Michał Boczek; Marek Kaluszka

On the Minkowski-Hölder type inequalities for generalized Sugeno integrals with an application

Kybernetika, Vol. 52 (2016), No. 3, 329–347

Persistent URL: http://dml.cz/dmlcz/145778

KYBERNETIKA — VOLUME 52 (2016), NUMBER 3, PAGES 329–347

ON THE MINKOWSKI-HÖLDER TYPE INEQUALITIES FOR GENERALIZED SUGENO INTEGRALS WITH AN APPLICATION

Michał Boczek and Marek Kaluszka

In this paper, we use a new method to obtain the necessary and sufficient condition guaranteeing the validity of the Minkowski-Hölder type inequality for the generalized upper Sugeno integral in the case of functions belonging to a wider class than the comonotone functions. As a by-product, we show that the Minkowski type inequality for seminormed fuzzy integral presented by Daraby and Ghadimi [11] is not true. Next, we study the Minkowski-Hölder inequality for the lower Sugeno integral and the class of μ-subadditive functions introduced in [20]. The results are applied to derive new metrics on the space of measurable functions in the setting of nonadditive measure theory. We also give a partial answer to the open problem 2.22 posed in [5].

Keywords: seminormed fuzzy integral, semicopula, monotone measure, Minkowski's inequality, Hölder's inequality, convergence in mean

Classification: 26E50, 28E10

1. INTRODUCTION

The concepts of fuzzy measures and the Sugeno integral were introduced by Sugeno in [34] as a tool for modelling nondeterministic problems. The study of inequalities for the Sugeno integral was initiated by Román-Flores et al. [30]. Since then, the fuzzy integral counterparts of several classical inequalities, including Chebyshev's, Minkowski's and Hölder's inequalities have been given by Agahi et al. [1], Klement et al. [23], Ouyang et al. [27, 28], Wu et al. [37] and many other researchers. Most of them deal with comonotone functions which highly limit the range of potential applications in probability, statistics, decision theory, risk theory and others.

Since many classical inequalities are free of the comonotonicity assumption, Agahi and Mesiar [2] asked whether one could omit it. They gave a version of the Cauchy-Schwarz inequality without the comonotonicity condition for two classes of Choquet-like integrals. In [20] the Chebyshev type inequalities were provided for positively dependent functions which form a wider class than the comonotone functions. The aim of this paper is to present another inequalities for nonadditive integrals without the comonotonicity condition.

DOI: 10.14736/kyb-2016-3-0329

The paper is organized as follows. In Section 2, we introduce a new concept, called $\star$-associativity, which extends the notion of comonotonicity. Next, we obtain the necessary and sufficient conditions ensuring that the Minkowski-Hölder type inequality holds for the generalized upper Sugeno integral and $\star$-associative functions. We give a counterexample showing that the Minkowski type inequality for seminormed fuzzy integral presented in [11, Theorem 3.1] is false. The sufficient conditions for subadditivity of some functionals based on the upper Sugeno integral are also provided. Section 3 presents the Minkowski-Hölder type inequality for the generalized lower Sugeno integral and μ-subadditive functions. The necessary and sufficient condition for subadditivity of the Sugeno integral with respect to a subadditive measure is given. Finally, in Section 4 we propose new metrics on the space of measurable functions when the involved measure is monotone. We also give a partial answer to the open problem posed by Borzová-Molnárová et al. [5].

2. INEQUALITIES FOR GENERALIZED UPPER SUGENO INTEGRAL

First, we introduce some basic definitions and properties. Let $(X, \mathcal{A})$ be a measurable space, where $\mathcal{A}$ is a σ-algebra of subsets of a nonempty set X. A *monotone measure* on $\mathcal{A}$ is a nondecreasing set function $\mu\colon \mathcal{A} \to [0,\infty]$ with $\mu(\emptyset) = 0$. We say that μ is *finite* if $\mu(X) < \infty$. A monotone measure μ is *continuous from below* if $\lim_{n\to\infty} \mu(A_n) = \mu\big(\lim_{n\to\infty} A_n\big)$ for all $A_n \in \mathcal{A}$ such that $A_n \subset A_{n+1}$, $n \in \mathbb{N}$.

Let $Y = [0, m)$ or $Y = [0, m]$, where $0 < m \leqslant \infty$; usually $Y = [0, 1]$, $Y = [0,\infty)$ or $Y = [0,\infty]$. The operator $\circ\colon Y^2 \to Y$ is said to be *nondecreasing* if $a \circ b \leqslant x \circ y$ for $a \leqslant x$, $b \leqslant y$. We say that $\circ\colon Y^2 \to Y$ is *right-continuous* if $\lim_{n\to\infty}(a_n \circ b_n) = a \circ b$ for all $a_n, b_n, a, b \in Y$ such that $b_n \searrow b$ and $a_n \searrow a$. Hereafter, $c_n \searrow c$ means that $\lim_{n\to\infty} c_n = c$ and $c_n > c_{n+1}$ for all n.

Recall that $f, g\colon X \to Y$ are *comonotone* on D if $\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geqslant 0$ for all $x, y \in D$. If f and g are comonotone on D, then for any $t \in Y$ either $(D \cap \{f \geqslant t\}) \subset (D \cap \{g \geqslant t\})$ or $(D \cap \{g \geqslant t\}) \subset (D \cap \{f \geqslant t\})$, where $\{f \geqslant t\} = \{x \in X\colon f(x) \geqslant t\}$.

Now we will generalize the concept of comonotonicity.

Definition 2.1. Given an operator $\star\colon Y^2 \to Y$, we say that $f, g\colon X \to Y$ are $\star$-*associated* on D if for any nonempty and measurable subset $A \subset D$,

$$\inf_{x\in A} \{f(x) \star g(x)\} = \inf_{x\in A} f(x) \star \inf_{x\in A} g(x). \tag{1}$$

From now on, $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $a_+ = a \vee 0$.

Example 2.2. Any functions $f, g\colon X \to Y$ are $\wedge$-associated on X.

Example 2.3. Any comonotone functions $f, g\colon X \to Y$ are $\star$-associated on X if the operator $\star$ is nondecreasing and right-continuous. Indeed, $\inf_{x\in A}\{f(x) \star g(x)\} \geqslant s \star t$ for all $A \subset X$, where $s = \inf_{x\in A} f(x)$ and $t = \inf_{x\in A} g(x)$. Let $\varepsilon > 0$, $A \subset X$ and $B = \{x \in A\colon f(x) < s+\varepsilon\}$ and $C = \{x \in A\colon g(x) < t+\varepsilon\}$. From the comonotonicity we obtain that $B \cap C \neq \emptyset$ as $B \subset C$ or $C \subset B$. Thus $\inf_{x\in A}\{f(x) \star g(x)\} \leqslant (s+\varepsilon) \star (t+\varepsilon)$. Because of the right-continuity of $\star$, we get the assertion.

Example 2.4. Let $f, g\colon X \to Y$ be measurable functions and $g = b\mathbb{1}_B$ for $b \in Y$, where $\mathbb{1}_B$ denotes the indicator of $B \subset X$, $B \cap \{f \geqslant b\} \neq B$ and $B \cap \{f \geqslant b\} \neq \{f \geqslant b\}$. Let $\star\colon Y^2 \to Y$ be a nondecreasing and right-continuous operator. If $x \star 0 = 0$ for all $x \in X$, then f, g are $\star$-associated on X, but not comonotone. Indeed, if $A \backslash B = \emptyset$, then $A \subset B$ and

$$\inf_{x\in A}\{f(x)\star g(x)\} = \inf_{x\in A}\{f(x)\star b\} = \inf_{x\in A} f(x)\star \inf_{x\in A} g(x).$$

If $A\backslash B \neq \emptyset$, then

$$\begin{aligned}\inf_{x\in A}\{f(x)\star g(x)\} &= \inf_{x\in A\cap B}\{f(x)\star b\}\wedge \inf_{x\in A\backslash B}\{f(x)\star 0\} = 0\\ &= \inf_{x\in A} f(x)\star \inf_{x\in A} g(x).\end{aligned}$$

Example 2.5. Suppose $\star$ is a nondecreasing operator such that $0 \star y = y \star 0 = 0$ for all $y \in Y$. Let $f = b\mathbb{1}_B + c\mathbb{1}_C$ and $g = b\mathbb{1}_B + c\mathbb{1}_D$, where $b, c \in Y$, $0 < b \wedge c$, and B, C are nonempty sets such that $B \cap C = \emptyset$ and $D = X\backslash(B \cup C) \neq \emptyset$. Clearly, f and g are $\star$-associated on X, but not comonotone.

Example 2.6. Functions f, g are $+$-associated if and only if they are comonotone. In fact, the condition (1) for $\star = +$ and $A = \{x, y\}$ is equivalent to $(a+b)\wedge 0 = (a\wedge 0)+(b\wedge 0)$ with $a = f(x) - f(y)$ and $b = g(x) - g(y)$, and this implies that $ab \geqslant 0$.

Open problem 1. Does there exist an operator $\circ \neq +$ such that the $\circ$-associativity property is equivalent to the comonotonicity property?

