Rafik Aramyan

Zoonoids with an equatorial characterization

Applications of Mathematics, Vol. 61 (2016), No. 4, 413–422

Persistent URL: http://dml.cz/dmlcz/145793

61 (2016) APPLICATIONS OF MATHEMATICS No. 4, 413–422

ZONOIDS WITH AN EQUATORIAL CHARACTERIZATION

Rafik Aramyan, Yerevan

(Received September 26, 2015)

Abstract. It is known that a local equatorial characterization of zonoids does not exist. The question arises: Is there a subclass of zonoids admitting a local equatorial characterization. In this article a sufficient condition is found for a centrally symmetric convex body to be a zonoid. The condition has a local equatorial description. Using the condition one can define a subclass of zonoids admitting a local equatorial characterization. It is also proved that a convex body whose boundary is an ellipsoid belongs to the class.

Keywords: integral geometry; convex body; zonoid; support function

MSC 2010: 53C45, 52A15, 53C65

1. Introduction

Zonotopes are convex bodies that are composed (in the sense of Minkowski addition) of line segments in $\mathbb{R}^n$ and so have a number of interesting symmetry properties. Zonoids are limits of zonotopes in the Hausdorff metric. Zonoids form a particularly important family of centrally symmetric convex bodies. Interest in zonoids arose from surprising connection between zonoids and analysis (positive definite functions, spherical Radon transforms), functional analysis (vector measures, subspaces of L^1), and stochastic and integral geometry (point processes, Crofton measures, stable laws) (see [9]).

We denote by $\mathbb{R}^n$ ($n \geqslant 2$) the Euclidean n-dimensional space. Let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ centered at the origin of $\mathbb{R}^n$ and let λ_{n-1} ($\lambda_1 \equiv \lambda$) be the spherical Lebesgue measure on $\mathbb{S}^{n-1}$ ($\lambda_k(\mathbb{S}^k) = \sigma_k$). Denote by $\mathbb{S}_\omega \subset \mathbb{S}^2$ the great circle with pole at $\omega \in \mathbb{S}^2$.

The most useful analytic description of a convex body (nonempty compact convex set) $\mathbb{K} \subset \mathbb{R}^n$ is the support function. The support function $H\colon \mathbb{R}^n \to (-\infty, \infty]$ of $\mathbb{K}$

is defined as

$$H(x) = \sup_{y\in\mathbb{K}} \langle y, x\rangle,\ x \in \mathbb{R}^n.$$

Here and below $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product in $\mathbb{R}^n$. The support function of $\mathbb{K}$ is positively homogeneous and convex. From now on we consider the support function H of a convex body as a function on the unit sphere $\mathbb{S}^{n-1}$ (because of the positive homogeneity of H, the values on $\mathbb{S}^{n-1}$ determine H completely).

It is well known (see [4]) that a convex body $\mathbb{K}$ is determined uniquely by its support function.

A convex body $\mathbb{K}$ is k-smooth if its support function H belongs to $\mathbb{C}^k(\mathbb{S}^{n-1})$, where $\mathbb{C}^k(\mathbb{S}^{n-1})$ denotes the space of k times continuously differentiable functions defined on $\mathbb{S}^{n-1}$.

The class of origin-symmetric convex bodies $\mathbb{K}$ in $\mathbb{R}^n$ symmetric with respect to the origin will be denoted by $\mathcal{K}_0^n$ (so called *centred* bodies).

It is known that (see [7]) *the support function H of a sufficiently smooth origin-symmetric convex body $\mathbb{K}\in\mathcal{K}_0^n$ has the following representation*

$$H(\xi) = \int_{\mathbb{S}^{n-1}} |\langle\xi,\Omega\rangle| h(\Omega)\lambda_{n-1}(\mathrm{d}\Omega), \quad \xi\in\mathbb{S}^{n-1}, \tag{1.1}$$

with an even continuous function $h(\cdot)$, not necessarily nonnegative, called the generating density of $\mathbb{K}$. *Such bodies (whose support functions have the integral representation* (1.1) *with a signed even measure) are called generalized zonoids. If h is a positive function on $\mathbb{S}^{n-1}$, the centrally symmetric convex body $\mathbb{K}$ is a zonoid.*

The problem of geometric characterization of zonoids was posed by W. Blaschke. Later the problem was posed repeatedly (see [8] for the history of the problem).

