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NORMAL NUMBER CONSTRUCTIONS FOR CANTOR SERIES WITH SLOWLY GROWING BASES

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Abstract. Let $Q = (q_n)_{n=1}^{\infty}$ be a sequence of bases with $q_i \geqslant 2$. In the case when the q_i are slowly growing and satisfy some additional weak conditions, we provide a construction of a number whose Q-Cantor series expansion is both Q-normal and Q-distribution normal. Moreover, this construction will result in a computable number provided we have some additional conditions on the computability of Q, and from this construction we can provide computable constructions of numbers with atypical normality properties.

Keywords: Cantor series; normal number

MSC 2010: 11K16, 11A63

1. Introduction

A real number x has a unique base b expansion of the form

$$x = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{b^n}, \tag{1.1}$$

where $a_0 = \lfloor x \rfloor$ and the digits a_n satisfy $a_n \in \{0, 1, 2, \ldots, b-1\}$ and $a_n \neq b-1$ infinitely often. This number is said to be normal to base b if for every finite sequence $(c_j)_{j=1}^{k}$ with $c_j \in \{0, 1, 2, \ldots, b-1\}$, we have

$$\lim_{n\to\infty} \frac{\#\{1 \leqslant i \leqslant n\colon\ c_j = a_{i+j-1},\ 1 \leqslant j \leqslant k\}}{n} = \frac{1}{b^k}.$$

This definition says that a number is normal when each string of digits appears with the frequency one would expect if the digits were chosen at random. Equivalently, one could say that the sequence $(b^k x)_{k=0}^{\infty}$ is uniformly distributed modulo 1.

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Although almost all real numbers are normal to base b, very few examples of such numbers are known, and those examples that are known are numbers that were explicitly constructed to be normal. One of the very first such constructions was due to Champernowne (see [7]), who showed that the number

$$0.12345678910111213141516\ldots,$$

formed by concatenating all the integers, is normal to base 10.

There are, of course, many different ways of representing a real number, such as continued fraction expansions and beta expansions, each with their own definitions of normality. Here, we are interested in the Q-Cantor series expansion. The study of normal numbers and other statistical properties of real numbers with respect to large classes of Cantor series expansions was first done by Erdős and Rényi in [8] and [9], by Rényi in [16], [15], and [14], and by Turán in [18].

The Q-Cantor series expansions, first studied by Cantor in [6], are a natural generalization of the b-ary expansions.[1] A basic sequence is a sequence of integers greater than or equal to 2. Given a basic sequence $Q = (q_n)_{n=1}^{\infty}$, the *Q-Cantor series expansion* of a real number x is the (unique)[2] expansion of the form

$$x = E_0 + \sum_{n=1}^{\infty} \frac{E_n}{q_1 q_2 \cdots q_n}, \tag{1.2}$$

where $E_0 = \lfloor x \rfloor$ and E_n is in $\{0, 1, \ldots, q_n - 1\}$ for $n \geqslant 1$ with $E_n \neq q_n - 1$ infinitely often. We abbreviate (1.2) with the notation $x = E_0.E_1E_2E_3\ldots$ with respect to Q.

Definitions of normality for Q-Cantor series require a few more definitions. Given a block of digits $B = [B_1, B_2, B_3, \ldots, B_k]$, define

$$N_n^Q(B, x) = \#\{1 \leqslant i \leqslant n\colon\ E_{i+j-1} = B_j,\ 1 \leqslant j \leqslant k\},$$

so that $N_n^Q(B, x)$ counts the number of times a given block of digits appears in the Q-cantor expansion for x up to the nth place. Moreover let

$$I_i(B) = \begin{cases} 1, & B_j < q_{i+j-1},\ 1 \leqslant j \leqslant k, \\ 0, & \text{otherwise}, \end{cases}$$

so that $I_i(B)$ detects whether or not the digit block B can even occur at the ith place in the Q-Cantor expansion for some point x.

[1] Cantor's motivation to study the Cantor series expansions was to extend the well known proof of the irrationality of the number $\mathrm{e} = \sum 1/n!$ to a larger class of numbers. Results along these lines may be found in the monograph of Galambos [10].

[2] Uniqueness can be proven in the same way as for the b-ary expansions.

We let $|B| = k$ denote the length of the block B. For a block B of length k define

$$Q_n(B) = \sum_{i=1}^{n} \frac{I_i(B)}{q_i q_{i+1} \cdots q_{i+k-1}},$$

which may be interpreted as the expected number of times to see the block B in the first n digits of a Q-Cantor series expansion if every digit E_i is chosen at random from the set $\{0, 1, \ldots, q_i - 1\}$. We also define

$$T_{Q,n}(x) = q_n q_{n-1} \ldots q_1 x \pmod 1.$$

A real number x is *Q-normal*[3] if for all blocks B regardless of length such that $\lim_{n\to\infty} Q_n(B) = \infty$, we also have

$$\lim_{n\to\infty} \frac{N_n^Q(B, x)}{Q_n(B)} = 1.$$

Let $\mathscr{N}(Q)$ be the set of Q-normal numbers. As with the definition for base b normality, this says, in essence, that the number of times a block of digits appears is the expected frequency if the digits were chosen at random. In fact, if we let $q_i = b$ for all i, this definition is precisely the definition for a base b normal number. The real number x is *Q-ratio normal* (here we write $x \in \mathscr{RN}(Q)$) if for all blocks B_1 and B_2 of equal length where $\lim_{n\to\infty} Q_n(B_1) = \lim_{n\to\infty} Q_n(B_2) = \infty$ we have