Now we are ready to present the Minkowski-Hölder type inequality for the *generalized upper Sugeno integral* of the form

$$\int_{\circ,D} f\,\mathrm{d}\mu := \sup_{t\in Y}\big\{t\circ\mu\big(D\cap\{f\geqslant t\}\big)\big\}, \tag{2}$$

where $f\colon X \to Y$ is a measurable function, μ is a monotone measure on $\mathcal{A}$ and $\circ\colon Y \times \mu(\mathcal{A}) \to [0,\infty]$ is a nondecreasing operator. Hereafter $\mu(\mathcal{A}) = \{\mu(A)\colon A \in \mathcal{A}\}$. The functional in (2) is the universal integral in the sense of Definition 2.5 in [23] if $\circ$ is the pseudomultiplication function (see [23, Definition 2.3]). Put $\mu(\mathcal{A}\cap D) = \{\mu(A\cap D)\colon A \in \mathcal{A}\}$. The following theorem gives an answer to open problems from [1, 20] and [28].

Theorem 2.7. Let $Y = [0, \bar{y}]$, where $0 < \bar{y} \leqslant \infty$. Assume the operators $\star, \Diamond\colon Y^2 \to Y$ and $\circ_i\colon Y \times \mu(\mathcal{A}) \to Y$ are nondecreasing and $0 \circ_i x = y \circ_i 0 = 0$ for all $x \in \mu(\mathcal{A})$, $y \in Y$, and $i = 1, 2, 3$. Suppose $\phi_i\colon Y \to Y$ are increasing and $\phi_i\big(Y\big) = Y$ for all i. Suppose also that f and g are $\star$-associated on $D \subset X$. Then the Minkowski-Hölder type inequality

$$\phi_1^{-1}\Big(\int_{\circ_1,D}\phi_1(f\star g)\,\mathrm{d}\mu\Big) \leqslant \phi_2^{-1}\Big(\int_{\circ_2,D}\phi_2(f)\,\mathrm{d}\mu\Big)\Diamond\,\phi_3^{-1}\Big(\int_{\circ_3,D}\phi_3(g)\,\mathrm{d}\mu\Big) \tag{3}$$

is satisfied if and only if for all $a, b \in Y$ and $c \in \mu(\mathcal{A}\cap D)$

$$\phi_1^{-1}\big(\phi_1(a\star b)\circ_1 c\big) \leqslant \phi_2^{-1}\big(\phi_2(a)\circ_2 c\big)\Diamond\,\phi_3^{-1}\big(\phi_3(b)\circ_3 c\big). \tag{4}$$

Proof. Arguing as in the proof of Lemma 3.8 in [33], we can show that

$$\int_{\circ_i,D} f\,\mathrm{d}\mu = \sup_{A\subset D, A\in\mathcal{A}} \left\{ \inf_{x\in A} f(x) \circ_i \mu(A) \right\} \tag{5}$$

for all i (see also [4, Theorem 2.2]). To shorten the notation, we write $\sup_A$ instead of $\sup_{A\subset D, A\in\mathcal{A}}$. From the continuity of ϕ_1 and (5) we get

$$L := \phi_1^{-1}\Big(\int_{\circ_1,D} \phi_1(f\star g)\,\mathrm{d}\mu \Big) = \sup_A \phi_1^{-1}\Big(\phi_1\big(\inf_{x\in A} \{f(x)\star g(x)\}\big) \circ_1 \mu(A) \Big).$$

Since f and g are $\star$-associated, we have

$$L = \sup_A \phi_1^{-1}\Big(\phi_1\big(\inf_{x\in A} f(x) \star \inf_{x\in A} g(x)\big) \circ_1 \mu(A) \Big).$$

Combining (4) with the monotonicity of $\Diamond$ and ϕ_i^{-1} yields

$$\begin{aligned} L &\leqslant \sup_A \Big\{ \phi_2^{-1}\Big(\phi_2\big(\inf_{x\in A} f(x)\big) \circ_2 \mu(A)\Big) \Diamond \phi_3^{-1}\Big(\phi_3\big(\inf_{x\in A} g(x)\big) \circ_3 \mu(A)\Big) \Big\} \\ &\leqslant \Big(\sup_A \phi_2^{-1}\Big(\phi_2\big(\inf_{x\in A} f(x)\big) \circ_2 \mu(A)\Big)\Big) \Diamond \Big(\sup_A \phi_3^{-1}\Big(\phi_3\big(\inf_{x\in A} g(x)\big) \circ_3 \mu(A)\Big)\Big) \\ &= \phi_2^{-1}\Big(\int_{\circ_2,D} \phi_2(f)\,\mathrm{d}\mu \Big) \Diamond \phi_3^{-1}\Big(\int_{\circ_3,D} \phi_3(g)\,\mathrm{d}\mu \Big). \end{aligned}$$

To obtain the necessary condition (4), put $f = a\mathbb{1}_A$ and $g = b\mathbb{1}_A$ in (3), where $c = \mu(A) \leqslant \mu(D)$ and $a, b \in Y$. □

Observe that the assumption $0 \circ_i x = y \circ_i 0 = 0$ is used only in the proof of the necessity of condition (4). Moreover, the condition (4) is sufficient for inequality (3) to hold if we set $Y = \mathbb{R}$ in (2) and both f and g are bounded from below.

Example 2.8. Let $a \star b = a \Diamond b = a + b - ab$, where $a, b \in Y = [0,1]$ and let $\circ_i = \cdot$ for all i. Put $\phi_i(x) = x^{p_i}$ and $c_i = c^{1/p_i}$, where $p_i > 0$ for all i. The condition (4) takes the form

$$0 \leqslant a(c_2 - c_1) + b(c_3 - c_1) + ab(c_1 - c_2c_3) \tag{6}$$

and holds if and only if $p_1 \leqslant p_j$ for $j = 2, 3$; in order to see this, put $a = 1$, $b = 0$ as well as $a = 0$, $b = 1$ in (6) and observe that

$$a(c_2 - c_1) + b(c_3 - c_1) + ab(c_1 - c_2c_3) \geqslant ab\big(c_2 - c_1 + c_3(1 - c_2)\big) \geqslant 0.$$

Example 2.9. Let $\phi_i(x) = x$ for all i and let $\star$ and $\Diamond$ be the drastic t-norm and the drastic t-conorm, respectively, i.e. $a \star b = (a \wedge b)\mathbb{1}_{\{a\vee b=1\}}$ and $a \Diamond b = (a \vee b)\mathbb{1}_{\{a\wedge b=0\}} + \mathbb{1}_{\{a\wedge b>0\}}$ for $a, b \in Y = [0,1]$ (see [22]). Assume that $\circ_i = \circ$ for all i, where $\circ$ is a nondecreasing operator such that $0 \circ_i x = y \circ_i 0 = 0$ for all $x, y \in [0,1]$. Clearly, any comonotone functions $f, g\colon X \to [0,1]$ are $\star$-associated (see Example 2). To prove that the Minkowski-Hölder type inequality (3) holds for all comonotone functions, it is enough to show that for all $a, b, c \in [0,1]$

$$(a \star b) \circ c \leqslant (a \circ c) \lozenge (b \circ c). \tag{7}$$

If $a \circ c > 0$ and $b \circ c > 0$, then (7) holds. If $a \circ c = 0$ or $b \circ c = 0$, then

$$(a \star b) \circ c \leqslant (a \circ c) \vee (b \circ c) = (a \circ c) \lozenge (b \circ c)$$

as $a \star b \leqslant a$ and $a \star b \leqslant b$.

We recall that the *Sugeno integral* and the *Shilkret integral* are given by

$$(S) \int_D f \, \mathrm{d}\mu := \sup_{y \in Y} \left\{ y \wedge \mu\big(D \cap \{f \geqslant y\}\big) \right\}, \tag{8}$$

$$(N) \int_D f \, \mathrm{d}\mu := \sup_{y \in Y} \left\{ y \cdot \mu\big(D \cap \{f \geqslant y\}\big) \right\}, \tag{9}$$

respectively, where $Y = [0, m]$ or $Y = [0, m)$ with $0 < m \leqslant \infty$ and the convention that $0 \cdot \infty = \infty \cdot 0 = 0$, see [32, 34, 35].

Corollary 2.10. Assume $\star\colon Y^2 \to Y$ is nondecreasing, $f, g\colon X \to Y$ are $\star$-associated on D and $\phi_i\colon Y \to Y$ are increasing functions such that $\phi_i\big(Y\big) = Y$ for $i = 1, 2, 3$. The following Minkowski-Hölder type inequality

$$\phi_1^{-1}\Big((S) \int_D \phi_1(f \star g) \, \mathrm{d}\mu\Big) \leqslant \phi_2^{-1}\Big((S) \int_D \phi_2(f) \, \mathrm{d}\mu\Big) \star \phi_3^{-1}\Big((S) \int_D \phi_3(g) \, \mathrm{d}\mu\Big)$$

holds true if and only if for $a, b \in Y$ and $c \in \mu(\mathcal{A} \cap D)$

$$(a \star b) \wedge \phi_1^{-1}(c) \leqslant \big(a \wedge \phi_2^{-1}(c)\big) \star \big(b \wedge \phi_3^{-1}(c)\big). \tag{10}$$

The above result generalizes Theorem 3.1 from [1] and Theorem 3.1 from [37]. In fact, since $a \vee b \leqslant a \star b$, we have $c \leqslant a \vee c \leqslant a \star c$, $c \leqslant c \star b$ and $c \leqslant c \star c$, so

$$(a \star b) \wedge c \leqslant (a \star b) \wedge (a \star c) \wedge (c \star b) \wedge (c \star c) = (a \wedge c) \star (b \wedge c).$$

It follows from the assumption $\phi_1 \geqslant \phi_j$ for $j = 2, 3$ that

$$(a \star b) \wedge \phi_1^{-1}(c) \leqslant \big(a \wedge \phi_1^{-1}(c)\big) \star \big(b \wedge \phi_1^{-1}(c)\big) \leqslant \big(a \wedge \phi_2^{-1}(c)\big) \star \big(b \wedge \phi_3^{-1}(c)\big).$$

Thus, the condition (10) holds.