W. Weil showed [10] that a local characterization of zonoids does not exist. Thus, no characterization of zonoids that involves only arbitrarily small neighborhoods of boundary points is possible. In 1977, W. Weil (see [10]) proposed the following conjecture about local equatorial characterization of zonoids. Let $\mathbb{K}\in\mathcal{K}_0^n$ be an origin-symmetric convex body and assume that for any equator $\mathbb{S}_\omega$, there exists a zonoid Z_ω and a neighborhood E_ω of $\mathbb{S}_\omega$ such that the boundaries of $\mathbb{K}$ and Z_ω coincide at all points where the exterior unit vector belongs to E_ω; then $\mathbb{K}$ is a zonoid. Affirmative answers for even dimensions were given independently by G. Panina [6] in 1988 and P. Goodey and W. Weil [3] in 1993. Recently, in 2008 F. Nazarov, D. Ryabogin, and A. Zvavitch (see [5]) showed that the answer to the conjecture in odd dimensions is negative.

The following question arises: Is there a subclass of zonoids admitting a local equatorial characterization?

Our main results are the following. In this article we find a new representation for the support function of a sufficiently smooth origin-symmetric convex body. Using the representation we propose a sufficient condition for an origin-symmetric convex body to be a zonoid. The condition has a local equatorial description. Using the condition, one can define a subclass of zonoids admitting a local equatorial characterization. It is also proved that a convex body whose surface is an ellipsoid belongs to the class.

Let $\mathbb{K}$ be an origin-symmetric convex body in $\mathbb{R}^3$ with sufficiently smooth boundary and with positive Gaussian curvature at every point of $\partial\mathbb{K}$ (to guarantee the existence of the expressions appearing below).

We need some notations. For $\Omega, \Phi \in \mathbb{S}^2$ ($\Omega \neq \Phi$), we denote by ω the unit vector perpendicular to Ω and Φ in the direction given by the right-hand rule (the direction of the cross product). Further, assuming that $s(\omega)$ is the point on $\partial\mathbb{K}$, the outer normal of which is ω, we denote by $k_1(\omega)$, $k_2(\omega)$ the principal normal curvatures of $\partial\mathbb{K}$ at $s(\omega)$ and let $k(\omega, \Omega)$ be the normal curvature at $s(\omega)$ in the direction Ω.

Theorem 1.1. *The support function H of a 2-smooth origin symmetric convex body $\mathbb{K} \in \mathcal{K}_0^3$ has the following representation*

$$H(\xi) = \frac{1}{4\pi^2} \int_{\mathbb{S}^2} |\langle \xi, \Omega \rangle| \int_{\mathbb{S}_\xi} \frac{\sqrt{k_1(\omega) k_2(\omega)}}{k^2(\omega, \Omega)} \lambda(\mathrm{d}\Phi) \lambda_2(\mathrm{d}\Omega), \quad \xi \in \mathbb{S}^2, \tag{1.2}$$

where ω is the unit vector in the direction of the cross product of Ω and Φ.

Note that for any $\Omega \in \mathbb{S}^2$ the expression

$$h(\Omega, \xi) = \int_{\mathbb{S}_\xi} \frac{\sqrt{k_1(\omega) k_2(\omega)}}{k^2(\omega, \Omega)} \lambda(\mathrm{d}\Phi) \tag{1.3}$$

has a local equatorial description. This means that for any $\Omega \in \mathbb{S}^2$ the expression $h(\Omega, \xi)$ depends on boundary of $\mathbb{K}$ which consists of points where the exterior unit vector belongs to a neighborhood of the equator $\mathbb{S}_\Omega$, since for each $\Phi \in \mathbb{S}_\xi$ the unit vector ω which is the direction of the cross product of Ω and Φ belongs to $\mathbb{S}_\Omega$. The last statement allows to propose a sufficient condition for a centrally symmetric convex body to be a zonoid (see Theorem 4.1 below). Using the condition we define a subclass of zonoids admitting a local equatorial characterization (see Definition 4.1 below).