$$\lim_{n\to\infty} \frac{N_n^Q(B_1, x)}{N_n^Q(B_2, x)} = 1.$$

A real number x is *Q-distribution normal* if the sequence $(T_{Q,n}(x))_{n=0}^{\infty}$ is uniformly distributed mod 1. Again, if $q_i = b$ for all i, this definition is precisely the equivalent definition for a base b normal number. Let $\mathscr{DN}(Q)$ be the set of Q-distribution normal numbers. The relationship between $\mathscr{N}(Q)$, $\mathscr{RN}(Q)$, and $\mathscr{DN}(Q)$ is discussed in [1] and [12] but is not fully understood: for example, unlike for base b expansions, there exist Q such that $\mathscr{N}(Q)$ and $\mathscr{DN}(Q)$ are not the same set, although it is not known for which Q this holds.

There are a number of classical results about base b normality which do not yet have analogues for Q-Cantor expansion normality. Even the simplest question, asking for an example of a Q-normal number for any reasonable Q, is unanswered in many cases. Altomare and Mance in [4] and Mance independently in [13] started with

[3] We choose to take a slightly different definition for Q-normality than is used elsewhere in the literature. Our definition is more appropriate for bounded basic sequences.

a set of data satisfying certain conditions and used this to generate both a sequence Q and a number x that was Q-normal. In their constructions, the sequence Q was constant for very long stretches at a time. Most other constructions that have been found thus far, such as those in [3], also put very stringent restrictions on what Q are allowed. Perhaps the most general Q-normal construction comes from [2]: there, the authors show that if Q is eventually periodic, then there is some integer b such that being Q-normal is equivalent to being base b normal, and thus constructions of base b normal numbers give Q-normal numbers in this case.

The first main result of this paper is the following, which provides a Q-normal number construction for a much broader set of basic sequences Q.

Theorem 1.1. *Let Q be a basic sequence that satisfies the following two conditions:*

- ▷ *Q is slowly growing; that is, if we let $q(n) = \max_{i \leqslant n} q_i$, then $q(n) = n^{o(1)}$; and,*
- ▷ *for or any block of digits B such that $\lim_{n\to\infty} Q_n(B) = \infty$ we have*

$$\lim_{n\to\infty} \frac{Q_n(B)}{n \log q(n)/ \log n} = \infty.$$

Then the number x_Q constructed in Section 2 is both Q-normal and Q-distribution normal.

The number x_Q that we construct is an explicit example. To define what we mean by an explicit example, we bring in some definitions from recursion theory. A real number x is *computable* if there exists $b \in \mathbb{N}$ with $b \geqslant 2$ and a total recursive function $f : \mathbb{N} \to \mathbb{N}$ that calculates the digits of x in base b. A sequence of real numbers (x_n) is *computable* if there exists a total recursive function $f : \mathbb{N}^2 \to \mathbb{Z}$ such that for all m, n we have that $(f(m, n) - 1)/m < x_n < (f(m, n) - 1)/m$.

Sierpiński gave an example of an absolutely normal number that is not computable in [17]. The authors feel that examples such as Sierpiński's are not fully explicit since they are not computable real numbers, unlike Champernowne's number. Turing gave the first example of a computable absolutely normal number in an unpublished manuscript. This paper may be found in his collected works [19]. See [5] by Becher, Figueira, and Picchi for further discussion.

We will also show the following:

Theorem 1.2. *If Q satisfies the conditions of Theorem 1.1 and $(q_n)_{n=1}^{\infty}$ and $(q(n))_{n=1}^{\infty}$ are computable sequences of integers, then x_Q is computable.*

In [12] the second author showed that for any basic sequence Q that is infinite in limit such that $Q_n(B) \to \infty$ for each admissable block B, the set $\mathscr{N}(Q) \setminus \mathscr{DN}(Q)$ is nonempty. He also showed that $\mathscr{RN}(Q) \setminus \mathscr{N}(Q)$ is nonempty only assuming Q is infinite in limit. In [1] the first and second authors improved this result and showed that if Q is infinite in limit, the set $\mathscr{RN}(Q) \cap \mathscr{DN}(Q) \setminus \mathscr{N}(Q)$ has full Hausdorff dimension.

Along these lines we will be able to provide constructions of computable real numbers that are in sets such as $\mathscr{N}(Q) \setminus \mathscr{DN}(Q)$ and $\mathscr{RN}(Q) \cap \mathscr{DN}(Q) \setminus \mathscr{N}(Q)$ under the same or slightly stronger assumptions than those of Theorem 1.2.

Notation. We will use asymptotic notations with their standard meaning. By $f(x) = O(g(x))$ or, equivalently, $f(x) \ll g(x)$, we mean that there is some constant C such that $|f(x)| \leqslant Cg(x)$. By $f(x) \asymp g(x)$, we mean $f(x) = O(g(x))$ and $g(x) = O(f(x))$. By $f(x) = o(g(x))$, we mean that $\lim_{x\to\infty}(f(x)/g(x)) = 0$. By $f(x) \sim g(x)$, we mean that $f(x) = g(x)(1 + o(1))$ or, equivalently, $\lim_{x\to\infty}(f(x)/g(x)) = 1$.

