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Diophantine Approximations of Infinite Series and Products

Ondřej Kolouch, Lukáš Novotný

Abstract. This survey paper presents some old and new results in Diophantine approximations. Some of these results improve Erdos' results on irrationality. The results in irrationality, transcendence and linear independence of infinite series and infinite products are put together with idea of irrational sequences and expressible sets.

1 Introduction

In number theory, the field of Diophantine approximation is the study of the approximation of real or complex numbers by rational or algebraic numbers. It has its early sources in astronomy, with the study of movement of the celestial bodies, and in the computations of π.

The first problem was to know how well a real number can be approximated by rational numbers. A real number x has a "good" rational approximation $\frac{p}{q}$ if the absolute value of the difference between x and $\frac{p}{q}$ may not decrease if $\frac{p}{q}$ is replaced by another rational number $\frac{p'}{q'}$ with $q' < q$.

The Diophantine approximations give methods how to find "the best" rational approximation of a given real number.

The techniques from Diophantine approximations have been vastly generalized, and today there are many applications to Diophantine equations, Diophantine inequalities, and Diophantine geometry.

2 Infinite series

A real number can be expressed as a sum of an infinite series. The character of a real number (rationality, irrationality or algebraicity and transcendence) depends on conditions for the infinite series representing the number.

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If the series converges very fast, it is easier to decide about its algebraic character. Erdős proved that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that
$$\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$$
then the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{a_n} \tag{1}$$
is irrational.

Erdős and Straus in [7] proved deeper results. They showed that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that
$$\limsup_{n\to\infty} \frac{a_n^2}{a_{n+1}} \leq 1,$$
and
$$\limsup_{n\to\infty} \frac{\operatorname{lcm}(a_1,\ldots,a_n)}{a_{n+1}} < \infty$$
then the series (1) is irrational. This holds except if $a_{n+1} = a_n^2 - a_n + 1$ for all n large enough, in which case
$$\sum_{n=1}^{\infty} \frac{1}{a_n} = \frac{1}{a_1} + \cdots + \frac{1}{a_{n_0-1}} + \frac{1}{a_{n_0} - 1}\,.$$

In 1975 Erdős [5] found a new result on the algebraic character of the series connected with its speed of convergence. Firstly, Erdős took an increasing sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ such that
$$a_n > n^{1+\varepsilon} \tag{2}$$
with $\varepsilon > 0$ holds for all sufficiently large n. This guarantees the convergence of the series (1). Erdős also assumed that the sequence $\{a_n\}_{n=1}^{\infty}$ satisfies the condition
$$\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty\,. \tag{3}$$
This condition tells that there exists a "quickly" divergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that for any given real number R there exists an index n_{k_0} such that for all $n_k > n_{k_0}$, $a_{n_k} > R^{2^{n_k}}$. Using these conditions Erdős proved that the sum of the series (1) is irrational.

Let us note that if we omit a finite number of the terms in the above sequence then it does not have any influence on the irrationality.

In the same paper Erdős proved that for every sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers (not necessarily monotonic) the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n} c_n}$$

is irrational.

Erdős also gave a definition of an *irrational sequence* of positive integers. A sequence $\{a_n\}_{n=1}^{\infty}$ is called irrational if for all sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n} \tag{4}$$

is an irrational number. If the sequence $\{a_n\}_{n=1}^{\infty}$ is not irrational then we call this sequence *rational sequence*.

This definition was extended for sequences $\{a_n\}_{n=1}^{\infty}$ of real numbers by Hančl [9] in 1993.

From the text above we know that the sequence $\{2^{2^n}\}_{n=1}^{\infty}$ is an irrational sequence.

On the other hand, the sequence $\{n!\}_{n=1}^{\infty}$ is rational. Take for example $c_n = n+2$ for all n. Then

$$\sum_{n=1}^{\infty} \frac{1}{a_n c_n} = \sum_{n=1}^{\infty} \frac{1}{n!(n+2)} = \sum_{n=1}^{\infty} \frac{n+2-1}{(n+2)!} = \sum_{n=1}^{\infty} \left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!} \right) = \frac{1}{2}$$

is a rational number.

Erdős [6] asked if the number

$$\sum_{n=1}^{\infty} \frac{1}{(2^{2^n}+1)c_n}$$

is irrational for all sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers.

