Y. Daikh; W. Chikouche

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Abstract. We are interested in the discretization of the heat equation with a diffusion coefficient depending on the space and time variables. The discretization relies on a spectral element method with respect to the space variables and Euler's implicit scheme with respect to the time variable. A detailed numerical analysis leads to optimal a priori error estimates.

Keywords: heat equation; diffusion coefficient; spectral element methods; a priori estimates

Classification: 35K05, 65N35, 35B45

1. Introduction

An impressive amount of work has been done concerning a priori and a posteriori analysis of parabolic type problems for finite element methods, see [6] and [1] for instance. An extension in spectral element method of some results obtained by Bergam et al. in [1] has been performed recently by N. Chorfi et al. in [4]. They were interested in a posteriori analysis of the spectral element discretization of the one-dimensional heat equation with constant diffusion coefficient. The spectral element method consists on approaching the solution of a partial differential equation by polynomial functions of high degree on each element of a decomposition.

In this paper, we are interested in the discretization of the heat equation with a diffusion coefficient depending on the space and time variables by an implicit Euler's scheme with respect to the time variable and spectral element method with respect to the space variables in a two- or three-dimensional bounded domain. For the space discretization, we consider a partition of the domain into rectangles in dimension 2 or rectangular parallelepipeds in dimension 3 which is conforming and without overlap. The discrete spaces are constructed from tensorized spaces of polynomials of the same high degree on each subdomain. The full discrete problem is then obtained by Galerkin method with numerical integration.

An outline of the paper is as follows.

- In Section 2, we present the linear heat equation and we study the continuous problem and its stability.

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- In Section 3, we describe its time semi-discretization and the corresponding stability property.
- Section 4 is devoted to the description of the space discretization of the problem by using spectral element method.
  The well-posedness of the corresponding problem in each section is proved.
- Optimal error estimates are proved in Section 5.

2. Position of the problem

Let Ω be a connected and bounded open set in $\mathbb{R}^d$ ($d = 1, 2$, or 3) with a Lipschitz-continuous boundary. Also let T be a fixed positive integer. We consider the heat equation

$$\begin{cases} \partial_t u - \operatorname{div}(\lambda \nabla u) = f & \text{in } \Omega \times]0, T[, \\ u = 0 & \text{on } \partial\Omega \times]0, T[, \\ u|_{t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where λ is a given continuous function on $\overline{\Omega} \times [0, T]$ satisfying for some positive constants $\lambda_{\min}$ and $\lambda_{\max}$,

$$\forall \boldsymbol{x} \in \overline{\Omega}, \forall t \in [0, T], \quad \lambda_{\min} \leq \lambda(\boldsymbol{x}, t) \leq \lambda_{\max}. \tag{2}$$

The data are the distribution f and the function u_0; the unknown is the function u.

As usual, we denote by $L^p(\Omega)$, $1 \leq p \leq \infty$, the Lebesgue spaces and by $H^s(\Omega)$, $s > 0$, the standard Sobolev spaces. The usual norm and seminorm of $H^s(\Omega)$ are denoted by $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ respectively. The space $H_0^1(\Omega)$ stands for the closure in $H^1(\Omega)$ of the space of infinitely differentiable functions with a compact support in Ω, and $H^{-1}(\Omega)$ stands for its dual space. For simplicity, we denote by $(\cdot, \cdot)$ the scalar product on $L^2(\Omega)$ and by $\|\cdot\|_{0,\Omega}$ the associated norm. By extension, the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$, is also denoted by $(\cdot, \cdot)$. We define $\mathcal{C}^0(0, T; L^2(\Omega))$, as the space of continuous functions in time, with values in $L^2(\Omega)$, and also $L^2(0, T; H_0^1(\Omega))$, respectively $L^2(0, T; H^{-1}(\Omega))$, as the space of square-integrable functions with values in $H_0^1(\Omega)$, respectively in $H^{-1}(\Omega)$.

The problem (1) admits the equivalent variational formulation.

Find u in $L^2(0, T; H_0^1(\Omega))$ such that $\partial_t u \in L^2(0, T; H^{-1}(\Omega))$ satisfying

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \tag{3}$$

and such that, for a.e. t in $]0, T[$,

$$\forall v \in H_0^1(\Omega), \quad (\partial_t u(t), v) + (\lambda(t) \nabla u(t), \nabla v) = (f(t), v). \tag{4}$$

It is well known [5, Chapter 3, §4] that, for any f in $L^2(0, T; H^{-1}(\Omega))$ and u_0 in $L^2(\Omega)$, problem (3)–(4) admits a unique solution u in $L^2(0, T; H_0^1(\Omega))$ such that

$\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, and this implies that $u \in \mathcal{C}^0(0,T;L^2(\Omega))$. Moreover, let us introduce the norm

(5) $$[[v]](t) = \left(\|v(t)\|^2_{0,\Omega} + \int_0^t \|\lambda^{\frac{1}{2}}(s)\nabla v(s)\|^2_{0,\Omega} \right)^{\frac{1}{2}}.$$

By taking v equal to $u(t)$ in (4) and integrating on the interval $]0,t[$, we easily derive the following estimate [1]: for all $t \in [0,T]$

(6) $$[[u]](t) \leq \left(\|u_0\|^2_{0,\Omega} + \frac{1}{\lambda_{\min}} \|f\|^2_{L^2(0,t;H^{-1}(\Omega))} \right)^{\frac{1}{2}}.$$

3. The time semi-discrete problem

In order to describe the time discretization of equation (1), we introduce a partition of the interval $[0,T]$ into subintervals $[t_{k-1},t_k]$, $1 \leq k \leq K$, such that $0 = t_0 < t_1 < \cdots < t_K = T$. We denote by $\tau_k := t_k - t_{k-1}$, by τ the $K-$tuple$(\tau_1,\ldots,\tau_K)$ and by $|\tau|$ the maximum of the τ_k, $1 \leq k \leq K$. We also define the regularity parameter

$$\sigma_\tau = \max_{2\leq k\leq K} \frac{\tau_k}{\tau_{k-1}}.$$

With each family $(v^k)_{0\leq k\leq K}$, we agree to associate the function v_τ on $[0,T]$ which is affine on each interval $[t_{k-1},t_k]$, $1 \leq k \leq K$, and equal to v^k at t_k, $0 \leq k \leq K$. Equivalently, this function can be written, for $1 \leq k \leq K$, as

(7) $$\forall t \in [t_{k-1},t_k], v_\tau(t) = v^k - \frac{t_k - t}{\tau_k}(v^k - v^{k-1}).$$

For simplicity, we introduce the notation $\lambda^k = \lambda(t_k)$ and $f^k = f(t_k)$, which obviously requires the continuity of f with respect to t. The semi-discrete problem issued from Euler's implicit scheme is now written as

$$\begin{cases} \dfrac{u^k - u^{k-1}}{\tau_k} - \operatorname{div}(\lambda^k \nabla u^k) = f^k & \text{in } \Omega,\ 1 \leq k \leq K, \\ u^k = 0 & \text{on } \partial\Omega,\ 1 \leq k \leq K, \\ u^0 = u_0 & \text{in } \Omega. \end{cases}$$

Equivalently, it admits the variational formulation.