2. Preliminary results

We need more notations. Let $\mathbb{K}$ be an origin-symmetric convex body in $\mathbb{R}^3$. For $\omega, \xi \in \mathbb{S}^2$ we denote by $(\widehat{\xi, \omega})$ the angle between ξ and ω, by $e(\omega, \xi)$ we denote the plane containing the origin of $\mathbb{R}^3$ and the directions ω and ξ, $\xi \neq \omega$. Denote by $\mathbb{K}(\omega, \xi)$ the projection of $\mathbb{K}$ onto $e(\omega, \xi)$ and let $R(\omega, \xi)$ be the curvature radius of $\partial\mathbb{K}(\omega, \xi)$ at the point whose outer normal direction is ω. Since $R(\omega, \xi_1) = R(\omega, \xi_2)$, where $\omega, \xi_1, \xi_2 \in \mathbb{S}^2$ and $\xi_2 \in e(\omega, \xi_1)$, if necessary we will assume that ξ is orthogonal to ω.

We need the following result from [2]: *for any 2-smooth origin-symmetric convex body $\mathbb{K}$, $\omega \in \mathbb{S}^2$, and $\psi \in \mathbb{S}_\omega$,*

$$R(\omega, \psi) = \frac{1}{\pi} \int_{\mathbb{S}_\omega} \cos^2(\psi - \nu) \frac{\sqrt{k_1 k_2}}{k^2(\omega, \nu)} \, \lambda(\mathrm{d}\nu), \tag{2.1}$$

where k_i, $i = 1, 2$ ($k_1 \geqslant k_2$) are the main normal curvatures of $\partial\mathbb{K}$ at the point with normal ω and $k(\omega, \nu)$ is the normal curvature at the same point in the direction $\nu \in \mathbb{S}_\omega$. Here for a given reference direction on $\mathbb{S}_\omega$ each $\nu \in \mathbb{S}_\omega$ determines an angle which we also denote by ν.

We will also need the following result, which corresponds to Theorem 2 from [1] with $n = 3$: *The support function of a 2-smooth origin-symmetric convex body $\mathbb{K} \in \mathcal{K}_0^3$ has the following representation. For $\xi \in \mathbb{S}^2$ (we choose ξ for the North Pole)*

$$H(\xi) = \frac{1}{4\pi} \int_{\mathbb{S}^2} R(\omega, \xi) \, \lambda_2(\mathrm{d}\omega). \tag{2.2}$$

The last statement follows from the following result for planar convex figures: *The support function of a 2-smooth origin-symmetric convex figure $\mathbb{K}$ has the following representation*

$$H(\xi) = \frac{1}{2} \int_0^\pi R(\psi) \sin(\widehat{\xi, \psi}) \, \mathrm{d}\psi, \quad \xi \in \mathbb{S}^1, \tag{2.3}$$

where $R(\psi)$ is the radius of curvature of $\partial\mathbb{K}$ at the point with normal ψ. Any $\psi \in \mathbb{S}^1$ determines an angle with respect to ξ which we also denote by ψ.

3. Proof of Theorem 1.1

Proof. Applying Fubini's theorem to (1.2), we obtain

$$H(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{S}_\xi}\int_{\mathbb{S}^2}|\langle\xi,\Omega\rangle|\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\lambda_2(\mathrm{d}\Omega)\,\lambda(\mathrm{d}\Phi). \tag{3.1}$$

For fixed $\Phi\in\mathbb{S}_\xi$ we use the usual spherical coordinates ν, u on $\mathbf{S}^2$ for points $\Omega=(\nu,u)$ based on the choice of Φ as the North Pole and ξ as the reference direction $(u=0)$ on the equator $\mathbf{S}_\Phi$. We have

$$H(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{S}_\xi}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2}|\langle\xi,\Omega\rangle|\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,(\nu,u))}\cos\nu\,\mathrm{d}\nu\,\mathrm{d}u\,\lambda(\mathrm{d}\Phi). \tag{3.2}$$

