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SADDLE POINT CRITERIA FOR SECOND ORDER η -APPROXIMATED VECTOR OPTIMIZATION PROBLEMS

ANURAG JAYSWAL, SHALINI JHA AND SARITA CHOUDHURY

The purpose of this paper is to apply second order η -approximation method introduced to optimization theory by Antczak [2] to obtain a new second order η -saddle point criteria for vector optimization problems involving second order invex functions. Therefore, a second order η -saddle point and the second order η -Lagrange function are defined for the second order η -approximated vector optimization problem constructed in this approach. Then, the equivalence between an (weak) efficient solution of the considered vector optimization problem and a second order η -saddle point of the second order η -Lagrangian in the associated second order η -approximated vector optimization problem is established under the assumption of second order invexity.

Keywords: efficient solution, second order η -approximation, saddle point criteria, optimality condition

Classification: 90C26, 90C29, 90C30, 90C46

1. INTRODUCTION

Many real life problems arising in several fields of science, engineering, economics, etc, are associated with mathematical optimization problems involving more than one objective functions to be optimized simultaneously. Such problems are known as multiobjective optimization problems or vector optimization problems. There are several methods to solve multicriteria decision making problems in which one of them is saddle point criteria. Lagrange multiplier and saddle point of Lagrange function have an important role in vector optimization problems which are studied by several authors, see for instance, [3, 4, 7, 8, 11, 12].

The concept of invexity was first introduced by Hanson [9]. Later, Bector and Bector [6] generalized the definition of an invex function to the concept of a second order invex function. Bazaraa et al. [5] and Luc [13] studied necessary conditions for optimality in a nonlinear vector optimization problem. Thereafter, Aghezzaf and Hachimi [1] developed second order necessary conditions for optimality in vector optimization problems with twice differentiable functions. Some other important concepts related to the second order optimality conditions are discussed in detail in Horst et al. [10].

A new approach for solution of the considered nonlinear differentiable vector optimization problem named as an (first order) η -approximation method was firstly given by Antczak [2, 3]. In this approach, an η -approximated vector optimization problem is constructed by modifying both objective and constraint functions of the given vector optimization problem at a fixed feasible point. Later, Antczak [3] defined η -Lagrange function and η -saddle point in an η -approximated vector optimization problem. Moreover, in [4], he introduced the so-called second order η -Lagrange function and second order η -saddle point in the second order η -approximated mathematical programming problem.

With the help of this concept, we use the η -approximation method to obtain a second order η -saddle point criteria for vector optimization problems involving second order invex functions. To do this, we use definitions of the second order η -Lagrange function and second order η -saddle point for the second order η -approximated vector optimization problem constructed in this approach. The main purpose of this paper is, therefore, to establish the equivalence between a second order η -saddle point in the second order η -approximated vector optimization problem and an efficient solution of the original vector optimization problem under assumption that all the objective and constraint functions in the considered vector optimization problem are second order invex with respect to same function η .

Finally, this paper is sectionally divided as follows: Section 2 includes some definitions and theorem on second order optimality conditions. In Section 3, the formulation of a second order η -approximated vector optimization problem is presented and the definition of a second order η -saddle point is derived for such a multiobjective programming problem. Section 4 includes the second order η -saddle point criteria for vector optimization problem (VOP). We conclude our paper in Section 5.

2. NOTATIONS AND PRELIMINARIES

Let $x, y \in R^n$. Then the following inequalities and equalities will be used:

- (i) $x = y \Leftrightarrow x_i = y_i, \forall i = 1, \dots, n;$
- (ii) $x < y \Leftrightarrow x_i < y_i, \forall i = 1, \dots, n;$
- (iii) $x \leq y \Leftrightarrow x_i \leq y_i, \forall i = 1, \dots, n;$
- (iv) $x \leq y \Leftrightarrow x_i \leq y_i, \forall i = 1, \dots, n,$ with strict inequality hold for at least one i .

Here, $x \not< y$ is the negation of $x < y$.

Throughout this paper, let X be a nonempty open subset of Euclidean space R^n and R_+^n denote the nonnegative orthant.

Definition 2.1. (Antczak [2]) A differentiable function $f : X \mapsto R^k$ is said to be invex at $u \in X$ on X with respect to $\eta : X \times X \mapsto R^n$, if

$$f(x) - f(u) \geq \nabla f(u)\eta(x, u), \forall x \in X.$$

If the above inequality holds for any $u \in X$, then f is said to be invex on X with respect to η .

Now, we shall give the definition of second order invexity for vector valued function by using the notion of second order invexity for scalar function defined in [4].

Definition 2.2. A function $f : X \mapsto R^k$ of C^2 -class is said to be (strictly) second order invex at $u \in X$ if, for all $x \in X$ and $x \neq u$, there exists $\eta : X \times X \mapsto R^n$ such that

$$f(x) - f(u) (>) \geq \nabla f(u)\eta(x, u) + p^T \nabla^2 f(u)\eta(x, u) - \frac{1}{2}p^T \nabla^2 f(u)p, \quad \forall p \in R^n, \quad (1)$$

where $\nabla f(u)$ is the Jacobian matrix and for any $r, s \in R^n$, we have

$$r^T \nabla^2 f(x)s = \begin{pmatrix} r^T \nabla^2 f_1(x)s \\ r^T \nabla^2 f_2(x)s \\ \vdots \\ r^T \nabla^2 f_k(x)s \end{pmatrix}$$

and the symbol “ T ” denotes the transpose operator. If inequality (1) holds for any $u \in X$, then f is said to be (strictly) second order invex on X with respect to η .

We consider the following nonlinear vector optimization problem:

$$\begin{aligned} \text{(VOP)} \quad & V\text{-min } f(x) = (f_1(x), \dots, f_k(x)) \\ \text{subject to} \quad & g(x) \leq 0, \\ & x \in X \end{aligned}$$

where $f : X \mapsto R^k$, and $g : X \mapsto R^m$ are functions of C^2 -class.