Suppose that $\mathrm{S}\colon [0,1]^2 \to [0,1]$ is a *semicopula* (also called a *t-seminorm*), i. e., a nondecreasing function with the neutral element equal to 1. It is clear that $\mathrm{S}(x, y) \leqslant x \wedge y$ and $\mathrm{S}(x, 0) = 0 = \mathrm{S}(0, x)$ for all $x, y \in [0, 1]$ (see [3, 13, 22]). We denote the class of all semicopulas by $\mathfrak{S}$. There are three important examples of semicopulas: M, Π and W, where $\mathrm{M}(a, b) = a \wedge b$, $\Pi(a, b) = ab$ and $\mathrm{W}(a, b) = (a + b - 1)_+$, usually called the *Łukasiewicz t-norm* [22].

Given $\mathrm{S} \in \mathfrak{S}$, the *seminormed fuzzy integral* is defined by

$$\int_{\mathrm{S},D} f \,\mathrm{d}\mu := \sup_{t\in[0,1]} \mathrm{S}\big(t, \mu(D \cap \{f \geqslant t\})\big),$$

see [27, 33]. Replacing semicopula S with M, we get the Sugeno integral (8) for $Y = [0, 1]$. Moreover, if $\mathrm{S} = \Pi$, then we get the *Shilkret integral* (9) for $Y = [0, 1]$.

Corollary 2.11. Let $\mathrm{S} \in \mathfrak{S}$ and $f, g\colon X \to [0, 1]$ be $\star$-associated, where $\star\colon [0, 1]^2 \to [0, 1]$ is a nondecreasing operator. Let $0 < p < \infty$ and $\mu(X) = 1$. The following inequality holds

$$\Big(\int_{\mathrm{S},D} (f \star g)^p \,\mathrm{d}\mu\Big)^{1/p} \leqslant \Big(\int_{\mathrm{S},D} f^p \,\mathrm{d}\mu\Big)^{1/p} \star \Big(\int_{\mathrm{S},D} g^p \,\mathrm{d}\mu\Big)^{1/p} \tag{11}$$

if and only if

$$\mathrm{S}\big((a \star b)^p, c\big)^{1/p} \leqslant \mathrm{S}\big(a^p, c\big)^{1/p} \star \mathrm{S}\big(b^p, c\big)^{1/p}$$

for $a, b \in [0, 1]$ and $c \in \mu(\mathcal{A} \cap D)$.

Daraby and Ghadimi [11] claim that the inequality (11) is satisfied for all comonotone functions if

$$\mathrm{S}(a \star b, c) \leqslant \big(\mathrm{S}(a, c) \star b\big) \wedge \big(a \star \mathrm{S}(b, c)\big), \quad a, b, c \in [0, 1], \tag{12}$$

under the assumption of continuity of monotone measure μ (see [11, Theorem 3.1]). We present a counterexample showing that this result is not true.

Counterexample 2.12. Put $A = X = [0, 1]$, $s = 1$, $\mathrm{T} = \mathrm{W}$, $a \star b = (a + b) \wedge 1$ and $f(x) = g(x) = 0.5\sqrt{x}$, $x \in [0, 1]$, in Theorem 3.1 from [11]. Clearly, f and g are comonotone. Let μ be the Lebesgue measure. Due to the property $a \star b = b \star a$, the condition (12) is satisfied if and only if

$$\mathrm{W}\big((a + b) \wedge 1, c\big) \leqslant \big(\mathrm{W}(a, c) + b\big) \wedge 1$$

for all $a, b, c \in [0, 1]$. Since $\mathrm{W} \leqslant 1$, it suffices to show that $\mathrm{W}\big((a+b)\wedge 1, c\big) \leqslant \mathrm{W}(a, c)+b$. In fact, if $a + b \leqslant 1$, then $\mathrm{W}(a + b, c) \leqslant \mathrm{W}(a, c) + b$ (see also [22, Remark 5.13 (iii)]). Otherwise,

$$c \leqslant (a + c - 1)_+ + (1 - a)_+ = (a + c - 1)_+ - (a + b - 1) + b \leqslant \mathrm{W}(a, c) + b.$$

Easy computations show that

$$\int_{\mathrm{W},X} f \,\mathrm{d}\mu = \sup_{t\in[0,1]} \big(t + \mu(\{f \geqslant t\}) - 1\big)_+ = \sup_{t\in[0,1]} \big(t - 4t^2\big)_+ = 0.0625,$$

$$\int_{\mathrm{W},X} (f \star g) \,\mathrm{d}\mu = \sup_{t\in[0,1]} \big(t + \mu(\{f \star g \geqslant t\}) - 1\big)_+ = \sup_{t\in[0,1]} \big(t - t^2\big)_+ = 0.25.$$

Hence, $0.25 = \int_{\mathrm{W},X} (f \star g) \,\mathrm{d}\mu > \int_{\mathrm{W},X} f \,\mathrm{d}\mu \star \int_{\mathrm{W},X} g \,\mathrm{d}\mu = 0.125$.

Now we focus on the subadditivity property of the generalized upper Sugeno integral (2), that is,

$$\int_{\circ,X}(f+g)\,\mathrm{d}\mu \leqslant \int_{\circ,X} f\,\mathrm{d}\mu + \int_{\circ,X} g\,\mathrm{d}\mu, \tag{13}$$

as this property is very important for applications. Let us recall that +-associativity is equivalent to comonotonicity, see Example 2.6.

Corollary 2.13. Let $Y=[0,m]$ or $Y=[0,m)$ for $0<m\leqslant\infty$ and let $\circ\colon Y^2\to Y$ be a nondecreasing operator such that $0\circ y = y\circ 0 = 0$ for all y. The functional (2) is subadditive for comonotone functions $f,g\colon X\to Y$ such that $f+g\in Y$ if and only if $(a+b)\circ c\leqslant (a\circ c)+(b\circ c)$ for $a,b\in Y$, $a+b\in Y$ and $c\in\mu(\mathcal{A})=Y$.

It follows from Corollary 2.13 that both the Sugeno integral (8) and the Shilkret integral (9) are subadditive for comonotone functions while the opposite-Sugeno integral $\int_{\mathrm{W},D} f\,\mathrm{d}\mu$ [18] is not.

Corollary 2.14. Let $\circ = \mathrm{S}\in\mathfrak{S}$. Then the subadditivity property (13) is fulfilled for any monotone measure μ such that $\mu(X)\leqslant 1$ and comonotone functions $f,g\colon X\to[0,1]$ such that $f+g\in[0,1]$ if and only if

$$\mathrm{S}\big(a+b,c\big)\leqslant \mathrm{S}(a,c)+\mathrm{S}(b,c) \tag{14}$$

for all $a,b,c\in[0,1]$, $a+b\in[0,1]$.

Borzová-Molnárová et al. [4] showed that the inequality (14) is satisfied for each semicopula with concave horizontal sections $x\mapsto \mathrm{S}(x,y)$. An example is the Marshall-Olkin semicopula $\mathrm{S}_{\alpha,\beta}(x,y)=(x^{1-\alpha}y)\wedge(xy^{1-\beta})$, where $\alpha,\beta\in[0,1]$. Observe that if $f=\mathbb{1}_A$ and $g=\mathbb{1}_B$ for $A\cup B=X$ and $A\cap B=\emptyset$, then the inequality (13) is of the form $\mu(X)\leqslant\mu(A)+\mu(B)$ for any semicopula S. Thus, the seminormed fuzzy integral is not subadditive if $\mu(A)+\mu(B)<\mu(X)$.

We say that $\mu\colon\mathcal{A}\to Y$ is *subadditive* if it is a monotone measure and $\mu(A\cup B)\leqslant \mu(A)+\mu(B)$ for all $A,B\in\mathcal{A}$. The class of subaditive measures is quite wide and includes the following monotone measures: λ-measure of Sugeno for $\lambda\in\big(-1/\mu(X),0\big)$ (see [35, Definition 4.3]); the *plausibility measure* [35]; the *coherent measure* $\mu(A)=\sup_{\mathsf{P}\in\mathcal{P}}\mathsf{P}(A)$, where $\mathcal{P}$ is a set of probability measures [15]; the *possibility measure* $\mu(A)=\sup_{x\in A}\psi(x)$, where $\psi\colon X\to Y$ [35], the *distortion measure* $\mu(A)=g\big(\mathsf{P}(A)\big)$, where P is probability measure and $g\colon[0,1]\to Y$ is such that $g(x+y)\leqslant g(x)+g(y)$ [31] and *uncertain measure* [25], among others.