Find $(u^k)_{0\leq k\leq K}$ in $L^2(\Omega) \times H^1_0(\Omega)^K$ satisfying

(8) $$u^0 = u_0 \text{ in } \Omega,$$

and such that, for $1 \leq k \leq K$,

(9) $$\forall v \in H^1_0(\Omega), \quad a^k(u^k,v) = L^k(v),$$

where the bilinear forms a^k, $1 \le k \le K$, are defined by

$$a^k(u,v) = (u,v) + \tau_k(\lambda^k \nabla u, \nabla v),$$

and the linear forms $L^k, 1 \le k \le K$, are defined as

$$L^k(v) = (u^{k-1}, v) + \tau_k(f^k, v).$$

The existence and uniqueness of a solution $(u^k)_{0\le k\le K}$ for any data f in $\mathcal{C}^0(0,T;H^{-1}(\Omega))$ and u_0 in $L^2(\Omega)$ is now a simple consequence of the Lax-Milgram lemma.

Moreover, by using the notation $\lambda^k_{\min} = \inf_{\boldsymbol{x}\in\overline{\Omega}} \lambda(\boldsymbol{x}, t_k)$ and taking $v = u^k$ in (9), we easily derive the following estimate

$$\|u^k\|^2_{0,\Omega} + \tau_k \big\|(\lambda^k)^{\frac{1}{2}} \nabla u^k\big\|^2_{0,\Omega} \le \|u^{k-1}\|^2_{0,\Omega} + \frac{\tau_k}{\lambda^k_{\min}} \|f^k\|^2_{-1,\Omega}. \tag{10}$$

We now define the norm on whole sequences v^ℓ, $0 \le \ell \le k$ by

$$[[(v^\ell)]]_k = \Big(\|v^k\|^2_{0,\Omega} + \sum_{\ell=1}^{k} \tau_\ell \big\|(\lambda^\ell)^{\frac{1}{2}} \nabla v^\ell\big\|^2_{0,\Omega} \Big)^{\frac{1}{2}}. \tag{11}$$

Proposition 3.1. *For any data f in $\mathcal{C}^0(0,T;H^{-1}(\Omega))$ and u_0 in $L^2(\Omega)$, problem* (8)–(9) *has a unique solution $(u^k)_{0\le k\le K}$, which satisfies for all k, $1 \le k \le K$,*

$$[[(u^\ell)]]_k \le \Big(\|u_0\|^2_{0,\Omega} + \sum_{\ell=1}^{k} \frac{\tau_\ell}{\lambda^\ell_{\min}} \|f^\ell\|^2_{-1,\Omega} \Big)^{\frac{1}{2}}. \tag{12}$$

Moreover, this solution is such that, for all k, $1 \le k \le K$,

$$\begin{aligned} &\Big(\sum_{\ell=1}^{k} \tau_\ell \Big\| \frac{u^\ell - u^{\ell-1}}{\tau_\ell} \Big\|^2_{-1,\Omega} \Big)^{\frac{1}{2}} \\ &\qquad \le \sqrt{2} \Big(\lambda_{\max} \|u_0\|^2_{0,\Omega} + \sum_{\ell=1}^{k} \tau_\ell \Big(1 + \frac{\lambda_{\max}}{\lambda^\ell_{\min}}\Big) \|f^\ell\|^2_{-1,\Omega} \Big)^{\frac{1}{2}}. \end{aligned} \tag{13}$$

Proof: By summing up estimate (10) on k, we derive (12), which is the semi-discrete analogue of (6). On the other hand, we derive from (9)

$$\Big\| \frac{u^\ell - u^{\ell-1}}{\tau_\ell} \Big\|_{-1,\Omega} = \sup_{v\in H^1_0(\Omega)} \frac{(f^\ell, v) - (\lambda^\ell \nabla u^\ell, \nabla v)}{|v|_{1,\Omega}},$$

which gives

$$\Big\| \frac{u^\ell - u^{\ell-1}}{\tau_\ell} \Big\|_{-1,\Omega} \le \|f^\ell\|_{-1,\Omega} + (\lambda_{\max})^{\frac{1}{2}} \big\|(\lambda^\ell)^{\frac{1}{2}} \nabla u^\ell\big\|_{0,\Omega}.$$

Multiplying the square of this inequality by τ_ℓ, summing on ℓ and using (12) leads to (13). □

The norm $[[(u^\ell)]]_k$ involved in this estimate is not equal to the norm $[[u_\tau]](t_k)$ (see (7) for the definition of the function u_τ). However, when u_0 is supposed to be in $H^1(\Omega)$, they are equivalent, as proven in the next lemma [1].

Lemma 3.2. *Assume that the function λ is continuously differentiable in time, with maximum value of $\partial_t\lambda$ on $\overline{\Omega}\times[0,T]$ denoted by $\mu_{\max}$. There exists a positive real number α_0, equal to $\lambda_{\min}/2\mu_{\max}$, such that the following equivalence property holds for $|\tau|\le\alpha_0$ and for any family $(v^\ell)_{0\le\ell\le K}$ in $H^1(\Omega)^{K+1}$*

$$\frac{1}{8}[[(v^\ell)]]_k^2 \le [[v_\tau]]^2(t_k) \le \frac{3}{4}\big(1+\frac{3}{2}\sigma_\tau\big)[[(v^\ell)]]_k^2 + \frac{3}{4}\tau_1\big\|(\lambda^1)^{\frac12}\nabla v^0\big\|_{0,\Omega}^2. \tag{14}$$

Proof: Owing to the definitions (5) and (11) of the norms, we have to compare the quantities

$$X_\ell = \int_{t_{\ell-1}}^{t_\ell} \big\|\lambda^{\frac12}(s)\nabla v_\tau(s)\big\|_{0,\Omega}^2\,ds \quad\text{and}\quad Y_\ell = \tau_\ell\big\|(\lambda^\ell)^{\frac12}\nabla v^\ell\big\|_{0,\Omega}^2.$$

Thanks to the definition of $\mu_{\max}$, we have the standard estimate

$$\forall s\in[t_{\ell-1},t_\ell],\quad \forall \boldsymbol{x}\in\Omega,\quad |\lambda(\boldsymbol{x},s)-\lambda^\ell(\boldsymbol{x})|\le\tau_\ell\,\mu_{\max},$$

so that, when $|\tau|\le\alpha_0$,

$$\forall s\in[t_{\ell-1},t_\ell],\quad \forall \boldsymbol{x}\in\Omega,\quad \frac12\le\frac{\lambda(\boldsymbol{x},s)}{\lambda^\ell(\boldsymbol{x})}\le\frac32\,. \tag{15}$$