Let D denote the set of all feasible solutions of the vector optimization problem (VOP), i. e.,

$$D = \{x \in X : g_j(x) \leq 0, j \in J = \{1, \dots, m\}\}.$$

Further, we denote the indexed set of constraints active at the given feasible point \bar{x} by

$$J(\bar{x}) = \{j \in J : g_j(\bar{x}) = 0\}.$$

Definition 2.3. (Antczak [2]) A feasible point $\bar{x} \in D$ is said to be an efficient solution of the vector optimization problem (VOP) if there exists no $y \in D$ such that

$$f(y) \leq f(\bar{x}).$$

Definition 2.4. (Antczak [2]) A feasible point $\bar{x} \in D$ is said to be a weak efficient solution of the vector optimization problem (VOP) if there exists no $y \in D$ such that

$$f(y) < f(\bar{x}).$$

Definition 2.5. (Aghezzaf and Hachimi [1]) A direction $d \in R^n$ is said to be a critical direction for a feasible point $\bar{x} \in D$ if it satisfy the following conditions:

(a) $\nabla f(\bar{x})d \leq 0,$

(b) $d^T \nabla f_i(\bar{x}) = 0$, for at least one $i \in I = \{1, 2, \dots, k\}$,

(c) $d^T \nabla g_j(\bar{x}) \leq 0$, $j \in J(\bar{x})$.

The set of all critical directions at \bar{x} is denoted by $A(\bar{x})$.

Motivated by Aghezzaf and Hachimi [1], we present the following modified version of the second order necessary conditions for efficiency of a feasible point \bar{x} in vector optimization problem (VOP).

Theorem 2.6. Let \bar{x} be an efficient solution of the vector optimization problem (VOP) at which the second order Abadie constraint qualification (ACQ) [1] is satisfied. Then, for every $d \in A(\bar{x})$, there exist $\bar{\nu} \in R_+^k$, $\bar{\mu} \in R_+^m$ such that

$$\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0, \tag{2}$$

$$\bar{\nu}^T d^T \nabla^2 f(\bar{x})d + \bar{\mu}^T d^T \nabla^2 g(\bar{x})d \geq 0, \tag{3}$$

$$\bar{\mu}^T g(\bar{x}) = 0, \tag{4}$$

$$\bar{\nu}^T \nabla f(\bar{x})d = 0, \tag{5}$$

$$\bar{\mu}_j d^T \nabla g_j(\bar{x}) = 0, \forall j \in J(\bar{x}), \tag{6}$$

$$\bar{\nu} \geq 0, \bar{\mu} \geq 0. \tag{7}$$

3. SECOND ORDER η -SADDLE POINT CRITERIA FOR THE SECOND ORDER η -APPROXIMATED VECTOR OPTIMIZATION PROBLEM

In [4], Antczak defined the so-called second order η -approximated optimization problem for a nonlinear mathematical programming problem. Now, we give his definition for the considered vector optimization problem (VOP).

Let \bar{x} be a feasible solution of the vector optimization problem (VOP). Then, a second order η -approximated vector optimization problem $(VOP)_\eta^2(\bar{x})$ corresponding to (VOP) is constructed as follows:

$$(VOP)_\eta^2(\bar{x}) \quad V\text{-min } F(x) = \left(f_1(\bar{x}) + \eta(x, \bar{x})^T \nabla f_1(\bar{x}) + \frac{1}{2} \eta(x, \bar{x})^T \nabla^2 f_1(\bar{x}) \eta(x, \bar{x}), \dots, \right. \\ \left. f_k(\bar{x}) + \eta(x, \bar{x})^T \nabla f_k(\bar{x}) + \frac{1}{2} \eta(x, \bar{x})^T \nabla^2 f_k(\bar{x}) \eta(x, \bar{x}) \right)$$

subject to $G(x) = g(\bar{x}) + \nabla g(\bar{x}) \eta(x, \bar{x}) + \frac{1}{2} \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x}) \leq 0$,

where f and g are defined as in vector optimization problem (VOP) and $\eta : X \times X \mapsto R^n$ is a bifunction. Let

$$D(\bar{x}) = \left\{ x \in X : g_j(\bar{x}) + \eta(x, \bar{x})^T \nabla g_j(\bar{x}) + \frac{1}{2} \eta(x, \bar{x})^T \nabla^2 g_j(\bar{x}) \eta(x, \bar{x}) \leq 0, \forall j \in J \right\}$$

denote the set of all feasible solutions of the second order η -approximated vector optimization problem $(VOP)_\eta^2(\bar{x})$.

Further, motivated also by Antczak [4], we define a second order η -Lagrange function and a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$.

Definition 3.1. The second order η -Lagrangian for $(\text{VOP})_\eta^2(\bar{x})$ is denoted by $L_\eta^2 : D(\bar{x}) \times R_+^k \times R_+^m \mapsto R^k$ and defined as

$$\begin{aligned} L_\eta^2(x, \nu, \mu) &= \text{diag } \nu f(\bar{x}) + \mu^T g(\bar{x})e + (\nu^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(x, \bar{x})e \\ &\quad + \frac{1}{2}(\nu^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \mu^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}))e \\ &= ((\nu_1 f_1(\bar{x}) + \mu^T g(\bar{x}) + (\nu^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(x, \bar{x}) \\ &\quad + \frac{1}{2}(\nu^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \mu^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})), \dots \\ &\quad \dots, (\nu_k f_k(\bar{x}) + \mu^T g(\bar{x}) + (\nu^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(x, \bar{x}) \\ &\quad + \frac{1}{2}(\nu^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \mu^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}))), \end{aligned}$$

where $\text{diag } \nu = \begin{pmatrix} \nu_1 & 0 & \dots & 0 \\ 0 & \nu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \nu_k \end{pmatrix}$, $\nu \in R_+^k$, $\mu \in R_+^m$, $e = (1, \dots, 1) \in R^k$.

Definition 3.2. Let $(\bar{x}, \bar{\nu}, \bar{\mu}) \in D(\bar{x}) \times R_+^k \times R_+^m$. Then $(\bar{x}, \bar{\nu}, \bar{\mu})$ is said to be a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ if the following inequalities hold:

- (i) $L_\eta^2(\bar{x}, \bar{\nu}, \mu) \leq L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu})$, $\forall \mu \in R_+^m$,
- (ii) $L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) \leq L_\eta^2(x, \bar{\nu}, \bar{\mu})$, $\forall x \in D(\bar{x})$.

Proposition 3.3. Let $(\bar{x}, \bar{\nu}, \bar{\mu}) \in D(\bar{x}) \times R_+^k \times R_+^m$ be a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ and $\bar{\nu} > 0$. Then \bar{x} is an efficient solution of the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$.