This question was partially answered by Duverney in [3]. Duverney proved that the number

$$\sum_{n=1}^{\infty} \frac{1}{(2^{2^n}+1)c_n}$$

is irrational for nondecreasing sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers such that

$$\log c_n = o(2^n)\,.$$

From the Duverney's result we know that the number

$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n}+1}$$

is irrational.

Another partial solution was given by Badea in [2], where he proved that if we have two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of positive integers such that

$$a_{n+1} > \frac{b_{n+1}}{b_n} a_n^2 - \frac{b_{n+1}}{b_n} a_n + 1$$

for all large n then the sum of the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n}$$

is an irrational number.

Erdős in [5] modified the condition (3). He made this condition stronger and he proved that if we replace this condition by

$$\limsup_{n\to\infty} a_n^{\frac{1}{t^n}} = \infty \quad \text{for all } t \in \mathbb{R}^+,$$

then the sum of series (1) is irrational, and moreover it is a Liouville number.

The speed of divergence of the sequence $\{a_n\}_{n=1}^{\infty}$ (the speed of convergence of the series (1)) can be expressed in a different way.

In another paper [4] Erdős proved that for an increasing sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers with

$$\lim_{n\to\infty} \frac{a_n}{a_1 \dots a_{n-1}} = \infty \tag{5}$$

the sum of series (1) is irrational.

This idea was improved by Romanian mathematician Sándor. In 1984 Sándor in [23] proved a criterion for irrationality of the series

$$\sum_{n=1}^{\infty} \frac{b_n}{a_n} . \tag{6}$$

If we have two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of positive integers such that

$$\limsup_{n\to\infty} \frac{a_n}{a_1 a_2 \dots a_{n-1}} \cdot \frac{1}{b_n} = \infty \tag{7}$$

and

$$\liminf_{n\to\infty} \frac{a_n}{a_{n-1}} \cdot \frac{b_{n-1}}{b_n} > 1 \tag{8}$$

then the sum of the series (6) is an irrational number.

Condition (7) in the theorem above tells us that there exists a subsequence in the sequence $\{a_n\}_{n=1}^{\infty}$ such that its speed of divergence is similar as in (5). Condition (8) guarantees the convergence of the series.

We can also ask, how big is the set of all real numbers in the form (4) for a given sequence $\{a_n\}_{n=1}^{\infty}$ and an arbitrary sequence $\{c_n\}_{n=1}^{\infty}$. Let $\{a_n\}_{n=1}^{\infty}$ be a given sequence of nonzero real numbers. Then the set of all real numbers x for which the sum (4) equals to x for some sequence $\{c_n\}_{n=1}^{\infty}$ is called the *expressible set* and we denote it

$$E_{\Sigma}\{a_n\}_{n=1}^{\infty} = \left\{ x \in \mathbb{R}; \exists \{c_n\}_{n=1}^{\infty} \subset \mathbb{Z}^+ : x = \sum_{n=1}^{\infty} \frac{1}{a_n c_n} \right\}.$$

It is obvious, that if expressible set $E_{\Sigma}\{a_n\}_{n=1}^{\infty}$ contains only irrational numbers, then the sequence $\{a_n\}_{n=1}^{\infty}$ is irrational.

In [8] Hančl proved, that if sequence of positive real numbers $\{a_n\}_{n=1}^\infty$ satisfies

$$\frac{1}{2a_n} \leq \sum_{i=n+1}^\infty \frac{1}{a_i} \tag{9}$$

then the expressible set

$$E_\Sigma\{a_n\}_{n=1}^\infty = \left(0, \sum_{n=1}^\infty \frac{1}{a_n}\right]$$

if $\sum_{n=1}^\infty \frac{1}{a_n}$ converges and

$$E_\Sigma\{a_n\}_{n=1}^\infty = (0, \infty)$$

if the sum diverges.

The condition (9) holds only for sequences slower than $\{3^n\}_{n=1}^\infty$. In [19] Hančl, Schinzel and Šustek showed, that for $A > 3$

$$\left(0, \frac{1}{(A-1)(\lceil A \rceil - 2)}\right] \subset E_\Sigma\{A^n\}_{n=1}^\infty .$$

This result holds only for geometric sequences, i.e. for sequences much slower than $\{2^{2^n}\}_{n=1}^\infty$.