2. The construction

We need some additional definitions. Given two blocks of integers $A = [a_1, a_2, \ldots, a_k]$ and $B = [b_1, b_2, \ldots, b_k]$ (which could be blocks of digits, E_i, or blocks of bases, q_i), we say that $A < B$ if $a_i < b_i$ for $1 \leqslant i \leqslant k$ (and we make an analogous definition for $A \leqslant B$).

For a given integer r, let n_r denote the smallest integer n such that $(q(n)^2+1)^r \leqslant n$. By the assumption that $q_n = n^{o(1)}$, the integer n_r always exists. Let $(N_r)_{r=0}^{\infty}$ be an increasing sequence of nonnegative integers defined so that $N_1 = 0$ and all the N_{r+1}'s are defined inductively as being the greatest integer less than n_{r+1} such that $N_{r+1} - N_r$ is divisible by r. In particular, we will have $N_r = n_r + O(r)$.

Divide the bases of Q from the $(N_r + 1)$st base to the (N_{r+1})th base into $(N_{r+1} - N_r)/r$ blocks of r consecutive bases, namely the blocks

$$R_{j,r} = [q_{N_r+jr+1}, q_{N_r+jr+2}, \ldots, q_{N_r+(j+1)r}], \quad 0 \leqslant j < (N_{r+1} - N_r)/r.$$

Let $R = [R_1, R_2, R_3, \ldots, R_r]$ be a block of r bases that equals the block $R_{j,r}$ for some j; and let $B_i = B_i(R)$, $1 \leqslant i \leqslant R_1R_2 \ldots R_r$, be the sequence of all the possible digit blocks of length r such that $B_i < R$, arranged in lexicographical order (i.e., starting with $B_1 = [0, 0, \ldots, 0]$, then $B_2 = [0, 0, \ldots, 0, 1]$ and so on, ending with $B_{R_1R_2\ldots R_{r-1}} = [R_1 - 1, R_2 - 1, \ldots, R_r - 1]$).

Let us define the number $x_Q \in [0, 1)$ by its digits in the following way. For any fixed $R = [R_1, R_2, \ldots, R_r]$, let j_1 be the smallest j such that $R = R_{j,r}$, j_2 be the next smallest j such that $R = R_{j,r}$, and so on. First, define the digits of x_Q corresponding to the bases $R_{j_1,r}$ to be B_1, so that $E_{N_r+j_1 r+i}(x) = 0$ for $1 \leqslant i \leqslant r$. Then, define the digits corresponding to the bases $R_{j_2,r}$ to be B_2, and so on, so the digits corresponding to $R_{j_i,r}$ will be $B_{i \pmod{R_1 R_2 \ldots R_r}}$.

3. Proof of Theorem 1.1

We focus first on showing that x_Q is Q-normal.

Let $B = [b_1, b_2, \ldots, b_k]$ be an arbitrary block of digits such that $Q_n(B) \to \infty$. To show that x is Q-normal, we must show that

$$N_n^Q(B, x) = Q_n(B)(1 + o(1)).$$

Let $N_n^{Q*}(B, x)$ be defined similarly to $N_n^Q(B, x)$, but have it only count those appearances of B which occur up to the nth place within the digits corresponding to a single $R_{j,r}$, and not beginning in some $R_{j,r}$ and terminating in a different $R_{j',r'}$. Likewise, let $Q_n^*(B)$ be defined by

$$\sum_{i=1}^{n}{}^{*} \frac{I_i(B)}{q_i q_{i+1} \cdots q_{i+k-1}},$$

where the starred sum only runs over those i for which $[q_i, q_{i+1}, \ldots, q_{i+k-1}]$ is a sub-block of $R_{j,r}$ for some j, r.

To prove that x_Q is Q-normal, it suffices to show the following three asymptotic equalities:

$$Q_n(B) \sim Q_n^*(B), \tag{3.1}$$

$$N_n^Q(B, x) = N_n^{Q*}(B, x) + o(Q_n(B)), \quad \text{and} \tag{3.2}$$

$$N_n^{Q*}(B, x) \sim Q_n^*(B). \tag{3.3}$$

Let n be a large integer, and let $r = r(n)$ be defined by $N_r < n \leqslant N_{r+1}$.

Proof of (3.1). The difference $Q_n(B) - Q_n^*(B)$ is at most the sum

$$\sum \frac{1}{q_i q_{i+1} \cdots q_{i+k-1}},$$

where the sum runs over all $i \leqslant n$ such that the sub-blocks $[q_i, q_{i+1}, \ldots, q_{i+k-1}]$ start in some R_{j_1,r_1} and end in another R_{j_2,r_2}. Each summand is at most $1/2^k$, so if we can show that the number of summands is $o(Q_n(B))$, we would have shown (3.1).