Proof. Suppose, contrary to the result, that \bar{x} is not an efficient solution of $(\text{VOP})_\eta^2(\bar{x})$. Then, there exists a point $u \in D(\bar{x})$ such that

$$\begin{aligned} F(u) &\leq F(\bar{x}) \\ \Rightarrow f_i(\bar{x}) + \eta(u, \bar{x})^T \nabla f_i(\bar{x}) + \frac{1}{2}\eta(u, \bar{x})^T \nabla^2 f_i(\bar{x})\eta(u, \bar{x}) \\ &\leq f_i(\bar{x}) + \eta(\bar{x}, \bar{x})^T \nabla f_i(\bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 f_i(\bar{x})\eta(\bar{x}, \bar{x}) \\ \Rightarrow \eta(u, \bar{x})^T \nabla f_i(\bar{x}) + \frac{1}{2}\eta(u, \bar{x})^T \nabla^2 f_i(\bar{x})\eta(u, \bar{x}) \\ &\leq \eta(\bar{x}, \bar{x})^T \nabla f_i(\bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 f_i(\bar{x})\eta(\bar{x}, \bar{x}), \end{aligned}$$

for all $i \in I$, with strict inequality for at least one $i \in I$. Since $\bar{\nu} \in R_+^k$ and $\bar{\nu} > 0$, therefore, multiplying both sides of the above inequality by $\bar{\nu}_i$ and taking summation over $i \in I$, we get

$$\begin{aligned} & \bar{\nu}^T \nabla f(\bar{x}) \eta(u, \bar{x}) + \frac{1}{2} \bar{\nu}^T \eta(u, \bar{x})^T \nabla^2 f(\bar{x}) \eta(u, \bar{x}) \\ & < \bar{\nu}^T \nabla f(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\nu}^T \nabla f(\bar{x}) \eta(u, \bar{x}) e + \frac{1}{2} \bar{\nu}^T \eta(u, \bar{x})^T \nabla^2 f(\bar{x}) \eta(u, \bar{x}) e \\ & < \text{diag } \bar{\nu} f(\bar{x}) + \bar{\nu}^T \nabla f(\bar{x}) \eta(\bar{x}, \bar{x}) e + \frac{1}{2} \bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) e. \end{aligned} \quad (8)$$

Again, $u \in D(\bar{x})$ implies

$$\begin{aligned} & g_j(\bar{x}) + \eta(u, \bar{x})^T \nabla g_j(\bar{x}) + \frac{1}{2} \eta(u, \bar{x})^T \nabla^2 g_j(\bar{x}) \eta(u, \bar{x}) \leq 0, \quad \forall j \in J \\ \Rightarrow & g(\bar{x}) + \nabla g(\bar{x}) \eta(u, \bar{x}) + \frac{1}{2} \eta(u, \bar{x})^T \nabla^2 g(\bar{x}) \eta(u, \bar{x}) \leq 0. \end{aligned}$$

Since $\bar{\mu} \in R_+^m$, the above inequality yields

$$\bar{\mu}^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(u, \bar{x}) + \frac{1}{2} \eta(u, \bar{x})^T \nabla^2 g(\bar{x}) \eta(u, \bar{x})) \leq 0. \quad (9)$$

Again, \bar{x} is feasible in $(\text{VOP})_\eta^2(\bar{x})$. Thus, replacing u by \bar{x} in the above inequality, we obtain

$$\bar{\mu}^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \leq 0. \quad (10)$$

As $(\bar{x}, \bar{\nu}, \bar{\mu})$ is a second order η -saddle point in $(\text{VOP})_\eta^2(\bar{x})$, therefore

$$L_\eta^2(\bar{x}, \bar{\nu}, \mu) \leq L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}), \quad \forall \mu \in R_+^m.$$

By the definition of the second order η -Lagrange function, it follows that

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) e \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) e \\ \leq & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) e \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) e. \end{aligned}$$

This implies that

$$\begin{aligned} & \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \\ \leq & \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})). \end{aligned}$$

For $\mu = 0$, the above inequality reduces to

$$\bar{\mu}^T(g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \geq 0. \tag{11}$$

On combining inequalities (10) and (11), we get

$$\bar{\mu}^T \left(g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}) \right) = 0. \tag{12}$$

From (9) and (12), it follows that

$$\begin{aligned} & \bar{\mu}^T(g(\bar{x}) + \nabla g(\bar{x})\eta(u, \bar{x}) + \frac{1}{2}\eta(u, \bar{x})^T \nabla^2 g(\bar{x})\eta(u, \bar{x})) \\ & \leq \bar{\mu}^T(g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \\ \Rightarrow & \bar{\mu}^T(g(\bar{x}) + \nabla g(\bar{x})\eta(u, \bar{x}) + \frac{1}{2}\eta(u, \bar{x})^T \nabla^2 g(\bar{x})\eta(u, \bar{x}))e \\ & \leq \bar{\mu}^T(g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e. \end{aligned} \tag{13}$$

Adding (8) and (13), we get

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(u, \bar{x})e \\ & \quad + \frac{1}{2}(\bar{\nu}^T \eta(u, \bar{x})^T \nabla^2 f(\bar{x})\eta(u, \bar{x}) + \bar{\mu}^T \eta(u, \bar{x})^T \nabla^2 g(\bar{x})\eta(u, \bar{x}))e \\ < & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ & \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e, \end{aligned}$$

which by the definition of second order η -Lagrange function, yields

$$L_\eta^2(u, \bar{\nu}, \bar{\mu}) < L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}).$$

This contradicts inequality (ii) in the Definition 3.2 of a second order η -saddle point in $(\text{VOP})_\eta^2(\bar{x})$. This completes the proof. \square

Now, we give an example of a vector optimization problem with twice differentiable functions to illustrate the result established in Proposition 3.3.

Example 3.4. Let us consider the vector optimization problem:

$$\begin{aligned} (\text{VOP}) \quad & V\text{-min } f(x) = (xe^{x+x^2} + x^4, -\arctan x - x^2) \\ \text{subject to} \quad & g(x) = -xe^x \leq 0, \\ & x \in R, \end{aligned}$$

where $f : R \mapsto R^2$, $g : R \mapsto R$ are twice differentiable functions. The set of all feasible solutions of the vector optimization problem (VOP) is given by $D = \{x \in R : x \geq 0\}$.