It appears to be the case that in general calculating the set $E_\Sigma\{a_n\}_{n=1}^\infty$ is not easy. For this reason it is sometimes better to take a metrical approach and attempt to calculate the size of the expressible set, rather than describing it exactly. For sequences of positive integers faster than $\{2^{2^n}\}_{n=1}^\infty$ the expressible set has zero Lebesgue measure. In [10] Hančl and Filip proved, that if $\{a_n\}_{n=1}^\infty$ is a nondecreasing sequence of positive integers satisfying (2) such that

$$\limsup_{n\to\infty} a_n^{\frac{1}{T^n}} > 1$$

for $T > 3$, then Lebesgue measure of the expressible set $E_\Sigma\{a_n\}_{n=1}^\infty$ is zero.

In [18] we can find results related to the Lebesgue measure of expressible set. Let $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ be sequences of positive integers with $\{a_n\}_{n=1}^\infty$ nondecreasing and satisfying (2), such that

$$\limsup_{n\to\infty} a_n^{\frac{1}{3^n}} = \infty ,$$

and

$$b_n \leq 2^{\log_2^\alpha a_n}$$

for $0 < \alpha < 1$ and for every sufficiently large n. Then the Lebesgue measure of $E_\Sigma\left\{\frac{a_n}{b_n}\right\}_{n=1}^\infty$ is zero.

If the Lebesgue measure of expressible set is zero, we can study the Hausdorff measure or Hausdorff dimension of such sets, which need not to be zero. Hančl, Nair, Novotný and Šustek in [17] proved for a nondecreasing sequence of positive integers $\{a_n\}_{n=1}^\infty$ with (2) and

$$P = \sup\left\{R; \limsup_{n\to\infty} a_n^{\frac{1}{R^n}} = \infty\right\} > 3$$

that the Hausdorff dimension satisfies

$$\dim_H E_\Sigma\{a_n\}_{n=1}^\infty \le \frac{2}{P-1}\,.$$

So we have for example

$$\dim_H E_\Sigma\{2^{4^n}\}_{n=1}^\infty \le \frac{2}{3} \quad \text{and} \quad \dim_H E_\Sigma\{2^{n^n}\}_{n=1}^\infty = 0\,.$$

3 Infinite products

We can also express a real number as an infinite product. It is a well-known fact that for a sequence $\{a_n\}_{n=1}^\infty$ of positive real numbers the product

$$\prod_{n=1}^\infty (1+a_n) \tag{10}$$

converges if and only if the series

$$\sum_{n=1}^\infty a_n$$

converges.

Deciding whether a real number given as an infinite product (10) is rational or irrational is a very difficult problem similar to the same question for infinite series. It is easy to see that

$$\prod_{n=2}^\infty \left(1+\frac{2}{n^3-1}\right) = \frac{3}{2} \qquad \text{and} \qquad \prod_{n=1}^\infty \left(1+\frac{3}{4(2^n-1)}\right) = \frac{8}{3}\,.$$

For all positive integers $K>1$ we have

$$\prod_{n=1}^\infty \left(1+\frac{1}{K^{2^n}}\right) = \frac{K^2}{K^2-1}\,.$$

But one can prove that

$$\prod_{n=1}^\infty \left(1+\frac{1}{K^{2^n}-1}\right)$$

is irrational for all positive integers $K>1$. If we slightly change this infinite product, we get the following open problem. We do not know if for a positive integer $K>1$ the number

$$\prod_{n=1}^\infty \left(1+\frac{1}{(K^{2^n}+1)c_n}\right),$$

is irrational for all sequences $\{c_n\}_{n=1}^\infty$ of positive integers.

As in the case of infinite series, we can use the idea of Erdős for products and define Π-*irrational sequences*. A Π-irrational sequence is a sequence $\{a_n\}_{n=1}^{\infty}$ such that for all sequences $\{c_n\}_{n=1}^{\infty}$ of positive integers the product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{a_n c_n}\right) \tag{11}$$

is an irrational number.

Hančl and Kolouch in [11] gave the first result in this topic and proved that if $\{a_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of positive integers such that (2) and

$$\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$$

then the sequence $\{a_n\}_{n=1}^{\infty}$ is Π-irrational.