So let us count how many sub-blocks of Q of length k up to the nth place start in some R_{j_1,r_1} and end in another R_{j_2,r_2}. Clearly every sub-block that starts before the N_kth place satisfies this condition. Each remaining sub-block occurs starting in one of the last $k-1$ places of a block of the form $R_{j,r}$ with $r \geqslant k$. Thus, at worst, the number of such sub-blocks is at most

$$\begin{aligned}
(k-1)\left\lceil \frac{n-N_r}{r} \right\rceil + (k-1)\frac{N_r - N_{r-1}}{r-1} + \ldots + (k-1)\frac{N_{k+1}-N_k}{k} + N_k \\
\leqslant (k-1) + \frac{k-1}{r}n + \frac{k-1}{r(r-1)}N_r + \frac{k-1}{(r-1)(r-2)}N_{r-1} + \ldots \\
+ \frac{k-1}{(k+1)k}N_{k+1} + \frac{1}{k}N_k \\
\leqslant \frac{k-1}{r}n + (k-1)\sum_{i=2}^{r} \frac{N_i}{i(i-1)} + O_k(1).
\end{aligned}$$

By definition, we have that $n_i/n_{i-1} \geqslant 5$, so that $N_i/N_{i-1} \geqslant 4$ for sufficiently large i. Thus, there exists a uniform constant C such that

$$\frac{N_i}{i(i-1)} \leqslant \frac{1}{C}\frac{N_{i+1}}{(i+1)i}$$

for all $i \geqslant 0$. So we have

$$\sum_{i=2}^{r} \frac{N_i}{i(i-1)} \ll \frac{N_r}{r(r-1)} \ll \frac{n}{r(r-1)}.$$

Thus the number of these sub-blocks is at worst

$$\frac{k-1}{r}n + \frac{k-1}{r(r-1)}n + O_k(1) = O_k\Big(\frac{n}{r+1}\Big).$$

At this point, to show that this is $o(Q_n(B))$, we must show that $r \gg \log n / \log q(n)$.

Since we defined r by $n \leqslant N_{r+1}$, we have

$$n \leqslant (q(n_{r+1})^2 + 1)^{r+1} + O(r).$$

By taking logarithms, we obtain

$$r \gg \frac{\log n}{\log q(n_{r+1})}.$$

Since $n_{r+1} \geqslant N_{r+1} \geqslant n$ and $q(n)$ is a nondecreasing function, the desired asymptotic inequality follows. □

Proof of (3.2). The difference $N_n^Q(B,x) - N_n^{Q*}(B,x)$ is at most the number of sub-blocks of Q of length k up to the nth place that start in some R_{j_1,r_1} and end in another R_{j_2,r_2}. By the argument of the previous section, this difference is at most $o(Q_n(B))$. □

Proof of (3.3). Consider a sub-block $R = [R_1, R_2, \ldots, R_{r'}]$ with $r' \leqslant r$ and an integer i such that $1 \leqslant i \leqslant r' - k + 1$ and $B < [R_i, R_{i+1}, \ldots, R_{i+k-1}]$. Let $\mathcal{R} = R_1R_2\ldots R_{r'}$ and $\mathcal{R}_i = R_iR_{i+1}\ldots R_{i+k-1}$. For any $\mathcal{R}$ consecutive j's for which $R = R_{j,r'}$, the corresponding digits of x will run through all possible blocks of digits exactly once, and thus the digits of B appear in the ith place of these sub-blocks exactly $\mathcal{R}/\mathcal{R}_i$ times.

Let $J_{R,n}$ denote the number of j such that $R = R_{j,r'}$ with all the bases of $R_{j,r'}$ occurring before the nth place. (We will say that $R_{j,r'}$ occurs completely before the nth place.) By the argument of the previous paragraph, the number of times the digits B occur in the ith place of the blocks $R_{j,r'}$ is

$$\begin{aligned}\frac{\mathcal{R}}{\mathcal{R}_i}\Big(\frac{J_{R,n}}{\mathcal{R}} + O(1)\Big) &= \frac{1}{\mathcal{R}_i}J_{R,n} + O\Big(\frac{\mathcal{R}}{\mathcal{R}_i}\Big)\\ &= \frac{1}{\mathcal{R}_i}J_{R,n} + O(q(n)^{r'}),\end{aligned}$$

where the last equality comes from the fact that each base is at most $q(n)$.

Therefore, we have

$$N_n^{Q*}(B,x) = \sum_{k\leqslant r'\leqslant r}\sum_{|R|=r'}\sum_{\substack{1\leqslant i\leqslant r'-k+1\\ B<[R_i,R_{i+1},\ldots,R_{i+k-1}]}}\Big(\frac{1}{\mathcal{R}_i}J_{R,n} + O(q(n)^{r'})\Big),$$

where the second sum runs over all R such that $R = R_{j,r'}$ for some $R_{j,r'}$ that appears completely before the nth place. Let us treat the big-O term separately. We have

$$\begin{aligned}\sum_{k\leqslant r'\leqslant r}\sum_{|R|=r'}\sum_{\substack{1\leqslant i\leqslant r'-k+1\\ B<[R_i,R_{i+1},\ldots,R_{i+k-1}]}} q(n)^{r'} &\leqslant \sum_{k\leqslant r'\leqslant r}\sum_{|R|=r'} r'q(n)^{r'}\\ &\leqslant \sum_{k\leqslant r'\leqslant r} rq(n)^{2r'} \leqslant r^2q(n)^{2r}.\end{aligned}$$