Clearly, $\bar{x} = 0$ is a feasible solution of the considered (VOP). Let $\eta : R \times R \mapsto R$ be defined as

$$\eta(x, \bar{x}) = \frac{1}{2}(x + \bar{x}).$$

Now, the associated second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ is constructed as follows:

$$\begin{aligned} (\text{VOP})_\eta^2(\bar{x}) \quad & V\text{-min } F(x) = \left(\frac{1}{2}x + \frac{1}{4}x^2, -\frac{1}{2}x - \frac{1}{4}x^2 \right) \\ \text{subject to} \quad & G(x) = -\frac{1}{2}x - \frac{1}{4}x^2 \leq 0. \end{aligned}$$

The set of all feasible solutions of $(\text{VOP})_\eta^2(\bar{x})$ is $D(\bar{x}) = \{x \in R : x \leq -2 \vee x \geq 0\}$ and the second order η -approximated Lagrangian $L_\eta^2 : D(\bar{x}) \times R_+^2 \times R_+ \mapsto R$ in the problem $(\text{VOP})_\eta^2(\bar{x})$ is given by

$$L_\eta^2(x, \nu, \mu) = \left(\frac{1}{2}(\nu_1 - \nu_2 - \mu)x + \frac{1}{4}(\nu_1 - \nu_2 - \mu)x^2, \frac{1}{2}(\nu_1 - \nu_2 - \mu)x + \frac{1}{4}(\nu_1 - \nu_2 - \mu)x^2 \right).$$

Therefore, $(\bar{x}, \bar{\nu}, \bar{\mu}) = (0, (\bar{\nu}_1, \bar{\nu}_2), \bar{\mu})$ is a second order η -saddle point where $\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu} = 0$, since

$$\begin{aligned} & L_\eta^2(\bar{x}, \bar{\nu}, \mu) - L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) \\ &= \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x}^2 \right) \\ &\quad - \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x}^2 \right) \\ &= (0, 0) - (0, 0) = (0, 0), \quad \forall \mu \in R_+, \end{aligned}$$

and $L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) - L_\eta^2(x, \bar{\nu}, \bar{\mu})$

$$\begin{aligned} &= \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x}^2 \right) \\ &\quad - \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x^2 \right) \\ &= (0, 0) - (0, 0) = (0, 0), \quad \forall x \in D(\bar{x}). \end{aligned}$$

Hence, by Proposition 3.3, $\bar{x} = 0$ is an efficient solution of the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$, what it can be easily verified.

Proposition 3.5. Let \bar{x} be an efficient solution of the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ at which the second order (ACQ) is satisfied. Assume that the objective function F and the constraint function G in $(\text{VOP})_\eta^2(\bar{x})$ are second order invex at \bar{x} on $D(\bar{x})$ with respect to the same function $\tilde{\eta} : D(\bar{x}) \times D(\bar{x}) \mapsto R^n$ (not necessarily equal to η), satisfying the condition $\eta(\bar{x}, \bar{x}) = 0$. Then there exist $\bar{\mu} \in R_+^m, \bar{\nu} \in R_+^k$ such that $(\bar{x}, \bar{\nu}, \bar{\mu})$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$.

Proof. Since \bar{x} is an efficient solution of the second order η -approximated vector optimization problem $(\text{VOP})_{\eta}^2(\bar{x})$ at which second order (ACQ) is satisfied, therefore, the second order optimality conditions (2)–(7) are also satisfied at \bar{x} . We shall show that $(\bar{x}, \bar{\nu}, \bar{\mu})$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_{\eta}^2(\bar{x})$, i. e.,

- (i) $L_{\eta}^2(\bar{x}, \bar{\nu}, \mu) \leq L_{\eta}^2(\bar{x}, \bar{\nu}, \bar{\mu}), \forall \mu \in R_+^m,$
- (ii) $L_{\eta}^2(\bar{x}, \bar{\nu}, \bar{\mu}) \leq L_{\eta}^2(x, \bar{\nu}, \bar{\mu}), \forall x \in D(\bar{x}).$

(i) Since $\bar{x} \in D(\bar{x})$, therefore

$$g(\bar{x}) + \eta(\bar{x}, \bar{x})^T \nabla g(\bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x}) \leq 0. \tag{14}$$

Since $\mu \in R_+^m$, multiplying both sides of (14) by μ^T , it follows that

$$\mu^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \leq 0. \tag{15}$$

From the second order optimality condition (4), we have

$$\bar{\mu}^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) = 0. \tag{16}$$

On combining (15) and (16), we get

$$\begin{aligned} & \mu^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \\ & \leq \bar{\mu}^T (g(\bar{x}) + \nabla g(\bar{x}) \eta(\bar{x}, \bar{x}) + \frac{1}{2} \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \mu^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) e \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \mu^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) e \\ \leq & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) e \\ & \quad + \frac{1}{2} (\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) e, \end{aligned}$$

which, by the definition of the second order η -Lagrange function, yields

$$L_{\eta}^2(\bar{x}, \bar{\nu}, \mu) \leq L_{\eta}^2(\bar{x}, \bar{\nu}, \bar{\mu}), \forall \mu \in R_+^m. \tag{17}$$

(ii) Since F and G are second order invex at \bar{x} on $D(\bar{x})$ with respect to the same function $\tilde{\eta}$, therefore, we have

$$\begin{aligned} F(x) - F(\bar{x}) & \geq \nabla F(\bar{x}) \tilde{\eta}(x, \bar{x}) + p^T \nabla^2 F(\bar{x}) \tilde{\eta}(x, \bar{x}) - \frac{1}{2} p^T \nabla^2 F(\bar{x}) p \\ \text{and } G(x) - G(\bar{x}) & \geq \nabla G(\bar{x}) \tilde{\eta}(x, \bar{x}) + p^T \nabla^2 G(\bar{x}) \tilde{\eta}(x, \bar{x}) - \frac{1}{2} p^T \nabla^2 G(\bar{x}) p, \end{aligned}$$

for all $x \in D(\bar{x})$ and $p \in R^n$. Hence, the above two inequalities are also satisfied for $p = \tilde{\eta}(x, \bar{x})$, i. e.,

$$F(x) - F(\bar{x}) \geq \nabla F(\bar{x})\tilde{\eta}(x, \bar{x}) + \tilde{\eta}(x, \bar{x})^T \nabla^2 F(\bar{x})\tilde{\eta}(x, \bar{x}) - \frac{1}{2}\tilde{\eta}(x, \bar{x})^T \nabla^2 F(\bar{x})\tilde{\eta}(x, \bar{x})$$

$$\text{and } G(x) - G(\bar{x}) \geq \nabla G(\bar{x})\tilde{\eta}(x, \bar{x}) + \tilde{\eta}(x, \bar{x})^T \nabla^2 G(\bar{x})\tilde{\eta}(x, \bar{x}) - \frac{1}{2}\tilde{\eta}(x, \bar{x})^T \nabla^2 G(\bar{x})\tilde{\eta}(x, \bar{x}),$$

which reduce to

$$F(x) - F(\bar{x}) \geq \nabla F(\bar{x})\tilde{\eta}(x, \bar{x}) + \frac{1}{2}\tilde{\eta}(x, \bar{x})^T \nabla^2 F(\bar{x})\tilde{\eta}(x, \bar{x}) \tag{18}$$

$$\text{and } G(x) - G(\bar{x}) \geq \nabla G(\bar{x})\tilde{\eta}(x, \bar{x}) + \frac{1}{2}\tilde{\eta}(x, \bar{x})^T \nabla^2 G(\bar{x})\tilde{\eta}(x, \bar{x}). \tag{19}$$