4 Irrationality and transcendence

The irrationality and transcendence of infinite products has a great history. Badea [1] proved that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of positive integers such that

$$a_{n+1} > \frac{b_{n+1}}{b_n}a_n^2 + \frac{b_{n+1}(b_n-1)}{b_n}a_n + 1 - b_{n+1}$$

holds for every sufficiently large n then the number

$$\prod_{n=1}^{\infty}\left(1+\frac{b_n}{a_n}\right)$$

is irrational.

Some approximations of the numbers

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{q^n}\right)$$

can be found in the paper of Väänänen [24].

Nyblom [22] constructed a certain set of transcendental valued infinite products with the help of second order linear recurrence sequences. He proved that if we have a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers greater than one and such that

$$\liminf_{n\to\infty}\frac{a_{n+1}}{a_n^{\lambda+1}} > 2$$

for a fixed $\lambda > 2$ then the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{a_n}\right) \tag{12}$$

is a transcendental number.

Hančl and Kolouch in [11] proved, that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive integers satisfying (2) and

$$\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty\,,$$

then the number (12) is irrational. In the same paper they proved a more general result. They work with two sequences of positive integers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfying (2),

$$\limsup_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$$

and

$$b_n \le a_n^{\frac{1}{\log^{1+\varepsilon}\log a_n}}. \tag{13}$$

Under these conditions the number (12) is irrational.

Note that the upper bound in (13) satisfies

$$\log^K a_n < a_n^{\frac{1}{\log^{1+\varepsilon}\log a_n}} < a_n^{\delta}$$

for every $\delta, \varepsilon, K > 0$ and every large n.

From this result we can show that the number

$$\prod_{n=1}^{\infty}\Big(1 + \frac{1}{p_n^{2^n}}\Big),$$

where p_n is the n-th prime number, is irrational.

Hančl and Kolouch in [12] proved another results on irrationality of infinite products. Let $\varepsilon > 0$. Assume that $(a_{n,m})_{m,n\ge 1}$ and $(b_{n,m})_{m,n\ge 1}$ are two infinite matrices of positive integers. Suppose that the sequence $\{a_{n,1}\}_{n=1}^{\infty}$ is nondecreasing with

$$\limsup_{n\to\infty} a_{n,1}^{\frac{1}{n!}} = \infty$$

and for all sufficiently large n

$$n^{1+\varepsilon} \le a_{n,1},$$

$$\sum_{j=1}^{n} \frac{b_{n-j+1,j}}{a_{n-j+1,j}} \le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}} - 1}$$

and

$$\prod_{j=1}^{n} a_{n-j+1,j} \le a_{n,1}^{\frac{1}{\log^{3+\varepsilon}\log a_{n,1}} + n}.$$

Then the number

$$\prod_{m=1}^{\infty}\Bigg(1 + \sum_{n=1}^{\infty} \frac{b_{n,m}}{a_{n,m}}\Bigg)$$

is irrational.

As a consequence they got that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that

$$\lim_{n\to\infty} a_n^{\frac{1}{n!}} = \infty$$

then the number

$$\prod_{m=1}^{\infty}\Big(1+\sum_{n=0}^{\infty}\frac{1}{a_{n+m}+n}\Big)$$

is irrational.

We can illustrate this result on several examples of irrational numbers.

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{(n+1)!}}\right)=\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)!}}\right)$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{(n+1)!}+1}\right)=\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)!}+1}\right)$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{n}{2^{(n+1)!}+m}\right)=\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{n+m}{2^{(n+m)!}+m}\right)$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{1}{2^{n^n}}\right)=\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{1}{2^{(n+m)^{n+m}}}\right)$$

$$\prod_{m=1}^{\infty}\left(1+\sum_{n=m}^{\infty}\frac{n!}{2^{n^n}+m^n}\right)=\prod_{m=1}^{\infty}\left(1+\sum_{n=1}^{\infty}\frac{(n+m)!}{2^{(n+m)^{n+m}}+m^{n+m}}\right)$$

5 Linear and algebraic independence

The concept of irrationality can be extended and we can deal with linear independence of real numbers, especially with linear independence of real numbers in the form (10) and number 1 over rationals numbers.

In [13] it is shown that for a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers satisfying

$$1<\liminf_{n\to\infty} a_n^{\frac{1}{(K+2)^n}}<\limsup_{n\to\infty} a_n^{\frac{1}{(K+2)^n}}<\infty$$

the numbers

$$1,\quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{a_n+1}\Big),\quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{na_n+1}\Big),\quad \ldots,\quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{n^Ka_n+1}\Big)$$

are linearly independent over the rational numbers.