Therefore,

$$N_n^{Q*}(B,x) = \Bigg(\sum_{k\leqslant r'\leqslant r}\sum_{|R|=r'}\sum_{\substack{1\leqslant i\leqslant r'-k+1\\ B<[R_i,R_{i+1},\ldots,R_{i+k-1}]}}\frac{1}{\mathcal{R}_i}J_{R,n}\Bigg) + O(r^2q(n)^{2r}).$$

If we examine this triple sum carefully and recall the definition of $J_{R,n}$, we see that this is

$$\sum \frac{I_i(B)}{q_i q_{i+1} \cdots q_{i+k-1}},$$

where the sum runs over all i such that $[q_i, q_{i+1}, \ldots, q_{i+k-1}]$ is a sub-block of some $R_{j,r'}$ that appears completely before the nth place. This sum is $Q_n^*(B)$ up to $O(r)$ (to account for the possibility that n occurs in the middle of some sub-block $R_{j,r'}$).

Therefore,

$$N_n^{Q*}(B,x) = Q_n^*(B) + O(r) + O(r^2 q(n)^{2r}) = Q_n^*(B) + O(r^2 q(n)^{2r}).$$

For sufficiently large n (which in turn will give large r), we have

$$r^2 q(n)^{2r} \ll \frac{(q(n)^2+1)^r}{r} \ll \frac{n}{r} = o(Q_n(B)).$$

Since we already know that $Q_n(B) \sim Q_n^*(B)$, we therefore have

$$N_n^{Q*}(B,x) = Q_n^*(B) + o(Q_n^*(B)),$$

which completes the proof of Q-normality. $\square$

Proof of Q-distribution normality. Our goal now is to show that $(T_{Q,m}(x))_{m=0}^{\infty}$ is uniformly distributed. (The switch from labelling indices by n to labelling indices by m is intentional.) We have that

$$T_{Q,m}(x) = \sum_{i=1}^{\infty} \frac{E_{m+i}}{q_{m+1} q_{m+2} \cdots q_{m+i}},$$

where $(E_i)_{i=1}^{\infty}$ are the digits of x_Q.

Let x_m be defined by

$$x_m := \sum_{i=1}^{l(r(m))} \frac{E_{m+i}}{q_{m+1} q_{m+2} \cdots q_{m+i}},$$

where $l(y) = \lfloor \sqrt{y} \rfloor$. Since

$$|x_m - T_{Q,m}(x)| \leqslant \frac{1}{q_{m+1} q_{m+2} \cdots q_{m+l(r(m))}} \leqslant 2^{-l(r(m))},$$

which tends to 0 with m, we have that $(T_{Q,m}(x))_{n=0}^{\infty}$ is uniformly distributed if and only if $(x_m)_{m=0}^{\infty}$ is.

Let $\mathcal{I}$ be some interval in $[0,1)$. To complete the proof of Q-distribution normality, we must show that

$$\#\{0 \leqslant m \leqslant n\colon\ x_m \in \mathcal{I}\} = n\mathcal{I}(1+o(1)).$$

As we did earlier, consider a block $R = [R_1, R_2, \ldots, R_{r'}]$ with $r' \leqslant r$ and an integer i such that $1 \leqslant i \leqslant r' - l(r') + 1$. Let $\mathcal{R} = R_1 R_2 \ldots R_{r'}$ and $\mathcal{R}_i = R_i R_{i+1} \ldots R_{i+l(r')-1}$. Let $J_{R,n}$ denote the number of j such that $R = R_{j,r'}$ with $R_{j,r'}$ occurring completely before the nth place.

Suppose that $m \leqslant n$ and q_m appears in Q at precisely the ith place of a sub-block $R_{j,r'}$. Then x_m is a rational number with denominator $\mathcal{R}_i$. The number of distinct blocks of digits $B < [R_i, R_{i+1}, \ldots, R_{i+l(n)-1}]$ such that $x_m \in \mathcal{I}$ is $\mathcal{R}_i|\mathcal{I}| + O(1)$. And thus, by applying the same technique as in Proof of (3.3), we see that the number of times $x_m \in \mathcal{I}$ with m satisfying the above conditions is

$$\Big(\frac{J_{R,n}}{\mathcal{R}_i} + O(q(n)^{r'})\Big)(\mathcal{R}_i|\mathcal{I}| + O(1)) = J_{R,n}|\mathcal{I}| + O\Big(\frac{J_{R,n}}{\mathcal{R}_i}\Big) + O(q(n)^{r'}).$$

For any fixed small $\varepsilon > 0$, let r_ε be an integer large enough so that $|\mathcal{I}|2^{-r_\varepsilon} < \varepsilon$ and $l(r')/r' < \varepsilon$ for any $r' > r_\varepsilon$. Then we have

$$\begin{aligned}
&\#\{0 \leqslant m \leqslant n\colon\ x_m \in \mathcal{I}\} \\
&\qquad \geqslant \sum_{r_\varepsilon \leqslant r' \leqslant r} \sum_{|R|=r'} \sum_{1 \leqslant i \leqslant r'-l(r')+1} \Big(J_{R,n}|\mathcal{I}| + O\Big(\frac{J_{R,n}}{\mathcal{R}_i}\Big) + O(q(n)^{r'})\Big) \\
&\qquad = \sum_{r_\varepsilon \leqslant r' \leqslant r} \sum_{|R|=r'} \sum_{1 \leqslant i \leqslant r'-l(r')+1} (J_{R,n}|\mathcal{I}|(1+O(\varepsilon)) + O(q(n)^{r'})).
\end{aligned}$$

The inequality is due to not counting those i for which $i \geqslant r' - l(r') + 1$.