It can also be noted that, thanks to the definition of v_τ, and for a.e. $\boldsymbol{x}$ in Ω,

$$\int_{t_{\ell-1}}^{t_\ell} |\nabla v_\tau(\boldsymbol{x},s)|^2\,ds = \frac{\tau_\ell}{3}\big(|\nabla v^\ell(\boldsymbol{x})|^2+|\nabla v^{\ell-1}(\boldsymbol{x})|^2+\nabla v^\ell(\boldsymbol{x})\cdot\nabla v^{\ell-1}(\boldsymbol{x})\big). \tag{16}$$

By combining (15) and (16), we obtain

$$X_\ell \ge \frac{\tau_\ell}{6}\Big(\big\|(\lambda^\ell)^{\frac12}\nabla v^\ell\big\|_{0,\Omega}^2 + \big\|(\lambda^\ell)^{\frac12}\nabla v^{\ell-1}\big\|_{0,\Omega}^2 + ((\lambda^\ell)^{\frac12}\nabla v^\ell,(\lambda^\ell)^{\frac12}\nabla v^{\ell-1})\Big).$$

So using the inequality $ab\ge -\frac14 a^2-b^2$ yields

$$X_\ell \ge \frac{\tau_\ell}{8}\big\|(\lambda^\ell)^{\frac12}\nabla v^\ell\big\|_{0,\Omega}^2 = \frac18 Y_\ell,$$

whence the first inequality in (14) holds.

Similarly, by combining (15) and (16) and using the inequality $ab\le\frac12a^2+\frac12b^2$, we have

$$X_\ell \le \frac{3\tau_\ell}{4}\big(\big\|(\lambda^\ell)^{\frac12}\nabla v^\ell\big\|_{0,\Omega}^2 + \big\|(\lambda^\ell)^{\frac12}\nabla v^{\ell-1}\big\|_{0,\Omega}^2\big).$$

When $\ell = 1$, we keep this inequality without modification. When $\ell > 1$, we use an analogue of (15) to obtain

$$X_\ell \le \frac{3\tau_\ell}{4} \big\|(\lambda^\ell)^{\frac{1}{2}} \nabla v^\ell\big\|_{0,\Omega}^2 + \frac{3\tau_{\ell-1}}{4} \frac{3}{2} \sigma_\tau \big\|(\lambda^{\ell-1})^{\frac{1}{2}} \nabla v^{\ell-1}\big\|_{0,\Omega}^2.$$

By summing up the previous lines on ℓ, we derive the second inequality in (14). □

In order to state the a priori error estimate (see [1]), we observe that the family $(e^k)_{0\le k\le K}$, with $e^k = u(t_k) - u^k$, satisfies $e^0 = 0$ and also, by integrating $\partial_t u$ between t_{k-1} and t_k and using equation (9) and equation (4) at time $t = t_k$,

$$\forall v \in H_0^1(\Omega),\ (e^k, v) + \tau_k(\lambda^k \nabla e^k, \nabla v) = (e^{k-1}, v) + \tau_k(\epsilon^k, v),$$

where the consistency error ϵ^k is given by

$$(\epsilon^k, v) = \bigg(\frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} (\partial_t u)(s)\, ds - (\partial_t u)(t_k), v\bigg).$$

So, applying (12) to this new problem, we derive the estimate. Indeed, if the solution u is such that $\partial_t^2 u$ belongs to $L^2(0,T; H^{-1}(\Omega))$, thus for $1 \le k \le K$,

$$[[(u(t_\ell) - u^\ell)]]_k \le \frac{2}{3\lambda_{\min}^{\frac{1}{2}}} \Big(\max_{1\le\ell\le k} \tau_\ell\Big) \big\|\partial_t^2 u\big\|_{L^2(0,t_k;H^{-1}(\Omega))}^2.$$

Thanks to Lemma 3.2, this also induces a similar bound for the norm $[[u - u_\tau]](t_k)$.

4. The time and space discrete problem

From now on, we assume that Ω admits a partition without overlap into a finite number of subdomains

$$\overline{\Omega} = \cup_{r=1}^R \overline{\Omega}_r \quad \text{and} \quad \Omega_r \cap \Omega_{r'} = \emptyset, \quad 1 \le r < r' \le R,$$

which satisfy the further conditions:

(i) each Ω_r, $1 \le r \le R$, is a rectangle in dimension $d = 2$ or a rectangular parallelepiped in dimension $d = 3$;

(ii) the intersection of two subdomains $\overline{\Omega}_r$ and $\overline{\Omega}_{r'}$, $1 \le r < r' \le R$, if not empty, is either a vertex or a whole edge or a whole face of both Ω_r and $\Omega_{r'}$.

We introduce the space $\mathbb{P}_N(\Omega_r)$ of restrictions to Ω_r of polynomials with d variables and degree $\le N$ with respect to each variable. Relying on this definition, we introduce the discrete spaces, for an integer $N \ge 2$,

$$\mathbb{Y}_N = \{w \in L^2(\Omega) \ \mid \ w_r = w|_{\Omega_r} \in \mathbb{P}_N(\Omega_r), \quad r = 1, \dots, R\},$$
$$\mathbb{X}_N^0 = \mathbb{Y}_N \cap H_0^1(\Omega).$$

Setting $\xi_0 = -1$ and $\xi_N = 1$, we introduce the $N-1$ nodes ξ_j, $1 \le j \le N-1$, and the $N+1$ weights ρ_j, $0 \le j \le N$, of the Gauss-Lobatto quadrature formula on $\overline{\Lambda} := [-1, 1]$. We recall that the following equality holds

$$\forall \varphi_N \in \mathbb{P}_{2N-1}(\Lambda), \int_{-1}^{1} \varphi_N(\zeta)\, d\zeta = \sum_{j=0}^{N} \varphi_N(\xi_j)\rho_j. \tag{17}$$

We also recall [2, form. (13.20)] the following property, which is useful in what follows

$$\forall \varphi_N \in \mathbb{P}_N(\Lambda), \|\varphi_N\|^2_{L^2(\Lambda)} \le \sum_{j=0}^{N} \varphi_N^2(\xi_j)\rho_j \le 3\|\varphi_N\|^2_{L^2(\Lambda)}. \tag{18}$$

Denoting by F_r the affine mapping that sends Λ^d onto Ω_r, we introduce the local discrete products, defined on continuous functions u and v on $\overline{\Omega}_r$ by

$$(u, v)_N^r = \begin{cases} \frac{meas(\Omega_r)}{4} \sum_{i=0}^{N} \sum_{j=0}^{N} u \circ F_r(\xi_i, \xi_j) v \circ F_r(\xi_i, \xi_j)\rho_i\rho_j \\ \qquad\qquad\qquad \text{if } \ d = 2, \\ \frac{meas(\Omega_r)}{8} \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{p=0}^{N} u \circ F_r(\xi_i, \xi_j, \xi_p) v \circ F_r(\xi_i, \xi_j, \xi_p)\rho_i\rho_j\rho_p \\ \qquad\qquad\qquad \text{if } \ d = 3. \end{cases}$$