Since $\bar{\nu} \in R_+^k$ and $\bar{\mu} \in R_+^m$, therefore, multiplying both sides of (18) and (19) by $\bar{\nu}^T$ and $\bar{\mu}^T$, respectively, and adding them, we get

$$\begin{aligned} \bar{\nu}^T F(x) + \bar{\mu}^T G(x) - \bar{\nu}^T F(\bar{x}) - \bar{\mu}^T G(\bar{x}) &\geq (\bar{\nu}^T \nabla F(\bar{x}) + \bar{\mu}^T \nabla G(\bar{x}))\tilde{\eta}(x, \bar{x}) \\ &\quad + \frac{1}{2}(\bar{\nu}^T \tilde{\eta}(x, \bar{x})^T \nabla^2 F(\bar{x})\tilde{\eta}(x, \bar{x}) + \bar{\mu}^T \tilde{\eta}(x, \bar{x})^T \nabla^2 G(\bar{x})\tilde{\eta}(x, \bar{x})). \end{aligned}$$

Using the second order optimality conditions (2) and (3) in the above inequality, we get

$$\begin{aligned} \bar{\nu}^T F(x) + \bar{\mu}^T G(x) - \bar{\nu}^T F(\bar{x}) - \bar{\mu}^T G(\bar{x}) &\geq 0 \\ \Rightarrow \bar{\nu}^T F(x) + \bar{\mu}^T G(x) &\geq \bar{\nu}^T F(\bar{x}) + \bar{\mu}^T G(\bar{x}) \end{aligned}$$

$$\begin{aligned} \Rightarrow \bar{\nu}^T \left\{ f(\bar{x}) + \nabla f(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) \right\} \\ \quad + \bar{\mu}^T \left\{ g(\bar{x}) + \nabla g(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}) \right\} \\ \geq \bar{\nu}^T \left\{ f(\bar{x}) + \nabla f(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) \right\} \\ \quad + \bar{\mu}^T \left\{ g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}) \right\} \\ \Rightarrow \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) \\ \quad + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})) \\ \geq \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\ \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})). \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x})e \\ \quad + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}))e \\ \geq \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e, \end{aligned}$$

which, by the definition of the second order η -Lagrange function, yields

$$L_\eta^2(x, \bar{\nu}, \bar{\mu}) \geq L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}), \quad \forall x \in D(\bar{x}). \tag{20}$$

Thus, from (17) and (20), we get the required result. This completes the proof. \square

4. SECOND ORDER η -SADDLE POINT CRITERIA FOR VECTOR OPTIMIZATION PROBLEM

In this section, we prove the equivalence between an efficient solution of the vector optimization problem (VOP) and a second order η -saddle point in its associated second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$, under the assumption of second order invexity. Firstly, we prove an important lemma before obtaining the main results of this section.

Lemma 4.1. Let x be any feasible solution of the vector optimization problem (VOP). Assume that the constraint function g is second order invex at \bar{x} on the set of feasible solutions D . Then, x is also a feasible solution of the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$, i.e., $D \subset D(\bar{x})$.

Proof. Let $x \in D$. Then

$$g(x) \leq 0. \tag{21}$$

Since g is second order invex at \bar{x} on D with respect to η , by Definition 2.2, we have

$$g(x) - g(\bar{x}) \geq \nabla g(\bar{x})\eta(x, \bar{x}) + p^T \nabla^2 g(\bar{x})\eta(x, \bar{x}) - \frac{1}{2} p^T \nabla^2 g(\bar{x})p, \quad \forall p \in R^n, x \in D.$$

This is also true for $p = \eta(x, \bar{x})$. Therefore,

$$g(x) \geq g(\bar{x}) + \nabla g(\bar{x})\eta(x, \bar{x}) + \frac{1}{2} \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}). \tag{22}$$

From (21) and (22), it follows that $G(x) \leq 0$, i.e., $x \in D(\bar{x})$. Hence, $D \subset D(\bar{x})$. This completes the proof. \square

Theorem 4.2. Let \bar{x} be a feasible solution of the vector optimization problem (VOP). Assume that the objective function f and the constraint function g are second order invex at \bar{x} on D with respect to the same function η , satisfying the condition $\eta(\bar{x}, \bar{x}) = 0$. If $(\bar{x}, \bar{\nu}, \bar{\mu}) \in D(\bar{x}) \times R_+^k \times R_+^m$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ and $\bar{\nu} \neq 0$, then \bar{x} is a weak efficient solution of the considered vector optimization problem (VOP).

Proof. Since $(\bar{x}, \bar{\nu}, \bar{\mu})$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$, therefore

$$L_\eta^2(\bar{x}, \bar{\nu}, \mu) \leq L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}), \quad \forall \mu \in R_+^m.$$

By the definition of the second order η -Lagrange function, it follows that

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \mu^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ & \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x})^T + \mu^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e \\ \leq & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ & \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e. \end{aligned}$$

As $\mu \in R_+^m$, therefore for $\mu = 0$, the above inequality yields

$$\bar{\mu}^T \left\{ g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}) \right\} e \geq 0.$$

Since $\eta(\bar{x}, \bar{x}) = 0$, therefore, we have

$$\bar{\mu}^T g(\bar{x}) \geq 0. \tag{23}$$

According to the assumption g is second order invex at \bar{x} and \bar{x} is feasible in (VOP). Therefore, by Lemma 4.1, \bar{x} is also feasible in its second order η -approximated vector optimization problem (VOP) $_{\eta}^2(\bar{x})$, i. e., $\bar{x} \in D(\bar{x})$. Thus, for $\bar{\mu} \in R_+^m$, we have

$$\bar{\mu}^T \left\{ g(\bar{x}) + \nabla g(\bar{x})\eta(\bar{x}, \bar{x}) + \frac{1}{2}\eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}) \right\} e \leq 0.$$

Again, by the hypothesis $\eta(\bar{x}, \bar{x})=0$, we get

$$\bar{\mu}^T g(\bar{x}) \leq 0. \tag{24}$$

On combining inequalities (23) and (24), we get

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{25}$$

Suppose, contrary to the result, that \bar{x} is not a weak efficient solution of the vector optimization problem (VOP). Then, there exists a point $x \in D$ such that

$$\begin{aligned} & f(x) < f(\bar{x}) \\ \Rightarrow & f(x) - f(\bar{x}) < 0. \end{aligned}$$