Theorem 2.15. Suppose $Y=[0,m]$ or $Y=[0,m)$ with $0<m\leqslant\infty$ and suppose $\circ\colon Y^2\to Y$ is a nondecreasing operator such that $x\circ(y+z)\leqslant(x\circ y)+(x\circ z)$ for all $x,y,z\in Y$ such that $y+z\in Y$. Suppose also that $(ax)\circ y\leqslant a^q(x\circ y)^r$ for some $q,r>0$ and for all $x,y,z\in Y, a>1$ such that $ax\in Y$. Then for any $p>0$, any subadditive measure μ and any measurable functions $f,g\colon X\to\mathbb{R}$ such that $|f+g|^p$, $|f|^p$, $|g|^p\in Y$, we have

$$\Big(\int_{\circ,X}|f+g|^p\,\mathrm{d}\mu\Big)^{1/(pq+1)}\leqslant\Big(\int_{\circ,X}|f|^p\,\mathrm{d}\mu\Big)^{r/(pq+1)}+\Big(\int_{\circ,X}|g|^p\,\mathrm{d}\mu\Big)^{r/(pq+1)}. \tag{15}$$

P r o o f. Without loss of generality, assume that $\int_{\circ,X} |f|^p \,\mathrm{d}\mu + \int_{\circ,X} |g|^p \,\mathrm{d}\mu < \infty$. Clearly, $\{|f+g| \geqslant t^{1/p}\} \subset \{|f| \geqslant \lambda t^{1/p}\} \cup \{|g| \geqslant (1-\lambda)t^{1/p}\}$ for $t \in Y$ and $\lambda \in (0,1)$. Thus, by the subadditivity of μ and monotonicity of $\circ$, we have

$$t \circ \mu\big(\{|f+g|^p \geqslant t\}\big) \leqslant t \circ \Big(\mu\big(\{|f|^p \geqslant \lambda^p t\}\big) + \mu\big(\{|g|^p \geqslant (1-\lambda)^p t\}\big)\Big).$$

From the assumptions on $\circ$, we get

$$\begin{aligned}\int_{\circ,X} |f+g|^p \,\mathrm{d}\mu &\leqslant \sup_{t\in Y}\big\{t \circ \mu\big(\{|f|^p \geqslant \lambda^p t\}\big)\big\} + \sup_{t\in Y}\big\{t \circ \mu\big(\{|g|^p \geqslant (1-\lambda)^p t\}\big)\big\} \\ &\leqslant \sup_{y\in\lambda^p Y}\big\{\tfrac{y}{\lambda^p} \circ \mu\big(\{|f|^p \geqslant y\}\big)\big\} + \sup_{y\in(1-\lambda)^p Y}\Big\{\tfrac{y}{(1-\lambda)^p} \circ \mu\big(\{|g|^p \geqslant y\}\big)\Big\} \\ &\leqslant \lambda^{-pq}\Big(\int_{\circ,X} |f|^p \,\mathrm{d}\mu\Big)^r + (1-\lambda)^{-pq}\Big(\int_{\circ,X} |g|^p \,\mathrm{d}\mu\Big)^r,\end{aligned}$$

where $\lambda^p Y = \{\lambda^p y \colon y \in Y\} \subset Y$. If $\int_{\circ,X} |f|^p \,\mathrm{d}\mu = 0$ or $\int_{\circ,X} |g|^p \,\mathrm{d}\mu = 0$, we take the limit as λ approaches 0 or 1, respectively. Otherwise, we obtain (15) by minimizing the right-hand side with respect to λ. □

Corollary 2.16. Let $Y = [0,1]$, $Y = [0,\infty)$ or $Y = [0,\infty]$. If μ is subadditive, then for all measurable functions $f, g \colon X \to \mathbb{R}$ and $p > 0$ we have

$$\Big((S)\int_X |f+g|^p \,\mathrm{d}\mu\Big)^{1/(p+1)} \leqslant \Big((S)\int_X |f|^p \,\mathrm{d}\mu\Big)^{1/(p+1)} + \Big((S)\int_X |g|^p \,\mathrm{d}\mu\Big)^{1/(p+1)},$$
$$\Big((N)\int_X |f+g|^p \,\mathrm{d}\mu\Big)^{1/(p+1)} \leqslant \Big((N)\int_X |f|^p \,\mathrm{d}\mu\Big)^{1/(p+1)} + \Big((N)\int_X |g|^p \,\mathrm{d}\mu\Big)^{1/(p+1)},$$

where $|f+g|^p$, $|f|^p$, $|g|^p \in Y$ and the integrals are defined, respectively, by (8) and (9).

The next result deals with a modified Shilkret integral and follows from Theorem 2.15 and the inequality $(x+y)^s \leqslant x^s + y^s$ for $x, y \geqslant 0$ and $0 < s < 1$.

Corollary 2.17. Let $a \circ_q b = (ab)^q$ with $0 < q < 1$ and let $Y = [0,1]$ or $Y = [0,\infty)$. For any subadditive measure μ and any measurable functions $f, g \colon X \to \mathbb{R}$, we get

$$\Big(\int_{\circ_q,X} |f+g|^p \,\mathrm{d}\mu\Big)^{1/p} \leqslant \Big(\int_{\circ_q,X} |f|^p \,\mathrm{d}\mu\Big)^{1/p} + \Big(\int_{\circ_q,X} |g|^p \,\mathrm{d}\mu\Big)^{1/p},$$

where $p = 1/(1-q)$, and $|f+g|^p$, $|f|^p$, $|g|^p \in Y$.

Simple calculations show that

$$\Big(\int_{\circ_q,X} |f|^p \,\mathrm{d}\mu\Big)^{1/p} = \sup_{t\in Y}\Big\{t^q \mu\big(\{|f| \geqslant t\}\big)^{q/p}\Big\},$$

so this functional is similar to a quasi-norm in the Lorentz type capacity spaces [9].

Now, we analyze the subadditivity of the Shilkret integral. Recall that a monotone measure μ is *maxitive* if for all disjoint sets $A, B \in \mathcal{A}$ we have

$$\mu(A \cup B) = \mu(A) \vee \mu(B). \tag{16}$$

Observe that μ is maxitive if and only if (16) holds for all $A, B \in \mathcal{A}$. In fact, if μ is maxitive and $A \cap B \neq \emptyset$, then $\mu(A \cup B) = \mu(A) \vee \mu(C)$ and $\mu(A \cup B) = \mu(D) \vee \mu(B)$, where $C = B \backslash A$ and $D = A \backslash B$. This implies that $\mu(A \cup B) = \mu(A) \vee \mu(B) \vee \mu(C) \vee \mu(D) = \mu(A) \vee \mu(B)$, so (16) is satisfied. Clearly, any maxitive measure is subadditive.

The following result can be found in [8] (see also [32, pp. 112 − 113] and [9, Theorem 4.2]).

Theorem 2.18. Let $Y = [0, 1]$, $Y = [0, \infty)$ or $Y = [0, \infty]$. The Shilkret inegral (9) is subadditive for all measurable functions $f, g \colon X \to Y$ if and only if the monotone measure μ is maxitive.

P r o o f. Denote the Shilkret integral for short by $\mathbf{I}(f)$.

„$\Leftarrow$" We follow the proof of [8, 32]. If $\mathbf{I}(f) = \mathbf{I}(g) = 0$, then $\mathbf{I}(f + g) = 0$ as $\mu\big(\{f + g \geqslant t\}\big) \leqslant \mu\big(\{f \geqslant t/2\}\big) + \mu\big(\{g \geqslant t/2\}\big) = 0$ for all $t > 0$. Therefore, we assume that $0 < \mathbf{I}(f) + \mathbf{I}(g) < \infty$, without loss of generality. By maxitivity of μ, we have

$$\begin{aligned} t\mu(\{f + g \geqslant t\}) &\leqslant t\mu\big(\{f \geqslant \lambda t\} \cup \{g \geqslant (1 - \lambda)t\}\big) \\ &= t\mu\big(\{f \geqslant \lambda t\}\big) \vee t\mu\big(\{g \geqslant (1 - \lambda)t\}\big) \end{aligned}$$

with $\lambda = \mathbf{I}(f)/(\mathbf{I}(f) + \mathbf{I}(g))$. Hence,

$$\mathbf{I}(f + g) \leqslant \big((\mathbf{I}(f)/\lambda) \vee \big(\mathbf{I}(g)/(1 - \lambda)\big) = \mathbf{I}(f) + \mathbf{I}(g).$$

„$\Rightarrow$" Suppose μ is not maxitive, i. e. $\mu(A \cup B) > \mu(A) \vee \mu(B)$ for some disjoint sets $A, B \in \mathcal{A}$. Thus, there exists $\lambda \in (0, 1)$ such that $\lambda\mu(A \cup B) > \mu(A) \vee \mu(B)$. Putting $f = \mathbb{1}_A + \lambda \mathbb{1}_B$, $g = (1 - \lambda)\mathbb{1}_B$, we get

$$\begin{aligned} \mathbf{I}(f) + \mathbf{I}(g) &= \big((\lambda\mu(A \cup B)) \vee \mu(A)\big) + (1 - \lambda)\mu(B) \\ &< \lambda\mu(A \cup B) + (1 - \lambda)\mu(A \cup B) = \mathbf{I}(f + g), \end{aligned}$$

so the Shilkret integral is not subadditive. □

Subadditivity of the Sugeno integral will be examined in the next section.