The global product is then defined on continuous functions u and v on $\overline{\Omega}$ by

$$((u, v))_N = \sum_{r=1}^{R} (u|_{\Omega_r}, v|_{\Omega_r})_N^r.$$

We denote by i_N the interpolation operator at the nodes ξ_j, $0 \le j \le N$. We need the local Lagrange interpolation operators $\mathcal{I}_N^r$: for each function φ continuous on $\overline{\Omega}_r$, $\mathcal{I}_N^r\varphi$ belongs to $\mathbb{P}_N(\Omega_r)$ and is equal to φ at all nodes $F_r(\xi_i, \xi_j)$, $0 \le i, j \le N$ in dimension 2 and at $F_r(\xi_i, \xi_j, \xi_p)$, $0 \le i, j, p \le N$ in dimension 3. Finally, for each function φ continuous on $\overline{\Omega}$, $\mathcal{I}_N\varphi$ denotes the function equal to $\mathcal{I}_N^r\varphi$ on each Ω_r, $1 \le r \le R$.

The fully discrete problem is now constructed from (8)–(9) by using the Galerkin method combined with numerical integration. It reads as follows:

find $(u_N^k)_{0 \le k \le K}$ in $\mathbb{Y}_N \times (\mathbb{X}_N^0)^K$, satisfying

$$u_N^0 = \mathcal{I}_N u_0 \ \text{ in } \ \Omega, \tag{19}$$

and such that, for $1 \le k \le K$,

$$\forall v_N \in \mathbb{X}_N^0(\Omega), \quad a_N^k(u_N^k, v_N) = L_N^k(v_N), \tag{20}$$

where the bilinear forms $a_N^k(\cdot, \cdot)$, $1 \le k \le K$, are defined by

$$a_N^k(u_N, v_N) = ((u_N, v_N))_N + \tau_k((\lambda^k \nabla u_N, \nabla v_N))_N,$$

and the linear forms L_N^k are defined by

$$L_N^k(v_N) = ((u_N^{k-1}, v_N))_N + \tau_k((f^k, v_N))_N.$$

It follows from (18) combined with Cauchy-Schwarz inequalities, that the forms a_N^k and L_N^k are continuous on $\mathbb{X}_N^0 \times \mathbb{X}_N^0$ and $\mathbb{X}_N^0$ respectively, and a_N^k are coercive with norms bounded independently of N.

In all that follows, c stands for a generic constant which can vary from one line to the next one but is always independent of N. The proof of the next proposition is standard.

Proposition 4.1. *For any data f continuous on $\overline{\Omega} \times [0,T]$ and a continuous u_0 on $\overline{\Omega}$, problem (19)–(20) has a unique solution $(u_N^k)_{0\le k\le K}$ in $\mathbb{Y}_N \times (\mathbb{X}_N^0)^K$. Moreover this solution satisfies for a constant c independent of N*

$$(21)\quad [[(u_N^\ell)]]_k \le c\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac12}\left(\|\mathcal{I}_N u_0\|_{0,\Omega}^2+\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)\sum_{\ell=1}^k \frac{\tau_\ell}{\lambda_{\min}^\ell}\big\|\mathcal{I}_N f^\ell\big\|_{0,\Omega}^2\right)^{\frac12}.$$

Proof: Taking v_N equal to u_N^k in (20), we have thanks to Cauchy-Schwarz inequality

$$\begin{aligned}((u_N^k, u_N^k))_N + \tau_k((\lambda^k \nabla u_N^k, \nabla u_N^k))_N \le ((u_N^{k-1}, u_N^{k-1}))_N^{\frac12} \cdot ((u_N^k, u_N^k))_N^{\frac12}\\ + \tau_k((\mathcal{I}_N f^k, \mathcal{I}_N f^k))_N^{\frac12} \cdot ((u_N^k, u_N^k))_N^{\frac12}.\end{aligned}$$

Using (18), Poincaré-Friedrichs inequality and the inequality $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$, for each $\varepsilon > 0$, we obtain

$$\begin{aligned}&\frac12\big\|u_N^k\big\|_{0,\Omega}^2 + \tau_k((\lambda^k\nabla u_N^k, \nabla u_N^k))_N\\ &\qquad\le \frac12\big\|u_N^{k-1}\big\|_{0,\Omega}^2 + \frac{1}{2\varepsilon}\frac{c\tau_k}{\lambda_{\min}^k}\big\|\mathcal{I}_N f^k\big\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\tau_k\big\|(\lambda^k)^{\frac12}\nabla u_N^k\big\|_{0,\Omega}^2.\end{aligned}$$

Summing up on k, and using (19), we get

$$\begin{aligned}(22)\quad &\frac12\big\|u_N^k\big\|_{0,\Omega}^2 + \sum_{\ell=1}^k \tau_\ell((\lambda^\ell\nabla u_N^\ell, \nabla u_N^\ell))_N\\ &\qquad\le \frac12\big\|\mathcal{I}_N u_0\big\|_{0,\Omega}^2 + \frac{c}{2\varepsilon}\sum_{\ell=1}^k \frac{\tau_\ell}{\lambda_{\min}^\ell}\big\|\mathcal{I}_N f^\ell\big\|_{0,\Omega}^2 + \frac{\varepsilon}{2}\sum_{\ell=1}^k \tau_\ell\big\|(\lambda^\ell)^{\frac12}\nabla u_N^\ell\big\|_{0,\Omega}^2.\end{aligned}$$

On the other hand, thanks to (2) and (18), we have

$$\big\|(\lambda^\ell)^{\frac12}\nabla u_N^\ell\big\|_{0,\Omega}^2 \le \lambda_{\max}((\nabla u_N^\ell, \nabla u_N^\ell))_N \le \frac{\lambda_{\max}}{\lambda_{\min}}((\lambda^\ell\nabla u_N^\ell, \nabla u_N^\ell))_N, \quad 1\le \ell\le k,$$

so, (22) implies that

$$\frac{1}{2}\|u_N^k\|_{0,\Omega}^2 + \sum_{\ell=1}^{k}\tau_\ell\|(\lambda^\ell)^{\frac{1}{2}}\nabla u_N^\ell\|_{0,\Omega}^2 \leq \frac{1}{2}\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)\Big(\|\mathcal{I}_N u_0\|_{0,\Omega}^2 + \frac{c}{\varepsilon}\sum_{\ell=1}^{k}\frac{\tau_\ell}{\lambda_{\min}^\ell}\|\mathcal{I}_N f^\ell\|_{0,\Omega}^2 + \varepsilon\sum_{\ell=1}^{k}\tau_\ell\|(\lambda^\ell)^{\frac{1}{2}}\nabla u_N^\ell\|_{0,\Omega}^2\Big).$$

Finally, we choose $\varepsilon = \frac{\lambda_{\min}}{\lambda_{\max}+\lambda_{\min}}$ to obtain (21). □

5. Error estimate

We now wish to establish the error estimate between the solution $(u^k)_{0\leq k\leq K}$ of problem (8)–(9) and the solution $(u_N^k)_{0\leq k\leq K}$ of problem (19)–(20).