Since $\bar{\nu} \in R_+^k$ and $\bar{\nu} \neq 0$, therefore, the above inequality implies

$$\bar{\nu}^T f(x) - \bar{\nu}^T f(\bar{x}) < 0. \tag{26}$$

Again $x \in D \subset D(\bar{x})$, from inequality (ii) in the definition of second order η -saddle point, we have

$$L_{\eta}^2(\bar{x}, \bar{\nu}, \bar{\mu}) \leq L_{\eta}^2(x, \bar{\nu}, \bar{\mu}).$$

By the definition of the second order η -Lagrange function, it follows that

$$\begin{aligned}
 & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\
 & \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e \\
 & \leq \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x})e \\
 & \quad + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}))e \\
 \Rightarrow & (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\
 & \quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \\
 & \leq (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) \\
 & \quad + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})).
 \end{aligned} \tag{27}$$

On the other hand, by the second order invexity of g at \bar{x} on D with respect to η , we have

$$g(x) - g(\bar{x}) \geq \nabla g(\bar{x})\eta(x, \bar{x}) + p^T \nabla^2 g(\bar{x})\eta(x, \bar{x}) - \frac{1}{2}p^T \nabla^2 g(\bar{x})p, \quad \forall p \in R^n.$$

Taking $p = \eta(x, \bar{x})$ in the above inequality, we get

$$g(x) - g(\bar{x}) \geq \nabla g(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}).$$

Since $\bar{\mu} \in R_+^m$, the above inequality implies that

$$\bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}) \geq \bar{\mu}^T \nabla g(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x}). \tag{28}$$

Also, by the second order invexity of f at \bar{x} on D with respect to η , we have

$$f(x) - f(\bar{x}) \geq \nabla f(\bar{x})\eta(x, \bar{x}) + p^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) - \frac{1}{2}p^T \nabla^2 f(\bar{x})p, \quad \forall p \in R^n.$$

Again, taking $p = \eta(x, \bar{x})$ in the above inequality, we get

$$f(x) - f(\bar{x}) \geq \nabla f(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}).$$

Since $\bar{\nu} \in R_+^k$, therefore, the above inequality implies that

$$\bar{\nu}^T f(x) - \bar{\nu}^T f(\bar{x}) \geq \bar{\nu}^T \nabla f(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}). \tag{29}$$

Adding (28) and (29), we get

$$\begin{aligned}
 & (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})) \\
 & \leq \bar{\nu}^T f(x) - \bar{\nu}^T f(\bar{x}) + \bar{\mu}^T g(x) - \bar{\mu}^T g(\bar{x}).
 \end{aligned}$$

Using the feasibility of x in (VOP) and (25), the above inequality reduces to

$$\begin{aligned}
 (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})) \\
 \leq \bar{\nu}^T f(x) - \bar{\nu}^T f(\bar{x}).
 \end{aligned}$$

From inequality (26), it follows that

$$\begin{aligned}
 (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) \\
 + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})) < 0.
 \end{aligned}$$

By the hypothesis $\eta(\bar{x}, \bar{x}) = 0$, the above inequality implies

$$\begin{aligned}
 (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(x, \bar{x})) \\
 < (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\
 + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})),
 \end{aligned}$$

which contradicts (27). Therefore, \bar{x} is a weak efficient solution of the vector optimization (VOP). This completes the proof. □

Now, we give an example of a nonlinear vector optimization problem with twice differentiable functions to illustrate the result established in Theorem 4.2.

Example 4.3. Let $X = (-1, 1)$. We consider the vector optimization problem:

$$\begin{aligned}
 \text{(VOP)} \quad & V\text{-min } f(x) = (\arcsin x + (\arcsin x)^2 + (\arcsin x)^3 + (\arcsin x)^4 \\
 & \quad \quad \quad + (\arcsin x)^5, x^4 \arcsin x + (\arcsin x)^2) \\
 \text{subject to} \quad & g(x) = (1 + x^4)(\arctan x)^2 - \arctan x \leq 0, \\
 & x \in X,
 \end{aligned}$$

where $f : X \mapsto R^2$, $g : X \mapsto R$ are twice differentiable functions. The set of all feasible solutions of (VOP) is given by

$$D = \{x \in X : (1 + x^4)(\arctan x)^2 - \arctan x \leq 0\}.$$

Clearly, $\bar{x} = 0$ is a feasible solution of the given vector optimization problem (VOP). Let $\eta : X \times X \mapsto R$ be defined as

$$\eta(x, \bar{x}) = \arcsin x - \arcsin \bar{x}.$$

Therefore, $\eta(\bar{x}, \bar{x}) = 0$.

Here, one can easily verify that f and g are second order invex at \bar{x} on D with respect to η as shown below. We have,

$$\begin{aligned} f_1(x) - f_1(\bar{x}) - \nabla f_1(\bar{x})\eta(x, \bar{x}) - p^T \nabla^2 f_1(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}p^T \nabla^2 f_1(\bar{x})p \\ = \arcsin x + (\arcsin x)^2 + (\arcsin x)^3 + (\arcsin x)^4 + (\arcsin x)^5 \\ - \arcsin x(1 + 2p) + p^2 \\ \geq 0, \forall p \in R. \end{aligned}$$

and $f_2(x) - f_2(\bar{x}) - \nabla f_2(\bar{x})\eta(x, \bar{x}) - p^T \nabla^2 f_2(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}p^T \nabla^2 f_2(\bar{x})p$

$$\begin{aligned} = x^4 \arcsin x + (\arcsin x)^2 - 2p \arcsin x + p^2 \\ \geq 0, \forall p \in R. \end{aligned}$$

Hence, f is a second order invex function with respect to η on D . Similarly,

$$\begin{aligned} g(x) - g(\bar{x}) - \nabla g(\bar{x})\eta(x, \bar{x}) - p^T \nabla^2 g(\bar{x})\eta(x, \bar{x}) + \frac{1}{2}p^T \nabla^2 g(\bar{x})p \\ = (1 + x^4)(\arctan x)^2 - \arctan x - \arcsin x(-1 + 2p) + p^2 \\ \geq 0, \forall p \in R. \end{aligned}$$

Thus, g is also a second order invex function with respect to η on D .

