles of linear algebra as a specifications of the case of smooth manifold. It allows to introduce on smooth manifold a metric by a natural way. The transfer of geometric structure arising in the linear spaces of associative algebras to a smooth manifold is also presented.

Key words: Geometric structures, bundles of linear algebra, vector bundles on smooth manifolds.

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1 Introduction

Bundles of linear algebra are specifications of vector bundles on smooth manifolds in which the standard layer is a linear algebra. This specification allows for a smooth manifold introduce some metric is naturally associated with a given algebra.

Consider a finite-dimensional associative (linear) algebra \mathbf{A} and let $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ any basis for the vector space of the algebra \mathbf{A} , and multiplication in this algebra is given by the structure tensor β , so that $\mathbf{e}_k \cdot \mathbf{e}_m = \beta_{km}^r \mathbf{e}_r$ [1]. Introduce necessary to further the concept of the determinant of an arbitrary element $\mathbf{a} \in \mathbf{A}$.

Definition 1 Let $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ an arbitrary fixed element and $\mathbf{x} \in \mathbf{A}$ an arbitrary current element. Then a *left determinant element* \mathbf{a} we call the determinant $\Delta_L(\mathbf{a}) = \det(a^k \beta_{km}^r)$, and *right determinant element* \mathbf{a} we call the determinant $\Delta_R(\mathbf{a}) = \det(a^m \beta_{km}^r)$.

Note that $\Delta_L(\mathbf{a})$ is the determinant of the system of linear scalar equations $a^k \beta_{km}^r x^m = b^r$ of the equivalent linear algebraic equations $\mathbf{a} \cdot \mathbf{x} = \mathbf{b}$, and $\Delta_R(\mathbf{a})$ is the determinant of the system of linear scalar equations $a^m \beta_{km}^r x^k = b^r$ of the equivalent linear algebraic equations $\mathbf{x} \cdot \mathbf{a} = \mathbf{b}$. It follows that inequality $\Delta_L(\mathbf{a}) \neq 0$ is a necessary and sufficient condition for unique solvability of the equation $\mathbf{a} \cdot \mathbf{x} = \mathbf{b}$, and inequality $\Delta_R(\mathbf{a}) \neq 0$ gives the necessary and sufficient conditions for the unique solvability of the equation $\mathbf{x} \cdot \mathbf{a} = \mathbf{b}$.

If algebra \mathbf{A} is unitary, that is, it has a unit, the inequality $\Delta_L(\mathbf{a}) \neq 0$ and $\Delta_R(\mathbf{a}) \neq 0$ the equivalent. In other words of inequality $\Delta_L(\mathbf{a}) \neq 0$ follows the inequality $\Delta_R(\mathbf{a}) \neq 0$ and, on the contrary, of inequality $\Delta_R(\mathbf{a}) \neq 0$ follows the inequality $\Delta_L(\mathbf{a}) \neq 0$. These inequalities give the conditions of reversibility element $\mathbf{a} \in \mathbf{A}$.

Theorem 1 (on the determinant) *For any elements $\mathbf{a}, \mathbf{b} \in \mathbf{A}$ of the identities:*

$$\Delta_L(\mathbf{a} \cdot \mathbf{b}) = \Delta_L(\mathbf{a})\Delta_L(\mathbf{b}) \quad \text{and} \quad \Delta_R(\mathbf{a} \cdot \mathbf{b}) = \Delta_R(\mathbf{a})\Delta_R(\mathbf{b}). \quad (1)$$

Proof To prove these identities we consider a system of nonlinear scalar equations

$$a^k b^m \beta_{km}^r \beta_{rw}^v x^w = c^v,$$

equivalent to the linear equation $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{x} = \mathbf{c}$. Then

$$\Delta_L(\mathbf{a} \cdot \mathbf{b}) = \det(a^k b^m \beta_{km}^r \beta_{rw}^v).$$

On the other hand, by virtue of the associative algebra \mathbf{A} , $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{x} = \mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{x})$, namely

$$a^k b^m \beta_{km}^r \beta_{rw}^v x^w = a^k b^s \beta_{km}^r \beta_{sw}^m x^w,$$

and that's why

$$\begin{aligned} \Delta_L(\mathbf{a} \cdot \mathbf{b}) &= \det(a^k b^m \beta_{km}^r \beta_{rw}^v) = \det(a^k \beta_{km}^r b^s \beta_{sw}^m) \\ &= \det(a^k \beta_{km}^r) \det(b^s \beta_{sw}^m) = \Delta_L(\mathbf{a})\Delta_L(\mathbf{b}). \end{aligned}$$

Similarly we can prove the second of identities (1). \square

Take the set of all elements of the algebra \mathbf{A} for which $\Delta_L(\mathbf{a}) \neq 0$, and denote this set $\mathfrak{R}_L(\mathbf{A})$. Theorem of determinant shows that the semigroup $\mathfrak{R}_L(\mathbf{A})$ under multiplication in \mathbf{A} , because if $\Delta_L(\mathbf{a}) \neq 0$ and $\Delta_L(\mathbf{b}) \neq 0$ so $\Delta_L(\mathbf{a} \cdot \mathbf{b}) \neq 0$. In the case where algebra \mathbf{A} is unitary, $\mathfrak{R}_L(\mathbf{A})$ is a group. This group has a subgroup $\mathfrak{J}_L(\mathbf{A})$, elements of which are characterized by the condition $\Delta_L(\mathbf{a}) = 1$.