Let $\Pi_N^{1,0}$ denote the orthogonal projection operator from $H_0^1(\Omega)$ onto $\mathbb{X}_N^0$ for the scalar product associated with the norm $|\cdot|_{1,\Omega}$. For $0\leq \ell\leq k, \Pi_N^{1,0}u^\ell$ will be denoted by p_N^ℓ.

Proposition 5.1. *Assume that f and u_0 are continuous on $\overline{\Omega}\times[0,T]$ and $\overline{\Omega}$ respectively. Then the following estimate holds for the error between the solution $(u^k)_{0\leq k\leq K}$ of problem (8)–(9) and the solution $(u_N^k)_{0\leq k\leq K}$ of problem (19)–(20)*

$$[[(u^\ell - u_N^\ell)]]_k \leq c\Big([[(u^\ell - p_{N-1}^\ell)]]_k + \Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac{1}{2}}\Big(\|u_0 - p_{N-1}^0\|_{0,\Omega} + \|u_0 - \mathcal{I}_N u_0\|_{0,\Omega} + \Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac{1}{2}}\sum_{\ell=1}^{k}\Big(\frac{\tau_\ell}{\lambda_{\min}^\ell}\Big)^{\frac{1}{2}}(E_{N,\ell}^{a,1}+E_{N,\ell}^{a,2}+E_{N,\ell}^{f})\Big)\Big), \tag{23}$$

where the quantities $E_{N,\ell}^{a,1}$, $E_{N,\ell}^{a,2}$ and $E_{N,\ell}^{f}$ are defined by

$$E_{N,\ell}^{a,1} = \sup_{v_N\in\mathbb{X}_N^0}\frac{(\frac{u^\ell-u^{\ell-1}}{\tau_\ell}, v_N) - ((\frac{p_{N-1}^\ell - p_{N-1}^{\ell-1}}{\tau_\ell}, v_N))_N}{|v_N|_{1,\Omega}},$$

$$E_{N,\ell}^{a,2} = \sup_{v_N\in\mathbb{X}_N^0}\frac{(\lambda^\ell\nabla u^\ell, \nabla v_N) - ((\lambda^\ell\nabla p_{N-1}^\ell, \nabla v_N))_N}{|v_N|_{1,\Omega}},$$

$$E_{N,\ell}^{f} = \sup_{v_N\in\mathbb{X}_N^0}\frac{(f^\ell, v_N) - ((f^\ell, v_N))_N}{|v_N|_{1,\Omega}}.$$

Proof: We have

$$[[(u^\ell - u_N^\ell)]]_k \le [[(u^\ell - p_{N-1}^\ell)]]_k + [[(u_N^\ell - p_{N-1}^\ell)]]_k,$$

so we have to estimate the term $[[(u_N^\ell - p_{N-1}^\ell)]]_k$. It follows from (9) and (20) that

$$((u_N^k - p_{N-1}^k, v_N))_N + \tau_k((\lambda^k \nabla(u_N^k - p_{N-1}^k), \nabla v_N))_N = ((u_N^{k-1} - p_{N-1}^{k-1}, v_N))_N + \tau_k M_N^k(v_N),$$

where M_N^k is the linear form on $\mathbb{X}_N^0$ defined by

$$\begin{aligned} M_N^k(v_N) = (&\frac{u^k - u^{k-1}}{\tau_k}, v_N) - ((\frac{p_{N-1}^k - p_{N-1}^{k-1}}{\tau_k}, v_N))_N \\ &+ (\lambda^k \nabla u^k, \nabla v_N) - ((\lambda^k \nabla p_{N-1}^k, \nabla v_N))_N \\ &+ ((f^k, v_N))_N - (f^k, v_N). \end{aligned} \tag{24}$$

Due to the Riesz's theorem, there exists a unique polynomial F_N^k in $\mathbb{X}_N^0$ such that

$$\forall v_N \in \mathbb{X}_N^0, \;\; M_N^k(v_N) = ((F_N^k, v_N))_N.$$

Thus the family $(u_N^k - p_{N-1}^k)_{0\le k\le K}$ is a solution of the discrete problem (19)-(20) with $\mathcal{I}_N u_0 - p_{N-1}^0$ instead of $\mathcal{I}_N u_0$ and F_N^k instead of f^k. So we proceed as in the proof of Proposition 4.1. Taking v_N equal to $u_N^k - p_{N-1}^k$, using Cauchy-Schwarz inequality and the fact that the form M_N^k is linear on the finite dimensional space $\mathbb{X}_N^0$, we get

$$\begin{aligned} &((u_N^k - p_{N-1}^k, u_N^k - p_{N-1}^k))_N + \tau_k((\lambda^k \nabla(u_N^k - p_{N-1}^k), \nabla(u_N^k - p_{N-1}^k)))_N \\ &\le ((u_N^{k-1} - p_{N-1}^{k-1}, u_N^{k-1} - p_{N-1}^{k-1}))_N^{\frac{1}{2}} \cdot ((u_N^k - p_{N-1}^k, u_N^k - p_{N-1}^k))_N^{\frac{1}{2}} \\ &\quad + \tau_k \sup_{v_N \in \mathbb{X}_N^0} \frac{((F_N^k, v_N))_N}{|v_N|_{1,\Omega}} \cdot |u_N^k - p_{N-1}^k|_{1,\Omega} \end{aligned}$$

using (18), Poincaré-Friedrichs inequality and the inequality $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$, for each $\varepsilon > 0$, summing up on k, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_N^k - p_{N-1}^k\|_{0,\Omega}^2 + \sum_{\ell=1}^k \tau_\ell((\lambda^\ell \nabla(u_N^\ell - p_{N-1}^\ell), \nabla(u_N^\ell - p_{N-1}^\ell)))_N \\ &\le \frac{1}{2}\|\mathcal{I}_N u_0 - p_{N-1}^0\|_{0,\Omega}^2 + \frac{1}{2\varepsilon}\sum_{\ell=1}^k \frac{\tau_\ell}{\lambda_{\min}^\ell}(\varphi^\ell)^2 \\ &\quad + \frac{\varepsilon}{2}\sum_{\ell=1}^k \tau_\ell \|(\lambda^\ell)^{\frac{1}{2}} \nabla(u_N^\ell - p_{N-1}^\ell)\|_{0,\Omega}^2, \end{aligned}$$

where $\varphi^\ell = \sup_{v_N \in \mathbb{X}_N^0} \frac{((F_N^\ell, v_N))_N}{|v_N|_{1,\Omega}}$. Consequently, following the same arguments as in the end of the proof of Proposition 4.1, we get

$$[[(u_N^\ell - p_{N-1}^\ell)]]_k \leq c\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac{1}{2}}\Big(\|\mathcal{I}_N u_0 - p_{N-1}^0\|_{0,\Omega}^2 + \Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)\sum_{\ell=1}^{k}\frac{\tau_\ell}{\lambda_{\min}^\ell}(\varphi^\ell)^2\Big)^{\frac{1}{2}}.$$