Again, by the work of Section 3.3, we know that the sum over $O(q(n)^{r'})$ will be at most $O(n/r) = o(n)$. Thus,

$$\begin{aligned}
\#\{0 \leqslant m \leqslant n\colon\ x_m \in \mathcal{I}\} &\geqslant o(n) + \sum_{r_\varepsilon \leqslant r' \leqslant r} \sum_{|R|=r'} \sum_{1 \leqslant i \leqslant r'-l(r')+1} (J_{R,n}|\mathcal{I}|(1+O(\varepsilon))) \\
&= o(n) + O(N_{r_\varepsilon}) + \sum_{1 \leqslant r' \leqslant r} \sum_{|R|=r'} \sum_{1 \leqslant i \leqslant r'-l(r')+1} (J_{R,n}|\mathcal{I}|(1+O(\varepsilon))) \\
&= o(n) + \sum_{1 \leqslant r' \leqslant r} \sum_{|R|=r'} (r' - l(r'))(J_{R,n}|\mathcal{I}|(1+O(\varepsilon))).
\end{aligned}$$

By the definition of r_ε, we have $r' - l(r') = r'(1+O(\varepsilon))$, so that

$$\begin{aligned}
\#\{0 \leqslant m \leqslant n\colon\ x_m \in \mathcal{I}\} &\geqslant o(n) + \sum_{1 \leqslant r' \leqslant r} \sum_{|R|=r'} (r' J_{R,n}|\mathcal{I}|(1+O(\varepsilon))) \\
&= o(n) + n|\mathcal{I}|(1+O(\varepsilon)) + O(r) \\
&= n|\mathcal{I}|(1+O(\varepsilon)+o(1)),
\end{aligned}$$

where again the $O(r)$ term comes from the fact that n could be in the middle of some term $R_{j,r}$. Since ε was arbitrary, the desired result follows. □

4. Computability

Proof of Theorem 1.2. Since $(q(n))$ is a computable sequence of integers, the sequence $((q(n)^r + 1)^r)_{n=1}^{\infty}$ is also a computable sequence of integers for a fixed integer r. Create a Turing machine $M\colon \mathbb{N} \to \mathbb{N}$ such that $M(r) = n_r$ as follows. Consider the Turing machine $L\colon \mathbb{N} \times \mathbb{N} \to \{0, 1\}$ such that $L(x, y) = 1$ if when $x \leqslant y$ and 0 otherwise. For input r have M at step n output n if $L((q(n)^2 + 1)^r, n) = 1$ and halt, otherwise increment n by 1. This process halts because n_r exists, and $M(r) = n_r$. Thus the sequence $(n_r)_{r=1}^{\infty}$ is a computable sequence of integers.

To see the sequence $(N_r)_{r=1}^{\infty}$ is a computable sequence of integers, consider the Turing machine $I\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that $I(n, r)$ is the greatest integer less than n_r such that $I(n, r) - n \equiv 0 \bmod r$. Construct I as follows. Consider the Turing machine $\mathrm{Rem}_r\colon \mathbb{N} \to \{0, 1\}$ such that $\mathrm{Rem}_r(n) \equiv n \bmod r$ with $0 \leqslant \mathrm{Rem}_r(n) < r$. Have I first compute $n - \mathrm{Rem}_r(n)$, and at step k check if $L(n - \mathrm{Rem}_r(n) + kr, n_r) = 0$. If $L(n - \mathrm{Rem}_r(n) + kr, n_r) = 0$, have I output $n - \mathrm{Rem}_r(n) + (k - 1)r$, otherwise increment k by 1. Finally construct the Turing machine $N\colon \mathbb{N} \to \mathbb{N}$ that on input r computes $I(I(\ldots I(I(0, 1), 2) \ldots r - 1), r)$. Then $N(r) = N_r$, so $(N_r)_{r=1}^{\infty}$ is a computable sequence of integers.

Now construct the Turing machine $E\colon \mathbb{N} \to \mathbb{N}$ with $E(n) = E_n$ as follows. First make the machine $r\colon \mathbb{N} \to \mathbb{N}$ such that $r(n)$ is the integer r such that $N_r \leqslant n < N_{r+1}$. Such a machine exists because the sequence of integers (N_r) is computable and the order relation on the integers is a computable relation. Construct a Turing machine $J_r\colon \mathbb{N}^r \times \mathbb{N} \to \mathbb{N}$ such that $J([R_1, R_2, \ldots, R_r], n)$ is the number of times the block $[R_1, R_2, \ldots, R_r]$ occurs at position t in Q with $t \equiv N_r \bmod r$, $t \geqslant N_r$, and $t \leqslant n$. Create a Turing machine $B_r\colon \mathbb{N}^r \times \mathbb{N} \to \mathbb{N}^r$ such that $B_r([R_1, R_2, \ldots, R_r], i)$ is the ith block B in the lexicographic ordering on $\mathbb{N}^r$ with $B < [R_1, R_2, \ldots, R_r]$. Finally let $R_r\colon \mathbb{N} \to \mathbb{N}^r$ be the Turing machine with $R_r(n) = [q_{N_r+jr+1}, q_{N_r+jr+2}, \ldots, q_{N_r+(j+1)r}]$ such that $N_r + jr + 1 \leqslant n \leqslant N_r + (j+1)r$. Then $E(n)$ is the $(n - N_{r(n)} \bmod r)$th element of $B_{r(n)}(R_{r(n)}(n), J_{r(n)}(R_{r(n)}(n), n))$. Thus the sequence (E_n) is a computable sequence. Since Q is also a computable sequence of integers, the real number $x_Q = \sum\limits_{i=1}^{\infty} E_i/(q_1 \ldots q_i)$ that was constructed in Section 2 is a computable real number. □