3. RESULTS FOR GENERALIZED LOWER SUGENO INTEGRAL

The *generalized lower Sugeno integral* of a measurable function $f \colon X \to Y$ on a set $D \in \mathcal{A}$ with respect to a monotone measure μ and a nondecreasing operator $\circ \colon Y \times \mu(\mathcal{A}) \to [0, \infty]$ is defined as

$$\underline{\int}_{\circ, D} f \, \mathrm{d}\mu := \inf_{t \in Y} \big\{ t \circ \mu\big(D \cap \{f > t\}\big) \big\}. \tag{17}$$

Observe that the functional (17) is the universal integral in the sense of Definition 2.5 in [23] if $a \circ 0 = a$ and $0 \circ b = b$ for all $a \in Y$ and $b \in \mu(\mathcal{A})$. Putting $\circ = \vee$ in (17) we obtain the *lower Sugeno integral* [26]

$$(S) \fint_D f \, \mathrm{d}\mu := \inf_{t \in Y} \left\{ t \vee \mu\big(D \cap \{f > t\}\big) \right\}. \tag{18}$$

Mimicking the proof of Theorem 5 in [21] and Theorem 9.1 in [35] one can show that for any $Y = [0, m] \subset [0, \infty]$ the integral (18) is equal to the Sugeno integral (8)

$$(S) \fint_D f \, \mathrm{d}\mu = (S) \int_D f \, \mathrm{d}\mu. \tag{19}$$

Open problem 2. Does there exist a pair of operators $(\triangledown, \triangle) \neq (\vee, \wedge)$ such that for all $f \colon X \to Y$

$$\fint_{\triangledown, D} f \, \mathrm{d}\mu = \int_{\triangle, D} f \, \mathrm{d}\mu \text{ ?}$$

We say that measurable functions $f, g \colon X \to Y$ are *μ-subadditive* for an operator $\triangledown \colon \mu(\mathcal{A})^2 \to \mu(\mathcal{A})$ and a set $D \in \mathcal{A}$ if for all $a, b \in Y$

$$\mu\Big(D \cap \big(\{f > a\} \cup \{g > b\}\big)\Big) \leqslant \mu\big(D \cap \{f > a\}\big) \triangledown \mu\big(D \cap \{g > b\}\big).$$

Observe that μ-subadditivity implies that $x \vee y \leqslant x \triangledown y$ for all $x, y \in \mu(\mathcal{A} \cap D)$.

Now, we present several examples of μ-subadditive functions.

Example 3.1. Any comonotone functions f, g are μ-subadditive with respect to an operator $\triangledown$ such that $x \vee y \leqslant x \triangledown y$ for all $x, y \in Y$. For instance, any t-semiconorm $\triangledown = \mathrm{S}^*$ on $Y = [0, 1]$ has this property (see [22]).

Example 3.2. Recall that μ is *submodular* if $\mu(A \cup B) \leqslant \mu(A) + \mu(B) - \mu(A \cap B)$ for all $A, B \in \mathcal{A}$. Let $D = X$, $x \triangledown y = 1 - (1 - x)(1 - y)$ for $x, y \in Y = [0, 1]$ and let μ be a submodular and monotone measure. Functions f, g are μ-subadditive if f, g are positive quadrant dependent [20], that is, $\mu\big(\{f > t\} \cap \{g > s\}\big) \geqslant \mu\big(\{f > t\}\big)\mu\big(\{g > s\}\big)$ for all $t, s \in Y$.

Example 3.3. Put $x \triangledown y = x + y$ for $x, y \in Y = [0, \infty]$. Then any functions f, g are μ-subadditive for a subadditive measure μ on X.

Suppose $\star, \Diamond \colon Y^2 \to Y$, and $\circ_i \colon Y \times \mu(\mathcal{A}) \to Y$, $i = 1, 2, 3$, are nondecreasing and $\Diamond$ is right-continuous. Suppose also that $\phi_i \colon Y \to Y$ is an increasing function and $\phi_i(Y) = Y$ for $i = 1, 2, 3$.

Theorem 3.4. Assume that for $a, b \in Y$ and $c, d \leqslant \mu(D)$, we have

$$\phi_1^{-1}\big(\phi_1(a \star b) \circ_1 (c \triangledown d)\big) \leqslant \phi_2^{-1}\big(\phi_2(a) \circ_2 c\big) \ \Diamond \ \phi_3^{-1}\big(\phi_3(b) \circ_3 d\big). \tag{20}$$

If f, g are μ-subadditive for $\triangledown$ and D, then

$$\phi_1^{-1}\bigg(\fint_{\circ_1, D} \phi_1(f \star g) \, \mathrm{d}\mu\bigg) \leqslant \phi_2^{-1}\bigg(\fint_{\circ_2, D} \phi_2(f) \, \mathrm{d}\mu\bigg) \ \Diamond \ \phi_3^{-1}\bigg(\fint_{\circ_3, D} \phi_3(g) \, \mathrm{d}\mu\bigg). \tag{21}$$

P r o o f. By the monotonicity of $\star$ and μ, for any $D \in \mathcal{A}$ we obtain

$$\mu\big(D \cap \{f \star g > a \star b\}\big) \leqslant \mu\Big(D \cap \big(\{f > a\} \cup \{g > b\}\big)\Big).$$

From μ-subadditivity of f, g and from the fact that $b \mapsto a \circ_1 b$ is a nondecreasing function we get

$$\begin{aligned} &\phi_1(a \star b) \circ_1 \mu\big(D \cap \{\phi_1(f \star g) > \phi_1(a \star b)\}\big) \\ &\quad \leqslant \phi_1(a \star b) \circ_1 \Big(\mu\big(D \cap \{\phi_2(f) > \phi_2(a)\}\big) \triangledown \mu\big(D \cap \{\phi_3(g) > \phi_3(b)\}\big)\Big). \end{aligned} \tag{22}$$

By (20) and (22)

$$\begin{aligned} &\phi_1^{-1}\Big(\phi_1(a \star b) \circ_1 \mu\big(D \cap \{\phi_1(f \star g) > \phi_1(a \star b)\}\big)\Big) \\ &\quad \leqslant \phi_2^{-1}\Big(\phi_2(a) \circ_2 \mu\big(D \cap \{\phi_2(f) > \phi_2(a)\}\big)\Big) \lozenge \phi_3^{-1}\Big(\phi_3(b) \circ_3 \mu\big(D \cap \{\phi_3(g) > \phi_3(b)\}\big)\Big). \end{aligned}$$

Since ϕ_1^{-1} is increasing, we have

$$\begin{aligned} &\phi_1^{-1}\Big(\fint_{\circ_1, D} \phi_1(f \star g)\, \mathrm{d}\mu\Big) \\ &\quad \leqslant \phi_2^{-1}\Big(\phi_2(a) \circ_2 \mu\big(D \cap \{\phi_2(f) > \phi_2(a)\}\big)\Big) \lozenge \phi_3^{-1}\Big(\phi_3(b) \circ_3 \mu\big(D \cap \{\phi_3(g) > \phi_3(b)\}\big)\Big) \end{aligned}$$

for all $a, b \in Y$. Taking the infimum over $a \in Y$, we get

$$\begin{aligned} &\phi_1^{-1}\Big(\fint_{\circ_1, D} \phi_1(f \star g)\, \mathrm{d}\mu\Big) \\ &\quad \leqslant \phi_2^{-1}\Big(\fint_{\circ_2, D} \phi_2(f)\, \mathrm{d}\mu\Big) \lozenge \phi_3^{-1}\Big(\phi_3(b) \circ_3 \mu\big(D \cap \{\phi_3(g) > \phi_3(b)\}\big)\Big). \end{aligned}$$

Proceeding similary with the infimum in $b \in Y$, we obtain (21). □

Example 3.5. We know from Example 3.1 that any comonotone functions $f, g\colon X \to Y$ are μ-subadditive with $\triangledown = \vee$. Put $Y = [0, \infty]$. If

$$(a \star b) \vee \big(\phi_1^{-1}(c) \vee \phi_1^{-1}(d)\big) \leqslant \big(a \vee \phi_2^{-1}(c)\big) \star \big(b \vee \phi_3^{-1}(d)\big), \tag{23}$$

then for all $D \in \mathcal{A}$ we get

$$\phi_1^{-1}\Big((S)\fint_D \phi_1(f \star g)\, \mathrm{d}\mu\Big) \leqslant \phi_2^{-1}\Big((S)\fint_D \phi_2(f)\, \mathrm{d}\mu\Big) \star \phi_3^{-1}\Big((S)\fint_D \phi_3(g)\, \mathrm{d}\mu\Big). \tag{24}$$

The inequality (23) is satisfied for any operator $\star$ such that $a \star b \geqslant a \vee b$ and functions $\phi_1 \geqslant \phi_i$, $i = 2, 3$. Indeed, combining $a_1 \star a_2 \leqslant \big(a_1 \vee \phi_2^{-1}(b_1)\big) \star \big(a_2 \vee \phi_3^{-1}(b_2)\big)$ with

$$\phi_1^{-1}(b_1) \vee \phi_1^{-1}(b_2) \leqslant \phi_1^{-1}(b_1) \star \phi_1^{-1}(b_2) \leqslant \big(a_1 \vee \phi_2^{-1}(b_1)\big) \star \big(a_2 \vee \phi_3^{-1}(b_2)\big)$$

yields (23). From (19) and (24) we can get a generalization of Theorem 3.1 in [37].