We conclude the proof thanks to (24). □

In order to estimate the term $E_{N,\ell}^{a,1}$, we denote by w^ℓ the quantity $\frac{u^\ell - u^{\ell-1}}{\tau_\ell}$ and we observe that $\frac{p_{N-1}^\ell - p_{N-1}^{\ell-1}}{\tau_\ell} = \Pi_{N-1}^{1,0} w^\ell$, so as a consequence of the exactness property (17), the terms $\big(\Pi_{N-1}^{1,0} w^\ell, v_N\big)_{0,\Omega}$ and $((\Pi_{N-1}^{1,0} w^\ell, v_N))_N$ coincide and thus, using Poincaré-Friedrichs inequality, we get

$$E_{N,\ell}^{a,1} \leq c\big\|w^\ell - \Pi_{N-1}^{1,0} w^\ell\big\|_{0,\Omega}. \tag{25}$$

Now, in order to evaluate the term $E_{N,\ell}^{a,2}$, we define $\tilde{N}$ as the integer part of $\frac{N-1}{2}$, so as a consequence of the exactness property (17), we have for any $v_N \in \mathbb{X}_N^0$

$$\begin{aligned}
&(\lambda^\ell \nabla u^\ell, \nabla v_N) - ((\lambda^\ell \nabla p_{N-1}^\ell, \nabla v_N))_N \\
&\quad = (\lambda^\ell(\nabla u^\ell - \nabla p_{N-1}^\ell), \nabla v_N) \\
&\qquad + \sum_{r=1}^{R}\Big(\int_{\Omega_r}(\lambda^\ell \nabla p_{N-1}^\ell - \mathcal{I}_{\tilde{N}}^r \lambda^\ell \nabla p_{\tilde{N}}^\ell)(\boldsymbol{x}) \cdot \nabla v_N(\boldsymbol{x})\, d\boldsymbol{x} \\
&\qquad + (\mathcal{I}_{\tilde{N}}^r \lambda^\ell \nabla p_{\tilde{N}}^\ell - \lambda^\ell \nabla p_{N-1}^\ell, \nabla v_N)_N^r\Big).
\end{aligned} \tag{26}$$

Due to Cauchy-Schwarz inequality and by using the notation $\lambda_{\max}^\ell := \sup_{\boldsymbol{x}\in\overline{\Omega}} \lambda(\boldsymbol{x}, t_\ell)$, the first term in the right hand side of (26) can be estimated as

$$(\lambda^\ell(\nabla u^\ell - \nabla p_{N-1}^\ell), \nabla v_N) \leq \lambda_{\max}^\ell |u^\ell - p_{N-1}^\ell|_{1,\Omega} |v_N|_{1,\Omega}.$$

Similar arguments also lead to

$$\begin{aligned}
&\sum_{r=1}^{R}\int_{\Omega_r}(\lambda^\ell \nabla p_{N-1}^\ell - \mathcal{I}_{\tilde{N}}^r \lambda^\ell \nabla p_{\tilde{N}}^\ell)(\boldsymbol{x}) \cdot \nabla v_N(\boldsymbol{x})\, d\boldsymbol{x} \\
&\qquad \leq \Big(\lambda_{\max}^\ell\big(|u^\ell - p_{N-1}^\ell|_{1,\Omega} + \sum_{r=1}^{R}\big|u^\ell - p_{\tilde{N}}^\ell\big|_{1,\Omega_r}\big) \\
&\qquad\qquad + \sum_{r=1}^{R}\big\|\lambda^\ell - \mathcal{I}_{\tilde{N}}^r \lambda^\ell\big\|_{\infty,\Omega_r}\big|p_{\tilde{N}}^\ell\big|_{1,\Omega_r}\Big)|v_N|_{1,\Omega}.
\end{aligned}$$

Now, thanks to Cauchy-Schwarz inequality and (18), the last term in the right hand side of (26) can be estimated as follows

$$\begin{aligned}\sum_{r=1}^{R}&(\mathcal{I}_{\tilde{N}}^{r}\lambda^{\ell}\nabla p_{\tilde{N}}^{\ell}-\lambda^{\ell}\nabla p_{N-1}^{\ell},\nabla v_{N})_{N}^{r}\\&\leq c\Big(\lambda_{\max}^{\ell}\big(|u^{\ell}-p_{N-1}^{\ell}|_{1,\Omega}+\sum_{r=1}^{R}\big|u^{\ell}-p_{\tilde{N}}^{\ell}\big|_{1,\Omega_r}\big)\\&\quad+\sum_{r=1}^{R}\big\|\lambda^{\ell}-\mathcal{I}_{\tilde{N}}^{r}\lambda^{\ell}\big\|_{\infty,\Omega_r}\big|p_{\tilde{N}}^{\ell}\big|_{1,\Omega}\Big)|v_N|_{1,\Omega}.\end{aligned}$$

So that

$$\begin{aligned}E_{N,\ell}^{a,2}\leq c\bigg(&\lambda_{\max}^{\ell}\Big(|u^{\ell}-p_{N-1}^{\ell}|_{1,\Omega}+\big|u^{\ell}-p_{\tilde{N}}^{\ell}\big|_{1,\Omega}\Big)\\&+\sum_{r=1}^{R}\big\|\lambda^{\ell}-\mathcal{I}_{\tilde{N}}^{r}\lambda^{\ell}\big\|_{\infty,\Omega_r}|u^{\ell}|_{1,\Omega}\bigg),\end{aligned}\tag{27}$$

since $\big|p_{\tilde{N}}^{\ell}\big|_{1,\Omega}=\big|\Pi_{\tilde{N}}^{1,0}u^{\ell}\big|_{1,\Omega}\leq\big\|\Pi_{\tilde{N}}^{1,0}\big\|_{\mathcal{L}(H_0^1(\Omega),\mathbb{X}_{\tilde{N}}^0(\Omega))}|u^{\ell}|_{1,\Omega}=|u^{\ell}|_{1,\Omega}$.