Now, the second order η -approximated vector optimization problem $(VOP)_\eta^2(\bar{x})$ is constructed as follows:

$$\begin{aligned} (VOP)_\eta^2(\bar{x}) \quad V\text{-min } F(x) &= (\arcsin x + (\arcsin x)^2, (\arcsin x)^2) \\ \text{subject to } \quad G(x) &= (\arcsin x)^2 - \arcsin x \leq 0. \end{aligned}$$

The set of all feasible solutions of $(VOP)_\eta^2(\bar{x})$ is

$$D(\bar{x}) = \{x \in X : (\arcsin x)^2 - \arcsin x \leq 0\}$$

and the second order η -approximated Lagrangian $L_\eta^2 : D(\bar{x}) \times R_+^2 \times R_+ \mapsto R$ in the problem $(VOP)_\eta^2(\bar{x})$ is given by

$$\begin{aligned} L_\eta^2(x, \nu, \mu) &= ((\nu_1 - \mu) \arcsin x + (\nu_1 + \nu_2 + \mu)(\arcsin x)^2, \\ &\quad (\nu_1 - \mu) \arcsin x + (\nu_1 + \nu_2 + \mu)(\arcsin x)^2), \end{aligned}$$

where $x \in D(\bar{x})$, $\nu = (\nu_1, \nu_2) \in R_+^2$ and $\mu \in R_+$. Therefore, $(\bar{x}, \bar{\nu}, \bar{\mu}) = (0, (\bar{\nu}_1, \bar{\nu}_2), \bar{\mu})$ is a second order η -saddle point, where $\bar{\nu}_1 = \bar{\mu}$, since

$$\begin{aligned} L_\eta^2(\bar{x}, \bar{\nu}, \mu) - L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) \\ = ((\bar{\nu}_1 - \mu) \arcsin \bar{x} + (\bar{\nu}_1 + \bar{\nu}_2 + \mu)(\arcsin \bar{x})^2, (\bar{\nu}_1 - \mu) \arcsin \bar{x} \\ + (\bar{\nu}_1 + \bar{\nu}_2 + \mu)(\arcsin \bar{x})^2) - ((\bar{\nu}_1 - \bar{\mu}) \arcsin \bar{x} + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin \bar{x})^2, \\ (\bar{\nu}_1 - \bar{\mu}) \arcsin \bar{x} + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin \bar{x})^2) \\ = (0, 0) - (0, 0) = (0, 0), \forall \mu \in R_+ \end{aligned}$$

$$\begin{aligned}
 &\text{and } L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) - L_\eta^2(x, \bar{\nu}, \bar{\mu}) \\
 &= ((\bar{\nu}_1 - \bar{\mu}) \arcsin \bar{x} + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin \bar{x})^2, (\bar{\nu}_1 - \bar{\mu}) \arcsin \bar{x} \\
 &\quad + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin \bar{x})^2) - ((\bar{\nu}_1 - \bar{\mu}) \arcsin x + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin x)^2, \\
 &\quad (\bar{\nu}_1 - \bar{\mu}) \arcsin x + (\bar{\nu}_1 + \bar{\nu}_2 + \bar{\mu})(\arcsin x)^2) \\
 &= (0, 0) - ((2\bar{\nu}_1 + \bar{\nu}_2)(\arcsin x)^2, (2\bar{\nu}_1 + \bar{\nu}_2)(\arcsin x)^2) \\
 &= -((2\bar{\nu}_1 + \bar{\nu}_2)(\arcsin x)^2, (2\bar{\nu}_1 + \bar{\nu}_2)(\arcsin x)^2) \\
 &\leq (0, 0), \forall x \in D(\bar{x}).
 \end{aligned}$$

Hence, by Theorem 4.2, $\bar{x} = 0$ is a weak efficient solution of the given vector optimization problem (VOP), what it can be easily verified.

Now, under stronger hypotheses, we prove the equivalence between a second order η -saddle point $(\bar{x}, \bar{\nu}, \bar{\mu})$ in the second order η -approximated vector optimization problem and an efficient solution \bar{x} of the considered vector optimization problem (VOP).

Theorem 4.4. Let \bar{x} be a feasible solution of the vector optimization problem (VOP). Assume that the objective function f is strictly second order invex and constraint function g is second order invex at \bar{x} on D with respect to the same function η , satisfying the condition $\eta(\bar{x}, \bar{x}) = 0$. If $(\bar{x}, \bar{\nu}, \bar{\mu}) \in D(\bar{x}) \times R_+^k \times R_+^m$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(VOP)_\eta^2(\bar{x})$ and $\bar{\nu} \neq 0$, then \bar{x} is an efficient solution of the considered vector optimization problem (VOP).

Proof. The proof follows the same lines as in Theorem 4.2. □

Theorem 4.5. Let \bar{x} be an efficient solution of the vector optimization problem (VOP) and the second order Abadie constraint qualification (ACQ) be satisfied at \bar{x} . If the bifunction η satisfies the condition $\eta(\bar{x}, \bar{x}) = 0$, then there exist $\bar{\nu} \in R_+^k, \bar{\mu} \in R_+^m$ such that $(\bar{x}, \bar{\nu}, \bar{\mu})$ is a second order η -saddle point in the second order η -approximated vector optimization problem $(VOP)_\eta^2(\bar{x})$.

Proof. Since \bar{x} is an efficient solution of the vector optimization problem (VOP), therefore it is feasible in vector optimization problem (VOP), i. e., $\bar{x} \in D$. This implies that

$$g(\bar{x}) \leq 0.$$

Since $\mu \in R_+^m$, therefore, from the above inequality, we get

$$\mu^T g(\bar{x}) \leq 0. \tag{30}$$

Furthermore, from Theorem 2.6, conditions (2)–(7) for (VOP) will be satisfied at \bar{x} . Thus, from condition (2), we can write

$$(\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) = 0, \forall x \in D(\bar{x}). \tag{31}$$

By hypothesis $\eta(\bar{x}, \bar{x}) = 0$, (31) can be rewritten as

$$(\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(x, \bar{x}) = (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}). \tag{32}$$

From condition (3), we have

$$0 \leq (\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x}) \eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x})).$$

Again, using the condition $\eta(\bar{x}, \bar{x}) = 0$, the above inequality implies

$$\begin{aligned} & \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \\ & \leq \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x}) \eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x})) \\ \Rightarrow & (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) \\ & + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \\ & \leq (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) + \\ & \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x}) \eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x})). \end{aligned}$$

Using (32) in the above inequality, we get

$$\begin{aligned} & (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) \\ & + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) \\ & \leq (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(x, \bar{x}) + \\ & \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x}) \eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x})). \end{aligned}$$