Example 3.6. Let $Y = [0,1]$, $D = X$ and $\mu(X) = 1$. Put $x \triangledown y = x + y - xy$ and $x \star y = x \Diamond y = (x+y) \wedge 1$, where $x, y \in Y$. If $\circ_i = \vee$ and $\phi_i(x) = x$, $i = 1,2,3$, then the condition (20) takes the form

$$\big((a+b) \wedge 1\big) \vee (c + d - cd) \leqslant (a \vee c + b \vee d) \wedge 1, \tag{25}$$

$a, b, c, d \in Y$. Since $(a+b) \wedge 1 \leqslant a + b \leqslant (a \vee c) + (b \vee d)$ and $c + d - cd \leqslant (a \vee c + b \vee d)$, the inequality (25) is true for all a, b, c, d. Hence, if $f, g \colon X \to [0,1]$ are positive quadrant dependent functions with respect to a submodular and monotone measure μ on X (see Example 3.2), then

$$(S)\!\!\!\!\!\!\;\;\;\int_X (f+g) \wedge 1 \,\mathrm{d}\mu \leqslant (S)\!\!\!\!\!\!\;\;\;\int_X f \,\mathrm{d}\mu + (S)\!\!\!\!\!\!\;\;\;\int_X g \,\mathrm{d}\mu.$$

Example 3.7. Suppose $Y = [0,1]$ and μ is a subadditive measure such that $\mu(\mathcal{A}) \subset Y$. From Theorem 3.4 with $a \star b = a \Diamond b = a \triangledown b = (a+b) \wedge 1$, $\circ_i = \vee$ and $\phi_i(x) = x$ for all i, it follows that

$$(S)\!\!\!\!\!\!\;\;\;\int_X (f+g) \,\mathrm{d}\mu \leqslant (S)\!\!\!\!\!\!\;\;\;\int_X f \,\mathrm{d}\mu + (S)\!\!\!\!\!\!\;\;\;\int_X g \,\mathrm{d}\mu$$

for all $f, g \colon X \to Y$ such that $f + g \leqslant 1$.

Example 3.8. Set $Y = [0, \infty]$ and $\triangledown = \star = \Diamond = +$. Let μ be a subadditive measure, $\circ_i = \vee$ and $\phi_i(x) = x$ for all i. Then

$$(S)\!\!\!\!\!\!\;\;\;\int_X (f+g) \,\mathrm{d}\mu \leqslant (S)\!\!\!\!\!\!\;\;\;\int_X f \,\mathrm{d}\mu + (S)\!\!\!\!\!\!\;\;\;\int_X g \,\mathrm{d}\mu$$

for all $f, g \colon X \to Y$.

Next, we prove that the subadditivity property of the Sugeno integral (8) with $Y = [0,1]$ or $Y = [0, \infty]$ is equivalent to the subadditivity of μ.

Theorem 3.9. Let $\mu(\mathcal{A}) \subset Y$. Then μ is subadditive if and only if

$$(S)\!\!\!\!\!\!\;\;\;\int_X (f+g) \,\mathrm{d}\mu \leqslant (S)\!\!\!\!\!\!\;\;\;\int_X f \,\mathrm{d}\mu + (S)\!\!\!\!\!\!\;\;\;\int_X g \,\mathrm{d}\mu \tag{26}$$

holds for all functions $f, g \colon X \to Y$ such that $f + g \in Y$.

Proof. If μ is subadditive, inequality (26) follows immediately from Examples 3.7 and 3.8. To see the converse, take $f = a\mathbb{1}_A$ and $g = a\mathbb{1}_B$, where $a \in Y$ and $A \cap B = \emptyset$. From the assumption $\mu(\mathcal{A}) \subset Y$ and (26), we have that there exists $a \geqslant \mu(A \cup B)$ such that

$$\mu(A \cup B) = a \wedge \mu(A \cup B) \leqslant \big(a \wedge \mu(A)\big) + \big(a \wedge \mu(B)\big) \leqslant \mu(A) + \mu(B).$$

If A and B are not disjoint sets, then

$$\mu(A \cup B) = \mu(A \cup (B \backslash A)) \leqslant \mu(A) + \mu(B \backslash A) \leqslant \mu(A) + \mu(B),$$

which completes the proof. □

Note that the subadditivity of the Sugeno integral from the viewpoint of subadditive transformations of aggregation functions is also investigated in [17].

Now, we show that from the Minkowski-Hölder type inequality for integral (2) one can obtain an inequality of the same type for integral (17) and vice versa. Suppose $Y = [0, m]$, $0 < m \leqslant \infty$. Let $h\colon Y \to Y$ be a decreasing function such that $h(Y) = Y$, $h(0) > 0$ and $h\big(m\big) = 0$. For instance, $h(x) = 1 - x$ for $Y = [0, 1]$ and $h(x) = 1/x$ for $Y = [0, \infty]$ under convention that $1/0 = \infty$ and $1/\infty = 0$. Suppose μ_h is a monotone measure on $(X, \mathcal{A})$ defined as $\mu_h(A) = h^{-1}\big(\mu(X\backslash A)\big)$. For a given operator $\circ\colon Y^2 \to Y$ let us define the operator

$$a \circ_h b = h^{-1}\big(h(a) \circ h(b)\big), \quad a, b \in Y.$$

For any measurable function $f\colon X \to Y$, we have

$$\begin{aligned} h^{-1}\bigg(⨍_{\circ,X} h(f)\,\mathrm{d}\mu\bigg) &= h^{-1}\Big(\inf_{y\in Y} \{h(y) \circ \mu(\{h(f) > h(y)\})\}\Big) \\ &= \inf_{y\in Y} \Big\{h^{-1}\Big(h(y) \circ \mu(\{f \leqslant y\})\Big)\Big\} = \int_{\circ_h,X} f\,\mathrm{d}\mu. \end{aligned} \tag{27}$$

Applying the formula (27) and Theorem 2.7 with $\star = \Diamond$ and $\phi_i(x) = x$ for all i gives the following Corollary.

Corollary 3.10. Assume $\circ\colon Y^2 \to Y$ is nondecreasing, $m \circ y = y \circ m = m$ for all $y \in Y = [0, m]$ and $f, g\colon X \to Y$ are $\star$-associated. The following inequality is satisfied

$$h^{-1}\bigg(⨍_{\circ,X} h(f \star g)\,\mathrm{d}\mu\bigg) \leqslant h^{-1}\bigg(⨍_{\circ,X} h(f)\,\mathrm{d}\mu\bigg) \star h^{-1}\bigg(⨍_{\circ,X} h(g)\,\mathrm{d}\mu\bigg)$$

if and only if $(a \star b) \circ_h c \leqslant (a \circ_h c) \star (b \circ_h c)$ for all $a, b, c \in Y$.

Example 3.11. From the well-known inequality $(x + y)/(1 + x + y) \leqslant \big(x/(1 + x)\big) + \big(y/(1 + y)\big)$ for $x, y \geqslant 0$, it follows that for all $a, b, c \geqslant 0$

$$\big((a + b)^{-1} + c^{-1}\big)^{-1} \leqslant \big(a^{-1} + c^{-1}\big)^{-1} + \big(b^{-1} + c^{-1}\big)^{-1}$$

with $1/0 = \infty$ and $1/\infty = 0$. This implies that the necessary and sufficient condition of Corollary 3.10 is satisfied for $Y = [0, \infty]$, $h(x) = x^{-1}$ and $\star = \circ = +$. Thus, for any comonotone functions $f, g\colon X \to Y$, we have

$$\bigg(⨍_{+,X} 1/(f + g)\,\mathrm{d}\mu\bigg)^{-1} \leqslant \bigg(⨍_{+,X} 1/f\,\mathrm{d}\mu\bigg)^{-1} + \bigg(⨍_{+,X} 1/g\,\mathrm{d}\mu\bigg)^{-1}.$$

The next result is an immediate consequence of Theorem 3.4 with $\phi_i(x) = x$ for all i and the formula

$$⨍_{\circ_h,X} f\,\mathrm{d}\mu_h = h^{-1}\bigg(\int_{\circ,X} h(f)\,\mathrm{d}\mu\bigg).$$

Corollary 3.12. Assume that $f, g\colon X \to Y$ are μ_h-subadditive for $\triangledown$, operator $\star$ is nondecreasing and right-continuous and $\circ$ is nondecreasing. Assume also that $(a \star b) \circ_h (c \triangledown d) \leqslant (a \circ_h c) \star (b \circ_h d)$ for all $a, b, c, d \in Y$. Then

$$h^{-1}\Big(\int_{\circ,X} h(f \star g)\,\mathrm{d}\mu\Big) \leqslant h^{-1}\Big(\int_{\circ,X} h(f)\,\mathrm{d}\mu\Big) \star h^{-1}\Big(\int_{\circ,X} h(g)\,\mathrm{d}\mu\Big).$$

Example 3.13. Suppose $\mu(A) = 1/\mu_h(X\backslash A)$ for $A \in \mathcal{A}$. From Example 3.8, formula (19) and Corollary 3.12 for $h(x) = 1/x$, $\circ_h = \vee$ and $\star = +$, it follows that for any measurable functions $f, g\colon X \to [0,\infty]$, we have

$$\Big((S)\int_X 1/(f+g)\,\mathrm{d}\mu\Big)^{-1} \leqslant \Big((S)\int_X 1/f\,\mathrm{d}\mu\Big)^{-1} + \Big((S)\int_X 1/g\,\mathrm{d}\mu\Big)^{-1},$$

with $(S)\int_X$ being the Sugeno integral (8) for $Y = [0,\infty]$ under the convention $1/\infty = 0$ and $1/0 = \infty$.