It remains to estimate the term $E_{N,\ell}^{f}$. Since f is only in the space $L^2(\Omega)$, we introduce the orthogonal projection operator Π_N^r from $L^2(\Omega_r)$ onto $\mathbb{P}_N(\Omega_r)$. Indeed, using (17) leads to, for any v_N in $\mathbb{X}_N^0$,

$$\begin{aligned}&\int_{\Omega_r}f^{\ell}(\boldsymbol{x})\cdot v_N(\boldsymbol{x})d\boldsymbol{x}-(f^{\ell},v_N)_N^r\\&\qquad=\int_{\Omega_r}(f^{\ell}-\Pi_{N-1}^{r}f^{\ell})(\boldsymbol{x})\cdot v_N(\boldsymbol{x})d\boldsymbol{x}-(\mathcal{I}_N^r f^{\ell}-\Pi_{N-1}^{r}f^{\ell},v_N)_N^r,\end{aligned}$$

so that, owing to (18) and Poincaré-Friedrichs inequality, we obtain

$$E_{N,\ell}^{f}\leq c\sum_{r=1}^{R}\Big(\big\|f^{\ell}-\Pi_{N-1}^{r}f^{\ell}\big\|_{0,\Omega_r}+\big\|f^{\ell}-\mathcal{I}_N^r f^{\ell}\big\|_{0,\Omega_r}\Big).\tag{28}$$

Now, to make complete the evaluation of $E_{N,\ell}^{a,1}$, $E_{N,\ell}^{a,2}$ and $E_{N,\ell}^{f}$, we need the following results. First, we recall from [2, Theorem 7.1 and Theorem 14.2] the approximation properties of the operators Π_N^r and $\mathcal{I}_N^r$, $1\leq r\leq R$: for any function φ in $H^s(\Omega_r)$, $s\geq 0$

$$\big\|\varphi-\Pi_N^r\varphi\big\|_{0,\Omega_r}\leq cN^{-s}\|\varphi\|_{s,\Omega_r},\tag{29}$$

and for any function φ in $H^s(\Omega_r)$, $s>\frac{d}{2}$

$$\big\|\varphi-\mathcal{I}_N^r\varphi\big\|_{0,\Omega_r}\leq cN^{-s}\|\varphi\|_{s,\Omega_r}.\tag{30}$$

The following result is derived from [3, Lemma VI.2.5] thanks to an interpolation argument, for any real number $s \geq 1$, and any function φ in $H_0^1(\Omega)$ such that each $\varphi|_{\Omega_r}$, $1 \leq r \leq R$, belongs to $H^s(\Omega_r)$

$$\|\varphi - \Pi_N^{1,0}\varphi\|_{1,\Omega} \leq cN^{1-s} \sum_{r=1}^{R} \|\varphi\|_{s,\Omega_r}. \tag{31}$$

Finally, in order to evaluate the term $\|\lambda^\ell - \mathcal{I}_N^r \lambda^\ell\|_{\infty,\Omega_r}$, we introduce the Gauss-Lobatto interpolation operator denoted by I_N: for any continuous function φ on Λ^d, $I_N\varphi$ is the only polynomial in $\mathbb{P}_N(\Lambda^d)$ which satisfies $(I_N\varphi)(\xi_i,\xi_j) = \varphi(\xi_i,\xi_j)$, $0 \leq i,j \leq N$ when $d = 2$, and $(I_N\varphi)(\xi_i,\xi_j,\xi_p) = \varphi(\xi_i,\xi_j,\xi_p)$, $0 \leq i,j,p \leq N$ when $d = 3$.

We have the identity

$$I_N = \begin{cases} i_N^x \circ i_N^y & \text{if} \;\; d = 2, \\ i_N^x \circ i_N^y \circ i_N^z & \text{if} \;\; d = 3, \end{cases}$$

where i_N^x (resp. i_N^y, i_N^z) denotes the Lagrange interpolation operator i_N (at the nodes ξ_j, $0 \leq j \leq N$) with respect to the variable x (resp. y, z).

We need the following results, derived from a Gagliardo-Nirenberg inequality.

Lemma 5.2. *For any function φ in $H^s(\Lambda^d)$, the following estimates hold:*

$$\|\varphi - i_N\varphi\|_{\infty,\Lambda^d} \leq cN^{\frac{1}{2}-s}\|\varphi\|_{s,\Lambda^d}, \quad \textit{for} \;\; d = 1 \;\; \textit{and} \;\; s > \frac{3}{4}, \tag{32}$$

and

$$\|\varphi - I_N\varphi\|_{\infty,\Lambda^d} \leq cN^{1-s}\|\varphi\|_{s,\Lambda^d}, \quad \textit{for} \;\; d = 2 \;\; \textit{and} \;\; s > \frac{5}{4}. \tag{33}$$

For the proof see Lemmas 2.2.3 and 2.2.5 in [7]. Using the same arguments, we can derive the analogue of this result in dimension 3.

Lemma 5.3. *For any real number $s > 2$, and for any function φ in $H^s(\Lambda^3)$, the following estimate holds*

$$\|\varphi - I_N\varphi\|_{\infty,\Lambda^3} \leq cN^{\frac{3}{2}-s}\|\varphi\|_{s,\Lambda^3}.$$

Proof: We note that we have the identity

$$\varphi - I_N\varphi = (\varphi - i_N^x \circ i_N^y\varphi) + (\varphi - i_N^z\varphi) - (id - i_N^x \circ i_N^y) \circ (id - i_N^z)\varphi,$$

so we have to estimate each term in the right hand side of this equality. A Gagliardo-Nirenberg inequality leads to, for any $\varepsilon > 0$

$$\begin{aligned} \|\varphi - i_N^x \circ i_N^y\varphi\|_{\infty,\Lambda^3} &= \|\varphi - i_N^x \circ i_N^y\varphi\|_{L^\infty(\Lambda;L^\infty(\Lambda^2))} \\ &\leq \|\varphi - i_N^x \circ i_N^y\varphi\|^{\frac{1}{2}}_{H^{\frac{1}{2}-\varepsilon}(\Lambda;L^\infty(\Lambda^2))} \|\varphi - i_N^x \circ i_N^y\varphi\|^{\frac{1}{2}}_{H^{\frac{1}{2}+\varepsilon}(\Lambda;L^\infty(\Lambda^2))}. \end{aligned}$$

It follows from (33) that for $s > \frac{7}{4}$

$$\|\varphi - i_N^x \circ i_N^y \varphi\|_{\infty,\Lambda^3} \le cN^{\frac{3}{2}-s} \|\varphi\|^{\frac{1}{2}}_{H^{\frac{1}{2}-\varepsilon}(\Lambda; H^{s-\frac{1}{2}+\varepsilon}(\Lambda^2))} \|\varphi\|^{\frac{1}{2}}_{H^{\frac{1}{2}+\varepsilon}(\Lambda; H^{s-\frac{1}{2}-\varepsilon}(\Lambda^2))}.$$