Using this theorem, we can now give computable examples of numbers that are normal of one type but not another, as in [1] and [12]. We will need the following definition and theorem from [12].

Let (P, Q) be a pair of basic sequences and suppose that $x = E_0.E_1E_2\ldots$ with respect to P. We define

$$\psi_{P,Q}(x) := \sum_{n=1}^{\infty} \frac{\min\{E_n, q_n - 1\}}{q_1 \ldots q_n}.$$

Theorem 4.1. *Suppose that $Q_1 = (q_{1,n}), Q_2 = (q_{2,n}), \ldots, Q_j = (q_{j,n})$ are basic sequences and infinite in limit. Set*

$$\Psi_j(x) = (\psi_{Q_{j-1},Q_j} \circ \psi_{Q_{j-2},Q_{j-1}} \circ \ldots \circ \psi_{Q_1,Q_2})(x).$$

If $x = E_0.E_1E_2\ldots$ with respect to Q_1 satisfies $E_n < \min\limits_{2\leqslant r\leqslant j}(q_{r,n} - 1)$ for infinitely many n, then for every block B

$$N_n^{Q_j}(B, \Psi_j(x)) = N_n^{Q_1}(B, x) + O(1).$$

We now state the following three theorems.

Theorem 4.2. *If Q is slowly growing, infinite in limit and the sequences (q_n) and $(q(n))$ are computable sequences of integers, then there is a computable real number in $\mathscr{N}(Q) \setminus \mathscr{DN}(Q)$.*

P r o o f. Let $p_n = \max\{\lfloor \log q_n \rfloor, 2\}$ and set $P = (p_n)$. By Theorem 1.2 there is a computable real number $x_Q \in \mathscr{N}(Q)$. Put $y = (\psi_{P,Q} \circ \psi_{Q,P})(x)$. Then y is Q-normal by Theorem 4.1, but $T_{Q,n}(y) \to 0$ so y is not Q-distribution normal. Furthermore, $E_{Q,n}(y) = \max\{E_{Q,n}(x), \lfloor \log q_n \rfloor, 2\}$, which is a computable sequence of integers. Therefore y is a computable real number. □

Theorem 4.3. *If Q is slowly growing, infinite in limit, and the sequences (q_n) and $(q(n))$ are computable sequences of integers, then there is a computable real number in $\mathscr{RN}(Q) \setminus \mathscr{N}(Q)$.*

P r o o f. Let $p_n = \max\{\lfloor q_n/2 \rfloor, 2\}$ and set $P = (p_n)$. The basic sequence P clearly has the same properties as Q. Let x_Q be a computable real number in $\mathscr{N}(P)$ and set $y = \psi_{P,Q}(x)$. The real number y is clearly computable, and by the calculations in [12] is in $\mathscr{RN}(Q) \setminus \mathscr{N}(Q)$. □

To prove the next result we will need the following definition and lemma. For a sequence of real numbers $X = (x_n)$ with $x_n \in [0, 1)$ and an interval $I \subseteq [0, 1]$, define $A_n(I, X) = \#\{i \leqslant n\colon\ x_i \in I\}$. We quote the following from [11].

Definition 4.1. Let $X = \{x_1, \dots, x_N\}$ be a finite sequence of real numbers. The number

$$D_N = D_N(X) = \sup_{0 \leqslant \alpha \leqslant \beta \leqslant 1} \left| \frac{A_N([\alpha, \beta), X)}{N} - (\beta - \alpha) \right|$$

is called the *discrepancy* of the sequence ω.

It is well known that a sequence X is uniformly distributed mod 1 if and only if $D_N(X) \to 0$.

Lemma 4.4. *Let $x_1, x_2, \dots, x_N$ and $y_1, y_2, \dots, y_N$ be two finite sequences in $[0, 1)$. Suppose $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N$ are nonnegative numbers such that $|x_n - y_n| \leqslant \varepsilon_n$ for $1 \leqslant n \leqslant N$. Then, for any $\varepsilon \geqslant 0$, we have*

$$|D_N(x_1, \dots, x_N) - D_N(y_1, \dots, y_N)| \leqslant 2\varepsilon + \frac{\overline{N}(\varepsilon)}{N},$$

where $\overline{N}(\varepsilon)$ denotes the number of n, $1 \leqslant n \leqslant N$, such that $\varepsilon_n > \varepsilon$.

