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# REMARKS ON $D$-INTEGRAL COMPLETE MULTIPARTITE GRAPHS

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*Abstract.* A graph is called distance integral (or $D$-integral) if all eigenvalues of its distance matrix are integers. In their study of $D$-integral complete multipartite graphs, Yang and Wang (2015) posed two questions on the existence of such graphs. We resolve these questions and present some further results on $D$-integral complete multipartite graphs. We give the first known distance integral complete multipartite graphs $K_{p_1,p_2,p_3}$ with $p_1 < p_2 < p_3$, and $K_{p_1,p_2,p_3,p_4}$ with $p_1 < p_2 < p_3 < p_4$, as well as the infinite classes of distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ with $s = 5, 6$.

*Keywords*: distance spectrum; integral graph; distance integral graph; complete multipartite graph

*MSC 2010*: 05C50

## 1. Introduction and preliminaries

The study of graphs with integral adjacency spectrum was initiated by Harary and Schwenk in 1974 (see [7]). A survey of papers up to 2002 appears in [3], but more than a hundred new studies on integral graphs have been published in the last ten years.

Let $G = (V, E)$ be a simple, connected graph with $n = |V|$ vertices. A distance matrix of $G$ is the $n \times n$ matrix $D$, indexed by $V$, such that $D_{u,v}$ is the distance between the vertices $u$ and $v$. Among the earliest users of a distance matrix in chemistry were Clark and Kettle in 1975 (see [4]). Topological indices based on the distance matrix, in particular its largest eigenvalue and its energy, play a significant role in research (see, for example, [5], [6], [8], [9], [13], [16]). A survey on the distance spectra of graphs appears in [2].

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The distance characteristic polynomial (or $D$-polynomial) of $G$ is $D_G(x) = |xI_n - D(G)|$. A graph $G$ is called $D$-integral if all the eigenvalues of its $D$-polynomial are integers. Distance integral graphs are studied only in [8], [11] in the case of some special, highly symmetric graphs, and in [10], [14], [15].

Complete multipartite graphs, in the case of integer distance spectrum, are studied in [14], [15]. In [15], Yang and Wang show that the $D$-characteristic polynomial of a complete multipartite graph $K_{p_1,p_2,\ldots,p_r}$ with $p_1+p_2+\ldots+p_r = n$ vertices is equal to

$$(1.1)\qquad P(K_{p_1,p_2,\ldots,p_r};x) = \prod_{i=1}^{r}(x+2)^{(p_i-1)}\prod_{i=1}^{r}(x-p_i+2)\biggl(1-\sum_{i=1}^{r}\frac{p_i}{x-p_i+2}\biggr).$$

If $p'_1, p'_2, \ldots, p'_s$ denote all the distinct integers among $p_1, p_2, \ldots, p_r$ and $a_i$, $i = 1, 2, \ldots, s$, denotes the multiplicity of $p'_i$ in the family $p_1, p_2, \ldots, p_r$, then $K_{p_1,p_2,\ldots,p_r}$ will also be denoted by $K_{a_1p'_1,a_2p'_2,\ldots,a_sp'_s}$.

In [15], the following sufficient and necessary conditions for complete $r$-partite graphs to be distance integral are given.

**Theorem 1.1** ([15], Theorem 2.6). *If a complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ on $n$ vertices is distance integral, then there exist integers $\mu_i$, $i = 1, 2, \ldots, s$, such that $-2 < p_1 - 2 < \mu_1 < p_2 - 2 < \mu_2 < \ldots < p_{s-1} - 2 < \mu_{s-1} < p_s - 2 < \mu_s < \infty$, and the numbers $a_1, a_2, \ldots, a_s$ defined by*

$$(1.2)\qquad a_k = \frac{\prod_{i=1}^{s}(\mu_i - p_k + 2)}{p_k\prod_{i=1,\, i\neq k}^{s}(p_i - p_k)}, \quad k = 1, 2, \ldots, s$$

*are positive integers.*

*Conversely, suppose that there exist integers $\mu_i$, $i = 1, 2, \ldots, s$, such that $-2 < p_1 - 2 < \mu_1 < p_2 - 2 < \mu_2 < \ldots < p_{s-1} - 2 < \mu_{s-1} < p_s - 2 < \mu_s < \infty$ and that the numbers $a_k$, in (1.2) are positive integers. Then the complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ is distance integral.*

**Corollary 1.1** ([15], Corollary 2.9). *For any positive integer $q$, the complete $r$-partite graph $K_{p_1q,p_2q,\ldots,p_rq} = K_{a_1p_1q,a_2p_2q,\ldots,a_sp_sq}$ is distance integral if and only if the complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ is distance integral.*