The same arguments lead to the following estimate for $s > \frac{7}{4}$

$$\begin{aligned}\|\varphi - i_N^z \varphi\|_{\infty,\Lambda^3} \le cN^{\frac{3}{2}-s} &\|\varphi\|^{\frac{1}{4}}_{H^{\frac{1}{2}-\varepsilon}(\Lambda; H^{\frac{1}{2}-\varepsilon}(\Lambda; H^{s-1+2\varepsilon}(\Lambda)))} \\ &\times \|\varphi\|^{\frac{1}{4}}_{H^{\frac{1}{2}-\varepsilon}(\Lambda; H^{\frac{1}{2}+\varepsilon}(\Lambda; H^{s-1}(\Lambda)))} \\ &\times \|\varphi\|^{\frac{1}{4}}_{H^{\frac{1}{2}+\varepsilon}(\Lambda; H^{\frac{1}{2}-\varepsilon}(\Lambda; H^{s-1}(\Lambda)))} \|\varphi\|^{\frac{1}{4}}_{H^{\frac{1}{2}+\varepsilon}(\Lambda; H^{\frac{1}{2}+\varepsilon}(\Lambda; H^{s-1-2\varepsilon}(\Lambda)))}.\end{aligned}$$

For the last term, using (33) and (32) respectively for $s' > \frac{5}{4}$ and $s > s' + \frac{3}{4}$, we obtain

$$\begin{aligned}\|(id - i_N^x \circ i_N^y) \circ (id - i_N^z)\varphi\|_{L^\infty(\Lambda; L^\infty(\Lambda^2))} &\le cN^{1-s'} \|(\varphi - i_N^z \varphi)\|_{L^\infty(\Lambda; H^{s'}(\Lambda^2))} \\ &\le cN^{\frac{3}{2}-s} \|\varphi\|_{H^{s-s'}(\Lambda; H^{s'}(\Lambda^2))}.\end{aligned}$$

We conclude by using the embeddings $H^s(\Lambda^2) \subset H^r(\Lambda; H^{s-r}(\Lambda))$ and $H^s(\Lambda^3) \subset H^r(\Lambda; H^{s-r}(\Lambda^2))$ for $0 \le r \le s$. □

Theorem 5.4. *Assume that the data f belong to $\mathcal{C}^0([0,T], H^\sigma(\Omega))$ for a real number $\sigma > \frac{d}{2}$, λ belongs to $\mathcal{C}^0([0,T], H^\nu(\Omega))$ for a real number $\nu > \frac{3d-1}{4}$, u_0 is continuous on $\overline{\Omega}$ and the solution $(u^k)_{0\le k\le K}$ of problem (8)–(9) is such that the restrictions $u^k|_{\Omega_r}$, $1 \le r \le R$, belong to $H^s(\Omega_r)$ for a real number $s \ge 1$. Then the following a priori error estimate holds*

$$\begin{aligned}[[(u^\ell - u_N^\ell)]]_k \le c \sum_{r=1}^R \Bigg(& N^{1-s}\Big(\|u^k\|_{s,\Omega_r} + (|\tau|\lambda_{\max})^{\frac{1}{2}} \sum_{\ell=1}^k \|u^\ell\|_{s,\Omega_r} \Big) \\ & + N^{1-s}\Big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac{1}{2}} \\ & \times \Big(\|u_0\|_{s,\Omega_r} + \Big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac{1}{2}} \Big(\frac{|\tau|}{\lambda_{\min}}\Big)^{\frac{1}{2}} \sum_{\ell=1}^k \big(\|w^\ell\|_{s,\Omega_r} + \lambda_{\max} \|u^\ell\|_{s,\Omega_r} \big) \Big) \\ & + \Big(1 + \frac{\lambda_{\max}}{\lambda_{\min}}\Big) \Big(\frac{|\tau|}{\lambda_{\min}}\Big)^{\frac{1}{2}} \lambda_{\max} \\ & \times \sum_{\ell=1}^k \Big(N^{\frac{d}{2}-\nu} \Big(\sum_{r=1}^R \sum_{\ell=1}^k \|\lambda^\ell\|_{H^\nu(\Omega_r)} \Big) \|u^\ell\|_{1,\Omega_r} + N^{-\sigma} \|f^\ell\|_{\sigma,\Omega_r} \Big) \Bigg).\end{aligned}$$

Note that this estimate is optimal in the sense that if λ is constant, we find the results obtained for the usual heat equation.

PROOF: The bound for the terms in the right hand sides of (25), (27) and (28) obviously follows from (29), (30), (31), Lemma 5.2 and Lemma 5.3.

Back to (23), by the definition of the norm $[[\cdot]]_k$, we observe that

$$[[(u^\ell - p^\ell_{N-1})]]_k \le \|u^k - p^k_{N-1}\|_{0,\Omega} + \sum_{\ell=1}^{k} \big(\tau_\ell \lambda^\ell_{\max}\big)^{\frac12} |u^\ell - p^\ell_{N-1}|_{1,\Omega}.$$

Using once more the approximation properties (30) and (31), we obtain

$$\begin{aligned}
[[(u^\ell - u^\ell_N)]]_k \le c\Bigg(& N^{1-s}\Big(\sum_{r=1}^{R} \|u^k\|_{s,\Omega_r} + \sum_{\ell=1}^{k} \big(\tau_\ell \lambda^\ell_{\max}\big)^{\frac12} \sum_{r=1}^{R} \|u^\ell\|_{s,\Omega_r}\Big) \\
& + N^{1-s}\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac12}\Big(\sum_{r=1}^{R}\|u_0\|_{s,\Omega_r} + \Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)^{\frac12}\sum_{\ell=1}^{k}\Big(\frac{\tau_\ell}{\lambda^\ell_{\min}}\Big)^{\frac12}\sum_{r=1}^{R}\|w^\ell\|_{s,\Omega_r}\Big) \\
& + (N^{1-s} + \tilde N^{1-s})\Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)\sum_{\ell=1}^{k}\Big(\frac{\tau_\ell}{\lambda^\ell_{\min}}\Big)^{\frac12}\lambda^\ell_{\max}\sum_{r=1}^{R}\|u^\ell\|_{s,\Omega_r} \\
& + \Big(1+\frac{\lambda_{\max}}{\lambda_{\min}}\Big)\sum_{\ell=1}^{k}\Big(\frac{\tau_\ell}{\lambda^\ell_{\min}}\Big)^{\frac12}\lambda^\ell_{\max}\Big(\tilde N^{\frac d2 - \nu}\Big(\sum_{r=1}^{R}\sum_{\ell=1}^{k}\|\lambda^\ell\|_{H^\nu(\Omega_r)}\Big)\sum_{r=1}^{R}\|u^\ell\|_{1,\Omega_r} \\
& + N^{-\sigma}\sum_{r=1}^{R}\|f^\ell\|_{H^\sigma(\Omega_r)}\Big)\Bigg),
\end{aligned}$$

taking into account the relationship between $\tilde N$ and N, we get the desired result. □

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Université de Jijel, Laboratoire de Mathématiques Pures et Appliquées, B.P. 98, Ouled Aissa, 18000 Jijel, Algeria

E-mail: yasmina-daikh@univ-jijel.dz
wided-chikouche@univ-jijel.dz

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