4. APPLICATION

As an application of the results of this paper, we provide new metrics in the space of $\mathcal{A}$-measurable functions $f\colon X \to \mathbb{R}$ defined on a fuzzy space $(X, \mathcal{A}, \mu)$. First, let us recall some facts. Taking $\circ = +$ and $Y = [0,\infty]$ in (17), we get the functional

$$d_F(\mathsf{X},\mathsf{Y}) = \inf_{\varepsilon \geqslant 0} \big\{\varepsilon + \mu\big(\{|\mathsf{X}-\mathsf{Y}| > \varepsilon\}\big)\big\}$$

on the space $L^0(X)$ of all random variables defined on a probability space $(X, \mathcal{A}, \mu)$. This functional was proposed by Fréchet [16] in order to metrize the convergence in measure μ (see also [7, p. 356] and [12, pp. 101-104]). The integral (17) with $\circ = \vee$ was introduced by Ky Fan [14]. He proved that $L^0(X)$ with the metric

$$d_{KF}(\mathsf{X},\mathsf{Y}) = \inf\big\{\varepsilon \geqslant 0\colon \mu\big(\{|\mathsf{X}-\mathsf{Y}| > \varepsilon\}\big) \leqslant \varepsilon\big\}$$

is a complete space. By (19) we have

$$d_{KF}(\mathsf{X},\mathsf{Y}) = (S)\!\!-\!\!\!\!\!\!\int_X |\mathsf{X}-\mathsf{Y}|\,\mathrm{d}\mu = (S)\int_X |\mathsf{X}-\mathsf{Y}|\,\mathrm{d}\mu.$$

Li [24] extended Ky Fan's result to cover the case of any continuous from below, finite, order continuous and subadditive measure μ. Some other results related to metrics determined by the Sugeno integral are presented in Borzová-Molnárová et al. [6].

Now we are ready to introduced new metrics. Given $p > 0$, let $Y = [0,\infty]$ and $\circ\colon Y^2 \to Y$ be a non-decreasing operator such that $x \circ (y+z) \leqslant (x \circ y) + (x \circ z)$ and $(ax) \circ y \leqslant a^p (x \circ y)$ for $x, y, z \in Y$ and $a > 1$. We also assume that if $1 \circ x \leqslant y$ for $0 < y < 1$, then $x \leqslant y$. For instance, $x \circ y = x^p \wedge y^u$ or $x \circ y = x^p y^u$, where $0 < u \leqslant 1$. Suppose μ is a subadditive measure and put

$$D_{\circ,p}(f,g) = \Big(\int_{\circ,X} |f-g|^p\,\mathrm{d}\mu\Big)^{1/(p^2+1)}.$$

As special cases we get

$$D_{\wedge,1}(f,g)=\left((S)\int_X |f-g|\,\mathrm{d}\mu\right)^{1/2},\quad D_{\cdot,1}(f,g)=\left((N)\int_X |f-g|\,\mathrm{d}\mu\right)^{1/2}. \tag{28}$$

Denote by $\mathcal{L}^p_\circ$ the class of measurable functions $f\colon X\to\mathbb{R}$ such that $D_{\circ,p}(f,0)<\infty$. Let $f\sim g$ mean that $\mu(\{|f-g|>0\})=0$ and let $L^p_\circ$ be the set of the equivalence classes in $\mathcal{L}^p_\circ$ determined by the equivalence relation $\sim$. If $[f]$ is the equivalence class containing f, define $d_{\circ,p}\big([f],[g]\big)=D_{\circ,p}(f,g)$.

Theorem 4.1. Suppose $\circ\colon Y^2\to Y$ is left-continuous in the second argument. If μ is subadditive and continuous from below, then $\big(L^p_\circ, d_{\circ,p}\big)$ is a complete metric space.

To prove Theorem 4.1 we need the monotone convergence theorem and Fatou's lemma for the integral (2). We recall that μ is *null-additive* if $\mu(A)=0$ implies $\mu(A\cup B)=\mu(B)$ for every $B\in\mathcal{A}$. Observe that if μ is subadditive, then it is also null-additive.

Lemma 4.2. (Monotone convergence) Let $\circ\colon Y^2\to Y$ be left-continuous in the second argument. If μ is a continuous from below, null-additive and monotone measure and if (f_n) is a sequence of functions $f_n\colon X\to Y$ which is nondecreasing and converges to f on $A^c=X\backslash A$ with $\mu(A)=0$, then $\lim_{n\to\infty}\int_{\circ,X} f_n\,\mathrm{d}\mu=\int_{\circ,X} f\,\mathrm{d}\mu$.

Proof. Measure μ is null-additive, so $\int_{\circ,X} g\,\mathrm{d}\mu=\int_{\circ,A^c} g\,\mathrm{d}\mu$ for any g. The rest of the proof is similar to that of Lemma 14 in [10]. □

Lemma 4.3. (Fatou) Suppose $\circ\colon Y^2\to Y$ is left-continuous in the second argument, $f_n\colon X\to Y$ for all n. If μ is a continuous from below, null-additive and monotone measure and $f(x)=\lim_{n\to\infty} f_n(x)$ for all $x\in A^c$ with $\mu(A)=0$, then $\int_{\circ,X} f\,\mathrm{d}\mu\leqslant\liminf_{n\to\infty}\int_{\circ,X} f_n\,\mathrm{d}\mu$.

Proof. The proof follows from Lemma 4.2 and standard arguments (see [19, Lemma 1.20]). □

Proof. [Proof of Theorem 4.1] Assume $c\circ d>0$ for some $c,d>0$; otherwise the result is trivial. We will show that $x\circ y>0$ for all $x,y>0$. In fact, suppose that $x\circ y=0$ for some $x,y>0$. Then $x\circ t\leqslant x\circ y=0$ for all $0\leqslant t\leqslant y$, so from the subadditivity of $t\mapsto x\circ t$ it easily follows that $x\circ t=0$ for all $t\in Y$. Next, by the monotonicity of $\circ$, we have $s\circ t\leqslant x\circ t=0$ for all $0\leqslant s\leqslant x$ and $(ax)\circ t\leqslant a^p(x\circ t)=0$ for any $a>1$. Hence so $s\circ t=0$ for all $s,t\in Y$, a contradiction.

Next, suppose $d_{\circ,p}\big([f],[g]\big)=0$. Hence, $\mu\big(\{|f-g|\geqslant t\}\big)=0$ for all $t>0$. Since μ is continuous from below, we have $\mu\big(\{|f-g|>0\}\big)=0$, so $f\sim g$. Clearly $d_{\circ,p}$ is symmetric and it follows from Theorem 2.15 for $r=1$ and $q=p$ that $d_{\circ,p}$ satisfies the triangle inequality. The proof of the completeness is a modified version of that of Lemma 1.31 in [19]. Given a Cauchy sequence $\big([f_n]\big)$, let $\big([f_{n(k)}]\big)$ be a subsequence such that $\big(d_{\circ,p}\big([f_{n(k+1)}],[f_{n(k)}]\big)\big)^{p^2+1}\leqslant 4^{-kp}$. Put $A_k=\{x\in X\colon |f_{n(k+1)}(x)-f_{n(k)}(x)|^p\geqslant 2^{-k}\}$.

Then $2^{-k} \circ \mu(A_k) \leqslant 4^{-kp}$ by the definition of $d_{\circ,p}$. From the property $(2x) \circ y \leqslant 2^p(x \circ y)$ we get

$$1 \circ \mu(A_k) \leqslant (2^p)^k \big(2^{-k} \circ \mu(A_k)\big) \leqslant 2^{-kp}.$$

By the assumption on $\circ$, $\mu(A_k) \leqslant 2^{-kp}$ for all k. Set $A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$. Since μ is subadditive, we have

$$\mu(A) \leqslant \mu\bigg(\bigcup_{k=i}^{\infty} A_k\bigg) \leqslant \sum_{k=i}^{\infty} \mu(A_k) \to 0 \;\text{ as } i \to \infty,$$

so $\mu(A) = 0$. Let $x \in A^c = X \backslash A$. Since $|f_{n(k+1)}(x) - f_{n(k)}(x)| < 2^{-k/p}$ for all large enough k, we have

$$\sup_{r \geqslant k} \big|f_{n(r)}(x) - f_{n(k)}(x)\big| \leqslant \sum_{r=k}^{\infty} \big|f_{n(r+1)}(x) - f_{n(r)}(x)\big| \leqslant \frac{2^{-k/p}}{1 - 2^{-1/p}}.$$

Thus, $\big(f_{n(k)}(x)\big)_{k=1}^{\infty}$ is a Cauchy sequence and $f_{n(k)}(x) \to f(x)$ for all $x \in A^c$, where f is some measurable function (if $A \neq \emptyset$, define f on A arbitrarily). We recall that any subadditive measure μ is also null-additive. By Lemma 4.3, we get

$$d_{\circ,p}\big([f],[f_n]\big) \leqslant \liminf_{k\to\infty} d_{\circ,p}\big([f_{n(k)}],[f_n]\big) \leqslant \sup_{m \geqslant n} d_{\circ,p}\big([f_m],[f_n]\big) \to 0, \; n \to \infty,$$

as $\big([f_n]\big)$ is a Cauchy sequence. This shows that $[f_n] \to [f]$ in metric $d_{\circ,p}$. □

Denote by L^1_N the set of all equivalence classes in $\mathcal{L}^1_\cdot$ determined by the equivalence relation $\sim$. Put $\|[f]\| = (N)\int_X |f| \,\mathrm{d}\mu$.