**Theorem 1.2** ([15], Theorem 3.2). *Let a complete $r$-partite graph $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ be distance integral with eigenvalues $\mu_i$. Let $\mu_i \geqslant 0$ and $p_i > 0$, $i = 1, 2, \ldots, s$, be integers such that $-2 < p_1 - 2 < \mu_1 < p_2 - 2 < \mu_2 < \ldots < p_{s-1} - 2 < \mu_{s-1} < p_s - 2 < \mu_s < \infty$ and let*

$$(1.3)\qquad a_k = \frac{\prod_{i=1}^{s}(\mu_i - p_k + 2)}{p_k\prod_{i=1,\, i\neq k}^{s}(p_i - p_k)}, \quad k = 1, 2, \ldots, s$$

*be positive integers. Then for*

$$(1.4)\quad b_k = \frac{\prod_{i=1}^{s-1}(\mu_i - p_k + 2)(\mu_s - p_k + 2 + rt)}{p_k \prod_{i=1,\, i\neq k}^{s}(p_i - p_k)}, \quad k = 1, 2, \ldots, s,$$

$$(1.5)\quad r = \mathrm{LCM}(r_1, r_2, \ldots, r_s), \quad r_k = \frac{p_k \prod_{i=1,\, i\neq k}^{s}(p_i - p_k)}{d_k}, \quad k = 1, 2, \ldots, s,$$

$$(1.6)\quad d_k = \mathrm{GCD}\Big(\prod_{i=1}^{s-1}(\mu_i - p_k + 2), p_k \prod_{i=1,\, i\neq k}^{s}(p_i - p_k)\Big), \quad k = 1, 2, \ldots, s,$$

*the complete $m$-partite graph $K_{p_1,p_2,\ldots,p_m} = K_{b_1p_1,b_2p_2,\ldots,b_sp_s}$ is distance integral for every nonnegative integer $t$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_{s-1}$, $\mu'_s = \mu_s + rt$.*

In [15], Yang and Wang concluded their study with the following questions. The first of them is answered affirmatively in [14], the other we answer affirmatively here.

**Question 1.1** ([15], Question 4.1)**.** Are there any distance integral complete $r$-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ for $s \geqslant 5$?

**Question 1.2** ([15], Question 4.2)**.** Are there any distance integral complete $r$-partite graphs $K_{p_1,p_2,\ldots,p_r} = K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ with $a_1 = a_2 = \ldots = a_s = 1$ for $s \geqslant 3$?

The rest of the present paper is organized as follows. In Section 2, we study complete multipartite graphs $K_{a_1p_1,a_2p_2}$ and give sufficient and necessary conditions for their distance integrality. Our conditions are more easily applicable than the conditions published in Theorem 3.1 of [15]. In Section 3, we give the first known distance integral complete multipartite graphs $K_{p_1,p_2,p_3}$ with $p_1 < p_2 < p_3$, and $K_{p_1,p_2,p_3,p_4}$ with $p_1 < p_2 < p_3 < p_4$. In Section 4, we give infinite classes of distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ with $s = 5, 6$, which are different from those of Yang and Wang in [14].

## 2. Distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2}$

Let us start with the definition of the join of graphs $G_1$ and $G_2$ and the notation of the spectrum of the adjacency matrix $A(G)$ of $G$ and the spectrum of the distance matrix $D(G)$ of $G$.

**Definition 2.1.** The join $G_1 \nabla G_2$ of graphs $G_1$ and $G_2$ is the graph obtained from the union of $G_1$ and $G_2$ by adding the edges joining every vertex of $G_1$ to every vertex of $G_2$.

**Definition 2.2.** Let $\lambda_1 < \lambda_2 < \ldots < \lambda_t$ be $t$ distinct eigenvalues of the adjacency matrix $A(G)$ of $G$ with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. The spectrum of $A(G)$ is also called the spectrum of $G$ and denoted by $\mathrm{Spec}(G) = \{\lambda_1^{(k_1)}, \lambda_2^{(k_2)}, \ldots, \lambda_t^{(k_t)}\}$.

**Definition 2.3.** Let $\mu_1 < \mu_2 < \ldots < \mu_t$ be $t$ distinct eigenvalues of the distance matrix $D(G)$ of $G$ with the corresponding multiplicities $k_1, k_2, \ldots, k_t$. The spectrum of $D(G)$ is also called the distance spectrum of $G$ and denoted by $\mathrm{Spec}_D(G) = \{\mu_1^{(k_1)}, \mu_2^{(k_2)}, \ldots, \mu_t^{(k_t)}\}$.