Applying the spherical cosine rule for the right spherical triangle (the vertices of the right spherical triangle are ξ, Ω and the direction $(0,u)$) we have

$$|\langle\xi,\Omega\rangle|=|\cos u||\cos\nu|. \tag{3.3}$$

Substituting the formula (3.3) into (3.2), we get

$$H(\xi)=\frac{1}{4\pi^2}\int_{\mathbb{S}_\xi}\int_0^{2\pi}|\cos u|\int_{-\pi/2}^{\pi/2}\cos^2\nu\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,(\nu,u))}\,\mathrm{d}\nu\,\mathrm{d}u\,\lambda(\mathrm{d}\Phi). \tag{3.4}$$

It follows from (2.1) that

$$\int_{-\pi/2}^{\pi/2}\cos^2\nu\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,(\nu,u))}\,\mathrm{d}\nu=\frac{\pi}{2}R(\omega,(0,u))=\frac{\pi}{2}R(\omega,\xi). \tag{3.5}$$

Substituting (3.5) into (3.4), we get

$$H(\xi)=\frac{1}{8\pi}\int_{\mathbb{S}_\xi}\int_0^{2\pi}|\cos u|R(\omega,\xi)\,\mathrm{d}u\,\lambda(\mathrm{d}\Phi). \tag{3.6}$$

Taking into account the symmetry of $\mathbb{K}$, we have

$$\int_0^{2\pi}|\cos u|R(\omega,\xi)\,\mathrm{d}u=2\int_{\pi/2}^{3\pi/2}|\cos u|R(\omega,\xi)\,\mathrm{d}u. \tag{3.7}$$

Assume that ω has the usual spherical coordinates $(0,\psi)$ with respect to Φ as the North Pole and ξ as the reference direction on $\mathbf{S}_\Phi$. After a change of variable $\psi=u-\pi/2$ in (3.7) and taking into account (2.2), we obtain

$$\int_{\pi/2}^{3\pi/2}|\cos u|R(\omega,\xi)\,\mathrm{d}u=\int_0^{\pi}R(\omega,\xi)\sin\psi\,\mathrm{d}\psi=2H(\xi), \tag{3.8}$$

since the values of the support functions in direction ξ are the same for $\mathbb{K}$ and the projection of $\mathbb{K}$ onto the plane perpendicular to Φ. Substituting (3.8) and (3.7) into (3.6), we obtain an identity and the Theorem 1 is proved. □

Note that for any $\xi \in \mathbb{S}^2$ the value $H(\xi)$ does not depend on the values of the inner integral of (1.2) for Ω perpendicular to ξ (see also (1.3)). Below we will assume that ξ is not perpendicular to Ω.

Theorem 4.1. *Let $\mathbb{K}$ be an origin-symmetric 2-smooth convex body in $\mathbb{R}^3$. If for any $\Omega \in \mathbb{S}^2$ the expression (see (1.2))*

$$h(\Omega,\xi) = \int_{\mathbb{S}_\xi} \frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\lambda(\mathrm{d}\Phi), \tag{4.1}$$

where ω is the unit vector in the direction of the cross product of Ω and Φ, does not depend on ξ then $\mathbb{K}$ is a zonoid.

P r o o f. Let for any $\Omega \in \mathbb{S}^2$ the expression

$$h(\Omega,\xi) = \int_{\mathbb{S}_\xi} \frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\lambda(\mathrm{d}\Phi) = h(\Omega), \tag{4.2}$$

does not depend on ξ. It follows from Theorem 1.1 that (1.2) becomes (1.1). Hence, the support function of $\mathbb{K}$ admits the zonoid representation (1.1) with positive generating density $h(\Omega) > 0$. □

Note that the converse statement is not true, otherwise there would exist a local equatorial characterization of zonoids which is not the case (see [5]).

We define the following subclass $\mathcal{EZ}$ of zonoids.