If on the linear space of the algebra \mathbf{A} as a metric form to take determinant $\Delta_L(\mathbf{x})$, then transform the vector space of the algebra \mathbf{A} , which are defined linear algebraic functions of the general form $\mathbf{a}' = \mathbf{a} \cdot \mathbf{x} \cdot \mathbf{b}^{-1}$, where $\mathbf{a}, \mathbf{b} \in \mathfrak{J}_L(\mathbf{A})$ and \mathbf{x} arbitrary element of the algebra, will retain this form, as

$$\Delta_L(\mathbf{x}') = \Delta_L(\mathbf{a} \cdot \mathbf{x} \cdot \mathbf{b}^{-1}) = \Delta_L(\mathbf{a})\Delta_L(\mathbf{x})\Delta_L(\mathbf{b}^{-1}) = \Delta_L(\mathbf{x}).$$

At the same time it is easy to see that the set of transformations linear space of algebra \mathbf{A} , determined by linear algebraic functions of general form, forms a group, so if $\mathbf{x}' = \mathbf{a}_1 \cdot \mathbf{x} \cdot \mathbf{b}_1^{-1}$ and $\mathbf{x}'' = \mathbf{a}_2 \cdot \mathbf{x}' \cdot \mathbf{b}_2^{-1}$, where $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2 \in \mathcal{J}_L(\mathbf{A})$

$$\mathbf{x}'' = \mathbf{a}_2 \cdot \mathbf{a}_1 \cdot \mathbf{x} \cdot \mathbf{b}_1^{-1} \cdot \mathbf{b}_2^{-1} = (\mathbf{a}_2 \cdot \mathbf{a}_1) \cdot \mathbf{x} \cdot (\mathbf{b}_2 \cdot \mathbf{b}_1),$$

where $(\mathbf{a}_2 \cdot \mathbf{a}_1), (\mathbf{b}_2 \cdot \mathbf{b}_1) \in \mathcal{J}_L(\mathbf{A})$; we denote this group $\mathbf{D}(\mathbf{A})$. Thus, in the linear space of the algebra \mathbf{A} geometric structure arises from a fundamental metric form $\Delta_L(\mathbf{x})$ and a group of $\mathbf{D}(\mathbf{A})$ as a group of motions. In this geometry, the value is accepted for any vector \mathbf{x} length

$$\|\mathbf{x}\| = \sqrt[n]{|\Delta_L(\mathbf{x})|}. \quad (2)$$

A corner is defined between the vectors \mathbf{x} and \mathbf{y} if they are connected by the transformation of a one-parameter subgroup of motions; in this case it is taken as an angle parameter that specifies the transformation taking \mathbf{x} to \mathbf{y} . This geometry will be called the *natural geometry of the linear algebra*.

We note here that in the case of complex and double numbers algebras, geometric structure defined above coincides with the Euclidean and pseudo-Euclidean plane geometry, and if you take the quaternion algebra, we obtain in the presented circuit geometry of four-dimensional Euclidean space. Thus, the natural geometry of linear algebra is, in a sense, a generalization of Euclidean and pseudo-Euclidean geometry.

Geometric structure arising in the linear spaces of associative algebras, is transferred to a smooth manifold, the same way as Euclidean or pseudo-Euclidean with linear space tolerated in a pseudo-Riemannian or new space. To do this, take a smooth manifold \mathbf{M} , $\dim \mathbf{M} = n$, and set on the manifold smooth field twice covariant and once contravariant tensor $\beta = \beta(\mathbf{x})$, $\mathbf{x} \in \mathbf{M}$. Then on each tangent space $\mathbf{T}_{\mathbf{x}}$ will be determined by the structure of linear algebra. In this algebra product of vectors $\xi, \eta \in \mathbf{T}_{\mathbf{x}}$ is defined using structural tensor as follows: if $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is a fixed basis of the tangent space $\mathbf{T}_{\mathbf{x}}$, and $\omega^1, \omega^2, \dots, \omega^n$ reciprocal basis cotangent space $\mathbf{T}_{\mathbf{x}}^*$, and $\beta = \beta_{km}^r \omega^k \otimes \omega^m \otimes \mathbf{e}_r$, $\xi = \xi^k \mathbf{e}_k$, $\eta = \eta^k \mathbf{e}_k$, the product of two vectors $\xi, \eta \in \mathbf{T}_{\mathbf{x}}$ is given by the formula:

$$\xi \cdot \eta = \beta(\xi, \eta) \equiv \beta_{km}^r \xi^k \eta^m \mathbf{e}_r. \quad (3)$$