Thus, it follows that

$$\begin{aligned} & \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) e \\ & + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) e \\ & \leq \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x}) e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(x, \bar{x}) e \\ & + \frac{1}{2}(\bar{\nu}^T \eta(x, \bar{x})^T \nabla^2 f(\bar{x}) \eta(x, \bar{x}) + \bar{\mu}^T \eta(x, \bar{x})^T \nabla^2 g(\bar{x}) \eta(x, \bar{x})) e. \end{aligned}$$

By the definition of second order η -Lagrange function, it follows that

$$L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) \leq L_\eta^2(x, \bar{\nu}, \bar{\mu}), \quad \forall x \in D(\bar{x}). \tag{33}$$

From the conditions (2) and (4), we have

$$\bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) = 0.$$

Again, using the hypothesis $\eta(\bar{x}, \bar{x}) = 0$ in the above inequality, we get

$$\begin{aligned} & \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x})) \eta(\bar{x}, \bar{x}) \\ & + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x}) \eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x}) \eta(\bar{x}, \bar{x})) = 0. \end{aligned} \tag{34}$$

From (30) and the hypothesis $\eta(\bar{x}, \bar{x}) = 0$, we have

$$\begin{aligned} &\mu^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\ &\quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \mu^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \leq 0. \end{aligned} \tag{35}$$

On combining inequalities (34) and (35), we obtain

$$\begin{aligned} &\mu^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\ &\quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \mu^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \\ &\leq \bar{\mu}^T g(\bar{x}) + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x}) \\ &\quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x})) \\ \Rightarrow &\text{diag } \bar{\nu} f(\bar{x}) + \mu^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ &\quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \mu^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e \\ &\leq \text{diag } \bar{\nu} f(\bar{x}) + \bar{\mu}^T g(\bar{x})e + (\bar{\nu}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}))\eta(\bar{x}, \bar{x})e \\ &\quad + \frac{1}{2}(\bar{\nu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 f(\bar{x})\eta(\bar{x}, \bar{x}) + \bar{\mu}^T \eta(\bar{x}, \bar{x})^T \nabla^2 g(\bar{x})\eta(\bar{x}, \bar{x}))e. \end{aligned}$$

By the definition of second order η -Lagrange function, it follows that

$$L_\eta^2(\bar{x}, \bar{\nu}, \mu) \leq L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}), \quad \forall \mu \in R_+^m. \tag{36}$$

From (33) and (36), we get the required result. This completes the proof. □

Now, we give an example of a nonlinear vector optimization problem with twice differentiable functions to illustrate the result established in Theorem 4.5.

Example 4.6. Let us consider the vector optimization problem:

$$\begin{aligned} \text{(VOP)} \quad &V\text{-min } f(x) = (x^4 + x^2 + x, -x^2 - x) \\ \text{subject to} \quad &g(x) = x^2 - x \leq 0, \\ &x \in R, \end{aligned}$$

where $f : R \mapsto R^2$, $g : R \mapsto R$. The set of all feasible solutions of (VOP) is given by $D = \{x \in R : 0 \leq x \leq 1\}$. Clearly, $\bar{x} = 0$ is an efficient solution of given vector optimization problem (VOP). Let $\eta : R \times R \mapsto R$ be defined as

$$\eta(x, \bar{x}) = \frac{1}{2}(x - \bar{x}).$$

Obviously, $\eta(\bar{x}, \bar{x}) = 0$. Now, the associated second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$ is constructed as follows:

$$\begin{aligned} \text{(VOP)}_\eta^2(\bar{x}) \quad &V\text{-min } F(x) = \left(\frac{1}{2}x + \frac{1}{4}x^2, -\frac{1}{2}x - \frac{1}{4}x^2\right) \\ \text{subject to} \quad &G(x) = -\frac{1}{2}x + \frac{1}{4}x^2 \leq 0. \end{aligned}$$

The set of all feasible solutions of $(\text{VOP})_\eta^2(\bar{x})$ is $D(\bar{x}) = \{x \in R : 0 \leq x \leq 2\}$ and the second order η -approximated Lagrangian $L_\eta^2 : D(\bar{x}) \times R_+^2 \times R_+ \mapsto R$ in the problem $(\text{VOP})_\eta^2(\bar{x})$ is given by

$$L_\eta^2(x, \nu, \mu) = \left(\frac{1}{2}(\nu_1 - \nu_2 - \mu)x + \frac{1}{4}(\nu_1 - \nu_2 + \mu)x^2, \frac{1}{2}(\nu_1 - \nu_2 - \mu)x + \frac{1}{4}(\nu_1 - \nu_2 + \mu)x^2\right).$$

Therefore, $(\bar{x}, \bar{\nu}, \bar{\mu}) = (0, (\bar{\nu}_1, \bar{\nu}_2), \bar{\mu})$, where $\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu} = 0$, is a second order η -saddle point in the second order η -approximated vector optimization problem $(\text{VOP})_\eta^2(\bar{x})$, since

$$\begin{aligned} &L_\eta^2(\bar{x}, \bar{\nu}, \mu) - L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) \\ &= \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \mu)\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \mu)\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \mu)\bar{x}^2\right) \\ &\quad - \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})\bar{x}^2\right) \\ &= (0, 0) - (0, 0) = (0, 0), \quad \forall \mu \in R_+, \end{aligned}$$

and $L_\eta^2(\bar{x}, \bar{\nu}, \bar{\mu}) - L_\eta^2(x, \bar{\nu}, \bar{\mu})$

$$\begin{aligned} &= \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})\bar{x}^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})\bar{x} + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})\bar{x}^2\right) \\ &\quad - \left(\frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})x^2, \frac{1}{2}(\bar{\nu}_1 - \bar{\nu}_2 - \bar{\mu})x + \frac{1}{4}(\bar{\nu}_1 - \bar{\nu}_2 + \bar{\mu})x^2\right) \\ &= (0, 0) - \frac{1}{2}(\bar{\mu}x^2, \bar{\mu}x^2) \\ &= -\frac{1}{2}(\bar{\mu}x^2, \bar{\mu}x^2) \leq (0, 0), \quad \forall x \in D(\bar{x}). \end{aligned}$$

5. CONCLUSION

In this paper, a new characterization of second order η -saddle point criteria has been established for nonlinear vector optimization problems with twice differentiable functions. Namely, the equivalence between an (weak) efficient solution in the original vector optimization problem and a second order η -saddle point of the η -Lagrange function in its associated second order η -approximated vector optimization problem has been proved. Then, under second order η -invexity hypotheses, a new characterization of (weak) efficient solutions in vector optimization has been presented. Furthermore, we have also given examples to show that, under suitable assumptions, the second order η -approximation approach is very useful to determine the (weak) efficient solutions of a nonlinear vector optimization problem.

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