Definition 4.1. We say that a zonoid $\mathbb{K} \in \mathcal{K}_0^3$ belongs to the subclass $\mathcal{EZ}$ if for any $\Omega \in \mathbb{S}^2$ the expression

$$h(\Omega,\xi) = \int_{\mathbb{S}_\xi} \frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\lambda(\mathrm{d}\Phi), \tag{4.3}$$

where ω is the unit vector in the direction of the cross product of Ω and Φ (see (1.3)), does not depend on $\xi \in \mathbb{S}^2$.

The condition (4.3) has a local equatorial description: for $\Omega \in \mathbb{S}^2$ it depends on the boundary of $\mathbb{K}$ which consists of points where the exterior unit vector belongs to a neighborhood of the equator $\mathbb{S}_\Omega$.

Let a zonoid $\mathbb{K} \in \mathcal{K}_0^3$ belong to the subclass $\mathcal{EZ}$. Then $h(\Omega,\xi) = h(\Omega)$, that has a local equatorial description, becomes the generating density of $\mathbb{K}$ and determines $\mathbb{K}$, since a zonoid is determined uniquely by its generating density. Thus the following statement is valid.

The subclass $\mathcal{EZ}$ of zonoids has the local equatorial description (4.3).

Also it follows from Definition 4.1 that the following statement is valid.

Let $\mathbb{K}$ be an origin-symmetric 2-smooth convex body and assume that for any equator $\mathbb{S}_\Omega$ ($\Omega \in \mathbb{S}^2$), there exists a $Z_\Omega \in \mathcal{EZ}$ and a neighborhood E_Ω of $\mathbb{S}_\Omega$ such that the boundaries of $\mathbb{K}$ and Z_Ω coincide at all points where the exterior unit vector belongs to E_Ω. Then $\mathbb{K} \in \mathcal{EZ}$.

5. A representation equivalent to (1.2)

Now we are going to show that a convex body whose boundary is an ellipsoid belongs to the subclass $\mathcal{EZ}$. First we find a representation for the support function of a smooth origin-symmetric convex body equivalent to (1.2). For a given $\Omega \in \mathbb{S}^2$ and a reference direction on $\mathbb{S}_\Omega \subset \mathbb{S}^2$ each $\Phi_1 \in \mathbb{S}_\Omega$ determines an angle which we denote by ϕ_1.

Theorem 5.1. *The support function $H(\cdot)$ of a 2-smooth origin symmetric convex body $\mathbb{K} \in \mathcal{K}_0^3$ has the following representation. For $\xi \in \mathbb{S}^2$,*

$$H(\xi) = \frac{1}{4\pi^2}\int_{\mathbb{S}^2} |\langle \xi, \Omega\rangle| \times \int_{\mathbb{S}_\Omega} \frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)} \frac{|\langle \xi, \Omega\rangle|}{\cos^2\phi_1 + \cos^2(\widehat{\xi,\Omega})\sin^2\phi_1}\,\lambda(\mathrm{d}\Phi_1)\lambda_2(\mathrm{d}\Omega), \tag{5.1}$$

where ω is the unit vector in the direction of the cross product of Ω and Φ_1. On $\mathbb{S}_\Omega$ we choose the direction perpendicular to ξ for the reference direction.

P r o o f. We consider the following correspondence $\Phi \to \Phi_1$, where $\Phi_1 \in \mathbb{S}_\Omega$, $\Phi \in \mathbb{S}_\xi$ and the plane containing Φ and ξ is perpendicular to the plane containing Φ_1 and Ω. Using the rule for the right spherical triangle, we have

$$\cos(\widehat{\xi,\Omega})\tan\phi_1 = \tan\phi, \tag{5.2}$$

and the corresponding Jacobian is

$$J = \frac{|\langle \xi, \Omega\rangle|}{\cos^2\phi_1 + \cos^2(\widehat{\xi,\Omega})\sin^2\phi_1}. \tag{5.3}$$