We can now prove the following theorem:

Theorem 4.5. *If Q is infinite in limit and computable and the sequence (L_n) defined in the proof of Theorem 1.2 is computable, then there is a computable real number in $\mathscr{RN}(Q) \cap \mathscr{DN}(Q) \setminus \mathscr{N}(Q)$.*

Proof. Let $P = (p_i)$ with $p_i = \lfloor \log i \rfloor + 2$. Note that P is slowly growing, computable, and the sequence $(p(n))$ is computable, so there is a computable real number $\xi \in \mathscr{N}(P)$ with $\xi = .F_1 F_2 \dots$ with respect to P. Fix a computable sequence of real numbers $X = (x_n)$ that is uniformly distributed modulo 1 (for example the Farey sequence). Define the sequences

$$\begin{aligned}
\nu_n &= \min\left\{t\colon \frac{\sum_{i=0}^{n-1} \log q_{L_{n-1}+i}}{\sum_{i=0}^{j-L_{n-1}-1} \log q_{L_{n-1}+i}} < \frac{1}{n},\ \forall j \geqslant t\right\};\\
\upsilon_{n,k} &= \min\left\{t\colon \frac{Q_n(B)}{\sum_{i=1}^{j} P_{i-k+1}(B)} < \frac{1}{n},\ \forall j \geqslant t \text{ and blocks } B \text{ of length } k\right\};\\
L_0 &= 0;\\
L_n &= \max\bigl\{\min\{t\colon \log(q_j) > n,\ \forall j \geqslant t\},\ L_{n-1} + n^2,\ L_{n-1} + \nu_n,\ \max_{k \leqslant n}\{\upsilon_{n,k}\}\bigr\}
\end{aligned}$$

and set $i(n) = \max\{j\colon L_j \leqslant n\}$. The sequence $(i(n))$ is computable since (L_n) is a computable sequence. Note that ν_n and $\upsilon_{n,k}$ exist since Q is infinite in limit and P is fully divergent. Define the set

$$S = \bigcup_{n=1}^{\infty} \{L_n, L_n + 1, \dots, L_n + n - 1\}.$$

Note that this set has density 0 since

$$\frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n}(L_i - L_{i-1})} \leqslant \frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i^2} \to 0 \quad \text{as } n \text{ goes to infinity.}$$

Define the sequence

$$E_n = \begin{cases} F_{n-L_i} & \text{if } n \in [L_i, L_i+1, \ldots, L_i + i\,], \\ \max\{\lfloor x_n q_n \rfloor, \{\log i(n)\}\} & \text{otherwise.} \end{cases}$$

We claim the real number $x = \sum\limits_{n=1}^{\infty} E_n/(q_1 \ldots q_n)$ is in $\mathscr{RN}(Q) \cap \mathscr{DN}(Q) \setminus \mathscr{N}(Q)$. Let B be a block of length k. Note that by the definition of L_n, there are only finitely many values $n \in \mathbb{N} \setminus S$ such that B occurs at position n in the Q-Cantor series expansion of x. This is because all digits E_n with $n \in \mathbb{N} \setminus S$ must be at least $\{\log i(n)\}$ and since $i(n)$ tends to infinity as n does. Thus if m is the maximum digit for the block B, we have that for $n \in \mathbb{N} \setminus S$ with $i(n) > m$ that $E_n > m$. Thus $N_n^Q(B,x) = \sum\limits_{i=1}^{i(n)} N_{i-k+1}^P(B,\xi) + O(1)$. So for any two blocks B_1 and B_2 of length k, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{N_n^Q(B_1,x)}{N_n^Q(B_2,x)} &= \lim_{n\to\infty} \frac{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_1,\xi) + O(1)}{\sum_{i=1}^{i(n)} N_{i-k+1}^P(B_2,\xi) + O(1)} \\ &= \lim_{n\to\infty} \frac{N_{n-k+1}^P(B_1,\xi)}{N_{n-k+1}^P(B_2,\xi)} = 1. \end{aligned}$$

Thus $x \in \mathscr{RN}(Q)$.

Consider the sequence $Y = (E_n/q_n)$. For $n \in \mathbb{N} \setminus S$, we have $|E_n/q_n - x_n| < 1/q_n$, which tends to 0 as n goes to infinity. We therefore have for $\varepsilon > 0$ that $\overline{N}(\varepsilon) = O(1) + \#S \cap \{1, \ldots, N\}$. Thus by Lemma 4.5

$$|D_N(X) - D_N(Y)| < 2\varepsilon + \frac{O(1)}{N} + \frac{\#S \cap \{1,\ldots,N\}}{N} < 3\varepsilon$$

if N is sufficiently large. Since the inequality holds for all $\varepsilon > 0$, we have that (E_n/q_n) is uniformly distributed mod 1. Thus $x \in \mathscr{DN}(Q)$.

Note that

$$\lim_{n\to\infty} \frac{N_n^Q(B,x)}{\sum_{i=1}^{i(n)} P_{i-k+1}(B)} = 1.$$

However,

$$\lim_{n\to\infty} \frac{Q_n(B)}{\sum_{i=1}^{i(n)} P_{i-k+1}(B)} = 0$$

by the definition of L_n, so $x \notin \mathscr{N}(Q)$.

Furthemore, the sequence E_n is computable because the sequences (F_n), (L_n) and $(i(n))$ are all computable. Thus x is a computable real number in $\mathscr{RN}(Q) \cap \mathscr{DN}(Q) \setminus \mathscr{N}(Q)$. $\square$

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