Corollary 4.4. If μ is maxitive, then L^1_N is a Banach space with the norm $\|\cdot\|$.

Proof. Any maxitive measure is subadditive and continuous from below [32]. Observe that $\|[f]-[g]\| = D_{\cdot,1}(f,g)^2$ (see (28)) and $\|[cf]\| = |c|\|[f]\|$ for $c \in \mathbb{R}$. The result follows immediately from Theorems 2.18 and 4.1. □

The next theorem is a counterpart of Theorem 2.21 in [5]

Theorem 4.5. Adopt the assumptions of Theorem 2.15 with $Y = [0,\infty]$ and some $p > 0$. If $[f_n], [f] \in L^p_\circ$ for all n and $\lim_{n\to\infty} d_{\circ,p}\big([f_n],[f]\big) = 0$, then $\int_{\circ,X} f_n^p \,\mathrm{d}\mu \to \int_{\circ,X} f^p \,\mathrm{d}\mu$ as $n \to \infty$.

Proof. From Theorem 2.15 we get $d_{\circ,p}\big([f_n],[0]\big) \leqslant d_{\circ,p}\big([f_n],[f]\big) + d_{\circ,p}\big([f],[0]\big)$ and $d_{\circ,p}\big([f],[0]\big) \leqslant d_{\circ,p}\big([f_n],[f]\big) + d_{\circ,p}\big([f_n],[0]\big)$. This implies

$$\bigg|\Big(\int_{\circ,X} f_n^p \,\mathrm{d}\mu\Big)^{1/(p^2+1)} - \Big(\int_{\circ,X} f^p \,\mathrm{d}\mu\Big)^{1/(p^2+1)}\bigg| \leqslant d_{\circ,p}\big([f_n],[f]\big),$$

which completes the proof. □

Theorem 4.5 gives a partial answer to the open problem 2.22 in [5] as there exist discontinuous and subadditive measures (see [32]).

5. ACKNOWLEDGEMENT

We are indebted to the referee for the references to works of Borzová-Molnárová et al. [6] and Greco et al. [17].

(Received July 7, 2015)

REFERENCES

[1] H. Agahi, R. Mesiar, and Y. Ouyang: General Minkowski type inequalities for Sugeno integrals. Fuzzy Sets and Systems *161* (2010), 708–715. DOI:10.1016/j.fss.2009.10.007

[2] H. Agahi and R. Mesiar: On Cauchy-Schwarz's inequality for Choquet-like integrals without the comonotonicity condition. Soft Computing *19* (2015), 1627–1634. DOI:10.1007/s00500-014-1578-0

[3] B. Bassan and F. Spizzichino: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. J. Multivariate Analysis *93* (2005), 313–339. DOI:10.1016/j.jmva.2004.04.002

[4] J. Borzová-Molnárová, L. Halčinová, and O. Hutník: The smallest semicopula-based universal integrals I: Properties and characterizations. Fuzzy Sets and Systems *271* (2015), 1–17.

[5] J. Borzová-Molnárová, L. Halčinová, and O. Hutník: The smallest semicopula-based universal integrals II: Convergence theorems. Fuzzy Sets and Systems *271* (2015), 18–30.

[6] J. Borzová-Molnárová, L. Halčinová, and O. Hutník: The smallest semicopula-based universal integrals III: Topology determined by the integral. Fuzzy Sets and Systems (2016).

[7] N. L. Carothers: Real Analysis. University Press, Cambridge 2000. DOI:10.1017/cbo9780511814228

[8] M. E. Cattaneo: On maxitive integration. Department of Statistics University of Munich 2013, http://www.stat.uni-muenchen.de.

[9] J. Cerdá: Lorentz capacity spaces. Contemporary Math. *445* (2007), 45–59. DOI:10.1090/conm/445/08592

[10] A. Chateauneuf, M. Grabisch, and A. Rico: Modeling attitudes toward uncertainty through the use of the Sugeno integral. J. Math. Econom. *44* (2008), 1084–1099. DOI:10.1016/j.jmateco.2007.09.003

[11] B. Daraby and F. Ghadimi: General Minkowski type and related inequalities for seminormed fuzzy integrals. Sahand Commun. Math. Analysis *1* (2014), 9–20.

[12] N. Dunford and J. T. Schwartz: Linear Operators, Part I General Theory. A Wiley Interscience Publishers, New York 1988.

[13] F. Durante and C. Sempi: Semicopulæ. Kybernetika *41* (2005), 315–328.

[14] K. Fan: Entfernung zweier zufälligen Grössen und die Konvergenz nach Wahrscheinlichkeit. Math. Zeitschrift *49* (1944), 681–683. DOI:10.1007/bf01174225

[15] H. Föllmer and A. Schied: Stochastic Finance. An Introduction In Discrete Time. De Gruyter, Berlin 2011. DOI:10.1515/9783110218053

[16] M. Fréchet: Sur divers modes de convergence d'une suite de fonctions d'une variable. Bull. Calcutta Math. Soc. *11* (1919-20), 187–206.

[17] S. Greco, R. Mesiar, F. Rindone, and L. Šipeky: Superadditive and subadditive transformations of integrals and aggregation functions. Fuzzy Sets and Systems *291* (2016), 40–53.

[18] H. Imaoka: On a subjective evaluation model by a generalized fuzzy integral. Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems *5* (1997), 517–529. DOI:10.1142/s0218488597000403

[19] O. Kallenberg: Foundations of Modern Probability. Second edition. Springer, Berlin 2002. DOI:10.1007/978-1-4757-4015-8

[20] M. Kaluszka, A. Okolewski, and M. Boczek: On Chebyshev type inequalities for generalized Sugeno integrals. Fuzzy Sets and Systems *244* (2014), 51–62. DOI:10.1016/j.fss.2013.10.015

[21] A. Kandel and W. J. Byatt: Fuzzy sets, fuzzy algebra, and fuzzy statistics. Proc. IEEE *66* (1978), 1619–1639. DOI:10.1109/proc.1978.11171

[22] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dordrecht 2000. DOI:10.1007/978-94-015-9540-7

[23] E. P. Klement, R. Mesiar, and E. Pap: A universal integral as common frame for Choquet and Sugeno integral. IEEE Trans. Fuzzy Systems *18* (2010), 178–187. DOI:10.1109/tfuzz.2009.2039367

[24] G. Li: A metric on space of measurable functions and the related convergence. Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems *20* (2012), 211–222. DOI:10.1142/s0218488512500109

[25] B. Liu: Uncertainty Theory. Fourth edition. Springer 2015. DOI:10.1007/978-3-662-44354-5

[26] T. Murofushi: A note on upper and lower Sugeno integrals. Fuzzy Sets and Systems *138* (2003), 551–558. DOI:10.1016/s0165-0114(02)00375-5

[27] Y. Ouyang and R. Mesiar: On the Chebyshev type inequality for seminormed fuzzy integral. Applied Math. Lett. *22* (2009), 1810–1815. DOI:10.1016/j.aml.2009.06.024

[28] Y. Ouyang, R. Mesiar, and H. Agahi: An inequality related to Minkowski type for Sugeno integrals. Inform. Sci. *180* (2010), 2793–2801. DOI:10.1016/j.ins.2010.03.018

[29] E. Pap, ed.: Handbook of Measure Theory. Elsevier Science, Amsterdam 2002.

[30] H. Román-Flores, A. Flores-Franulič, and Y. Chalco-Cano: The fuzzy integral for monotone functions. Applied Math. Comput. *185* (2007), 492–498. DOI:10.1016/j.amc.2006.07.066

[31] L. Rüschendorf: Mathematical Risk Analysis: Dependence, Risk Bounds, Optimal Allocations and Portfolios. Springer Science and Business Media, Berlin 2013. DOI:10.1007/978-3-642-33590-7

[32] N. Shilkret: Maxitive measure and integration. Indagationes Math. *33* (1971), 109–116. DOI:10.1016/s1385-7258(71)80017-3

[33] F. Suárez García and P. Gil Álvarez: Two families of fuzzy integrals. Fuzzy Sets and Systems *18* (1986), 67–81. DOI:10.1016/0165-0114(86)90028-x

[34] M. Sugeno: Theory of Fuzzy Integrals and its Applications. Ph.D. Dissertation, Tokyo Institute of Technology 1974.

[35] Z. Wang and G. Klir: Generalized Measure Theory. Springer, New York 2009. DOI:10.1007/978-0-387-76852-6

[36] Ch. Wu, X. Rena, and C. Wu: A note on the space of fuzzy measurable functions for a monotone measure. Fuzzy Sets and Systems *182* (2011), 2–12. DOI:10.1016/j.fss.2010.10.006

[37] L. Wu, J. Sun, X. Ye, and L. Zhu: Hölder type inequality for Sugeno integral. Fuzzy Sets and Systems *161* (2010), 2337–2347. DOI:10.1016/j.fss.2010.04.017

Michał Boczek, Institute of Mathematics, Lodz University of Technology, ul. Wólczańska 215, 90-924 Lodz. Poland.
e-mail: michboczek@gmail.com

Marek Kaluszka, Institute of Mathematics, Lodz University of Technology, ul. Wólczańska 215, 90-924 Lodz. Poland.
e-mail: marek.kaluszka@p.lodz.pl