After change of variables in (1.2) using (5.3), we obtain (5.1). □

Also note that one can change the variables in (4.3)

$$h(\Omega,\xi)=\int_{\mathbb{S}_\xi}\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\lambda(\mathrm{d}\Phi) \tag{5.4}$$
$$=\int_{\mathbb{S}_\Omega}\frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega,\Omega)}\,\frac{|\langle\xi,\Omega\rangle|}{\cos^2\phi_1+\cos^2(\widehat{\xi,\Omega})\sin^2\phi_1}\,\lambda(\mathrm{d}\Phi_1).$$

6. A convex body whose boundary is an ellipsoid belongs to the class $\mathcal{EZ}$

Let $\mathbb{K}\in\mathcal{K}_0^3$ be a convex body whose boundary is an ellipsoid with semi-principal axes of length a, b, c. The standard equation of $\partial\mathbb{K}$ centered at the origin of a Cartesian coordinate system and aligned with the coordinate axes is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{6.1}$$

Later for a fixed $\Omega\in\mathbb{S}^2$ we use the usual spherical coordinates ν, φ on $\mathbb{S}^2$ based on the choice of the z-axis direction as the zenith direction and the choice of the ξ axis direction as the azimuth reference:

$$\Omega=(\sin\nu\cos\varphi,\sin\nu\sin\varphi,\cos\nu).$$

Now for $\Phi_1\in\mathbb{S}_\Omega$ determined by the angle ϕ_1 (as the reference direction on $\mathbb{S}_\Omega$ we choose the direction of the trace of the plane containing $\mathbb{S}_\Omega$ and the plane containing the z-axis and Ω) we have

$$\Phi_1=(-\cos\phi_1\cos\nu\cos\varphi+\sin\phi_1\sin\varphi,-\cos\phi_1\cos\nu\sin\varphi-\sin\phi_1\cos\varphi,\cos\phi_1\sin\nu).$$

For ω which is the unit vector in the direction of the cross product of Ω and Φ_1 we have

$$\omega=(\sin\phi_1\cos\nu\cos\varphi+\cos\phi_1\sin\varphi,\sin\phi_1\cos\nu\sin\varphi-\cos\phi_1\cos\varphi,-\sin\phi_1\sin\nu),$$

which coordinates we denote by $\omega=(n_1,n_2,n_3)$.

Now we are going to calculate the expression (4.3) (see also (5.4)) for the ellipsoid $\partial\mathbb{K}$. Let $s(n_1,n_2,n_3)$ be the point on $\partial\mathbb{K}$, the outer normal of which is $\omega=(n_1,n_2,n_3)$. It is known that the Gauss curvature of the ellipsoid $\partial\mathbb{K}$ with the semi-principal axes of length a, b, c at $s(n_1,n_2,n_3)$ is

$$k_1(\omega)k_2(\omega)=\frac{(n_1^2a^2+n_2^2b^2+n_3^2c^2)^2}{a^2b^2c^2}. \tag{6.2}$$

One can calculate that the normal curvature in the direction Ω of the ellipsoid $\partial\mathbb{K}$ with the semi-principal axes of length a, b, c at $s(n_1, n_2, n_3) \in \partial\mathbb{K}$ is (note that Ω is perpendicular to ω)

$$k^2(\omega, \Omega) = (n_1^2 a^2 + n_2^2 b^2 + n_3^2 c^2)\left(\frac{\sin^2\nu\cos^2\varphi}{a^2} + \frac{\sin^2\nu\sin^2\varphi}{b^2} + \frac{\cos^2\nu}{c^2}\right)^2. \tag{6.3}$$

Substituting (6.3) and (6.2) into (5.4), we get

$$h(\Omega, \xi) = \int_{\mathbb{S}_\Omega} \frac{|\langle\xi, \Omega\rangle|(abc)^{-1}}{\cos^2\phi_1 + \cos^2(\widehat{\xi, \Omega})\sin^2\phi_1} \times \left(\frac{\sin^2\nu\cos^2\varphi}{a^2} + \frac{\sin^2\nu\sin^2\varphi}{b^2} + \frac{\cos^2\nu}{c^2}\right)^{-2} \lambda(\mathrm{d}\Phi_1) \tag{6.4}$$

which does not depend on ξ since (we assume ξ is not perpendicular to Ω)

$$\int_0^{2\pi} \frac{|\langle\xi, \Omega\rangle|}{\cos^2\phi_1 + \cos^2(\widehat{\xi, \Omega})\sin^2\phi_1}\,\mathrm{d}\phi_1 = 2\pi. \tag{6.5}$$