As a consequence the authors found a criterion for irrationality of infinite products. If the sequence of positive integers $\{a_n\}_{n=1}^{\infty}$ satisfies

$$1<\liminf_{n\to\infty} a_n^{\frac{1}{2^n}}<\limsup_{n\to\infty} a_n^{\frac{1}{2^n}}<\infty\,,$$

then the number (12) is irrational.

Some authors deal with algebraic independence of infinite products. Let F_n denote the n-th Fibonacci number. Luca and Tachiya [21] proved that if $d\geq 2$ is an integer and $\gamma\notin\{0,1\}$ is a rational number then the infinite products

$$\prod_{k=1}^{\infty}\Big(1+\frac{1}{F_{d^k}}\Big)\quad\text{and}\quad\prod_{\substack{k=1\\F_{d^k}\neq-\gamma}}^{\infty}\Big(1+\frac{\gamma}{F_{d^k}}\Big)$$

are algebraically independent over $\mathbb{Q}$.

Kurosawa, Tachiya and Tanaka in [20] showed that the numbers

$$\prod_{n=0}^{\infty}\Big(1+\frac{1}{F_{2\cdot 3^n}}\Big) \quad \text{and} \quad \prod_{n=0}^{\infty}\Big(1+\frac{1}{F_{2\cdot 5^n}}\Big)$$

are algebraically independent.

6 Linearly independent sequences and expressible set

We can join the concept of linear independence and the concept of irrational sequences and define *linearly independent sequences*. It means the sequences of positive real numbers $\{a_{1,n}\}_{n=1}^{\infty}, \{a_{2,n}\}_{n=1}^{\infty}, \dots, \{a_{K,n}\}_{n=1}^{\infty}$, where K is a positive integer, for which the numbers

$$1, \quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{a_{1,n}c_n}\Big), \quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{a_{2,n}c_n}\Big), \quad \dots, \quad \prod_{n=1}^{\infty}\Big(1+\frac{1}{a_{K,n}c_n}\Big)$$

are linearly independent over $\mathbb{Q}$ for any sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers.

It is easy to show that the concept of linearly independent sequences is a generalization of linear independence, irrationality and irrational sequences. If we set $c_n = 1$ for all n, we get the definition of linear independence over $\mathbb{Q}$. If we set K equal to 1, we obtain the concept of irrational sequences and at last if we set $K = 1$ and $c_n = 1$ for all n, we get irrationality. Some results in this topic can be found in works of Hančl, Kolouch, Korčeková and Novotný ([13], [15]), from where the following example of linearly independent sequences comes:

$$\left\{\frac{n^{6\cdot 9^n}+5}{n^{9^n}+3}\right\}_{n=1}^{\infty} \qquad \text{and} \qquad \left\{\frac{n^{3\cdot 9^n}+7}{n^{8^n}+5}\right\}_{n=1}^{\infty}.$$

The concept of irrational sequence is close to special subset of real numbers. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers. The set of all real numbers x, for which exists a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that the product (11) converges and is equal to x, is called Π-*expressible set*, thus

$$E_{\Pi}\{a_n\}_{n=1}^{\infty} = \left\{x \in \mathbb{R}; \exists c_n \in \mathbb{Z}^+ : x = \prod_{n=1}^{\infty}\Big(1+\frac{1}{a_n c_n}\Big)\right\}.$$

It is easy to see, that if $E_{\Pi}\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}\setminus\mathbb{Q}$, then the sequence $\{a_n\}_{n=1}^{\infty}$ is irrational. In [16] it is shown that if the sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers satisfies the conditions (2) and

$$1+\frac{1}{2a_n} \leq \prod_{j=n+1}^{\infty}\Big(1+\frac{1}{a_j}\Big),$$

then

$$E_{\Pi}\{a_n\}_{n=1}^{\infty} = \left(1, \prod_{j=1}^{\infty}\Big(1+\frac{1}{a_j}\Big)\right].$$

In the same work it is also shown, that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that the series (1) is convergent but not absolutely convergent, then

$$E_\Pi\{a_n\}_{n=1}^{\infty} = \mathbb{R}^+ .$$

We can mention that there is no known result about Π-expressible set having zero Lebesgue measure.

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