Introducing the product of the vectors of the tangent space at each point of a smooth manifold \mathbf{M} , we thus transform the tangent vector bundle of \mathbf{TM} in the tangent bundle of linear algebra, structure constants which will be coordinates structure tensor in touch point in space $\mathbf{T}_{\mathbf{x}}$. But you can do otherwise. If each tangent space we define the structure of algebra, isomorphic to a fixed algebra \mathbf{A} , we get the tangent bundle of linear algebra \mathbf{AM} , standard layer which will be given algebra. A cutset of this bundle form an infinite dimension linear algebra $\mathbf{A}(\mathbf{M})$ where the multiplication is defined by the formula (3). A restriction of this algebra in the fixed point $\mathbf{x} \in \mathbf{M}$ will give us an algebra $\mathbf{A}_{\mathbf{x}}$, isomorphic to the standard algebra \mathbf{A} .

Geometric structure arising on the tangent bundle of linear algebra, defined gauge group $\mathcal{J}(\mathbf{AM})$, which each point $\mathbf{x} \in \mathbf{M}$ is the group $\mathcal{J}(\mathbf{A}_{\mathbf{x}})$. In this

geometric structure of vectors of length at each point $x \in \mathbf{M}$ is calculated by the formula (2). Thus we can determine the length of the arc line on any smooth manifold integrating the lengths of infinitesimal elements tangent to this line.

Notice, that arc length is invariant under the action of the gauge group $\mathcal{J}(\mathbf{AM})$.

The gauge group $\mathcal{J}(\mathbf{AM})$ generates invariant extension of derivations of algebra $\mathbf{A}(\mathbf{M})$, (the general scheme of extensions differentiations are invariant under a gauge transformation [2]). If $\partial: \mathbf{A}(\mathbf{M}) \rightarrow \mathbf{A}(\mathbf{M})$ derivation (i.e. linear operator satisfying the Leibniz identity) its invariant extension is given by the formula:

$$\nabla\{\partial\}(\xi) = \partial\xi + \Gamma \cdot \xi - \xi \cdot \bar{\Gamma}, \quad (4)$$

where Γ and $\bar{\Gamma}$ are cutset of the bundle \mathbf{AM} . They are under the influence of the gauge group vary according to the following rules:

$$\Gamma' = a \cdot \Gamma \cdot a^{-1} - (\partial a) \cdot a^{-1} \quad \text{and} \quad \bar{\Gamma}' = b \cdot \bar{\Gamma} \cdot b^{-1} - b \cdot \partial b^{-1}.$$

The invariance of the formula (4) can be verified by direct calculation:

$$\begin{aligned} \nabla\{\partial\}(a \cdot \xi \cdot b^{-1}) &= \partial(a) \cdot \xi \cdot b^{-1} + a \cdot \partial(\xi) \cdot b^{-1} + a \cdot \xi \cdot \partial(b^{-1}) \\ &\quad + a \cdot \Gamma \cdot \xi \cdot b^{-1} - \partial(a) \cdot \xi \cdot b^{-1} - a \cdot \xi \cdot \bar{\Gamma} \cdot b^{-1} - a \cdot \xi \cdot \partial(b^{-1}) \\ &= a \cdot (\partial\xi + \Gamma \cdot \xi - \xi \cdot \bar{\Gamma}) \cdot b^{-1} = a \cdot (\nabla\{\partial\}(\xi)) \cdot b^{-1}. \end{aligned}$$

With invariant differentiation represented by the formula (4), we can determine the curvature and cross-section of the invariant. If there is a Lie algebra $\mathbf{D}(\mathbf{M})$ of derivations of the algebra $\mathbf{A}(\mathbf{M})$, then for any pair of derivations defined their commutator $[\partial_1, \partial_2] \in \mathbf{D}(\mathbf{M})$. Then the invariant curvature of the cutset can be entered by means of commutators invariant derivations according to the formula

$$\mathbf{K}\{\partial_1, \partial_2\} = [\nabla\{\partial_1\}, \nabla\{\partial_2\}] - \nabla\{[\partial_1, \partial_2]\}$$

similarly, the curvature is determined to field of spin [3].

References

- [1] Vishnevsky, V. V., Shirokov, A. P., Shurigin, V. V.: Spaces over Algebras. *Kazan University*, Kazan, 1965.
- [2] Dubrovin, B. A., Novikov, S. P., Fomenko, A. T.: Modern Geometry. *Nauka*, Moscow, 1978.
- [3] Penrose, R., Rindler, W.: Spinors and Space-time. *Mir*, Moscow, 1987.