Thus we have proved that for the ellipsoid $\partial\mathbb{K}$ the expression $h(\Omega, \xi)$ (see (5.4)) does not depend on the polar angle of ξ measured from Ω. It is clear that for the azimuth angle of orthogonal projection of ξ on the reference plane that passes through the origin and is orthogonal to Ω, measured from a fixed reference direction on that plane, one can find

$$h(\Omega, \xi) = \int_{\mathbb{S}_\Omega} \frac{|\langle\xi, \Omega\rangle|(abc)^{-1}}{\cos^2(\phi_1 - \phi_0) + \cos^2(\widehat{\xi, \Omega})\sin^2(\phi_1 - \phi_0)} \times \left(\frac{\sin^2\nu\cos^2\varphi}{a^2} + \frac{\sin^2\nu\sin^2\varphi}{b^2} + \frac{\cos^2\nu}{c^2}\right)^{-2} \mathrm{d}\Phi_1 \tag{6.6}$$

which does not depend on ϕ_0. Thus a convex body $\mathbb{K} \in \mathcal{K}_0^3$ whose boundary is an ellipsoid belongs to the subclass $\mathcal{EZ}$.

The following question is open: is there a zonoid whose boundary is not an ellipsoid and which belongs to the class $\mathcal{EZ}$?

References

[1] *R. H. Aramyan*: Reconstruction of centrally symmetric convex bodies in $\mathbb{R}^n$. Bul. Acad. Ştiinţe Repub. Mold., Mat. *65* (2011), 28–32.

[2] *R. H. Aramyan*: Measures in the space of planes and convex bodies. J. Contemp. Math. Anal., Armen. Acad. Sci. *47* (2012), 78–85; translation from Izv. Nats. Akad. Nauk Armen., Mat. *47* (2012), 19–30. (In Russian.)

[3] *P. Goodey, W. Weil*: Zonoids and generalisations. Handbook of Convex Geometry, Vol. A, B (P. M. Gruber et al., eds.). North-Holland, Amsterdam, 1993, pp. 1297–1326.

[4] *K. Leichtweiss*: Konvexe Mengen. Hochschulbücher für Mathematik 81, VEB Deutscher Verlag der Wissenschaften, Berlin, 1980. (In German.)

[5] *F. Nazarov, D. Ryabogin, A. Zvavitch*: On the local equatorial characterization of zonoids and intersection bodies. Adv. Math. *217* (2008), 1368–1380.

[6] *G. Yu. Panina*: Representation of an n-dimensional body in the form of a sum of $(n-1)$-dimensional bodies. Izv. Akad. Nauk Arm. SSR, Mat. *23* (1988), 385–395 (In Russian.); translation in Sov. J. Contemp. Math. Anal. *23* (1988), 91–103.

[7] *R. Schneider*: Über eine Integralgleichung in der Theorie der konvexen Körper. Math. Nachr. *44* (1970), 55–75. (In German.)

[8] *R. Schneider*: Convex Bodies: the Brunn-Minkowski Theory. Encyclopedia of Mathematics and Its Applications 44, Cambridge University Press, Cambridge, 1993.

[9] *R. Schneider, W. Weil*: Zonoids and Related Topics. Convexity and Its Applications. Birkhäuser, Basel, 1983, pp. 296–317.

[10] *W. Weil*: Blaschkes Problem der lokalen Charakterisierung von Zonoiden. Arch. Math. *29* (1977), 655–659. (In German.)

Author's address: *Rafik Aramyan*, Russian-Armenian University, Institute of Mathematics of National Academy of Sciences of RA, 24/5 Marshal Baghramian ave, Yerevan, 0019, Armenia, e-mail: `rafikaramyan@yahoo.com`.