The following theorem is useful for getting conditions for $D$-integrality of $K_{a_1p_1,a_2p_2}$.

**Theorem 2.1** ([12]). *For $i = 1, 2$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and the eigenvalues $\lambda_{i,1} = r_i \geqslant \ldots \geqslant \lambda_{i,n_i}$ of the adjacency matrix of $G_i$. The distance spectrum of $G_1 \nabla G_2$ consists of the eigenvalues $-\lambda_{i,j} - 2$ for $i = 1, 2$ and $j = 2, 3, \ldots, n_i$, and two further simple eigenvalues $n_1 + n_2 - 2 - (r_1 + r_2)/2 \pm \sqrt{(n_1 - n_2 - (r_1 - r_2)/2)^2 + n_1 n_2}$.*

It is clear that $K_{a_1p_1,a_2p_2} = K_{a_1p_1} \nabla K_{a_2p_2}$. Using the above theorem for $K_{a_1p_1}$, $K_{a_2p_2}$, we have the following theorem.

**Theorem 2.2.** *The graph $K_{a_1p_1,a_2p_2}$ is D-integral if and only if*

$$\frac{(a_1+1)p_1 + (a_2+1)p_2 - 4}{2} \pm \sqrt{\frac{((a_1+1)p_1 - (a_2+1)p_2)^2}{4} + a_1a_2p_1p_2}$$

*are integers and its distance spectrum is*

$$\left\{ \frac{(a_1+1)p_1 + (a_2+1)p_2 - 4}{2} \pm \sqrt{\frac{((a_1+1)p_1 - (a_2+1)p_2)^2}{4} + a_1a_2p_1p_2}, \right.$$
$$\left. (p_1 - 2)^{(a_1-1)}, (p_2 - 2)^{(a_2-1)}, (-2)^{(a_1p_1 - a_1 + a_2p_2 - a_2)} \right\}.$$

P r o o f. The $A$-spectrum of $K_{a_1p_1}$ is $\{p_1(a_1 - 1), 0^{(p_1a_1 - a_1)}, (-p_1)^{(a_1-1)}\}$ and the $A$-spectrum of $K_{a_2p_2}$ is $\{p_2(a_2 - 1), 0^{(p_2a_2 - a_2)}, (-p_2)^{(a_2-1)}\}$. Now it is sufficient to use Theorem 2.1. □

Using $(a_1, a_2) = (1, 1), (2, 1), (2, 2), (3, 1)$ in Theorem 2.2, we have the following corollary.

**Corollary 2.1.**
1. *The graph* $K_{p_1,p_2}$ *is D-integral if and only if* $p_1^2 - p_1p_2 + p_2^2$ *is a perfect square. Moreover, its distance spectrum is* $\{(-2)^{(p_1+p_2-2)}, p_1 + p_2 - 2 \pm \sqrt{p_1^2 - p_1p_2 + p_2^2}\}$.
2. *The only distance integral graph among stars is* $K_2$.
3. *The graph* $K_{2p_1,p_2}$ *is distance integral if and only if* $9p_1^2 - 4p_1p_2 + 4p_2^2$ *is a perfect square. Moreover, its distance spectrum is* $\{(-2)^{(2p_1+p_2-3)}, p_1 - 2, (3p_1 + 2p_2 - 4 \pm \sqrt{9p_1^2 - 4p_1p_2 + 4p_2^2})/2\}$.
4. *The graph* $K_{2p_1,2p_2}$ *is distance integral if and only if* $9p_1^2 - 2p_1p_2 + 9p_2^2$ *is a perfect square. Moreover, its distance spectrum is* $\{(-2)^{(2p_1+2p_2-4)}, p_1 - 2, p_2 - 2, (3p_1 + 3p_2 - 4 \pm \sqrt{9p_1^2 - 2p_1p_2 + 9p_2^2})/2\}$.
5. *The graph* $K_{3p_1,p_2}$ *is distance integral if and only if* $4p_1^2 - p_1p_2 + p_2^2$ *is a perfect square. Moreover, its distance spectrum is* $\{(-2)^{(3p_1+p_2-4)}, (p_1 - 2)^2, 2p_1 + p_2 - 2 \pm \sqrt{4p_1^2 - p_1p_2 + p_2^2}\}$.

The following corollary gives sufficient and necessary conditions for complete bipartite graphs to be $D$-integral.

**Corollary 2.2.** $K_{p_1,p_2}$ *is D-integral if and only if there exist integers* $k, u$ *and* $v$ *such that* $p_1 = k(v^2 + 2uv)$, $p_2 = k(v^2 - u^2)$, *or* $p_1 = k(v^2 - u^2)$, $p_2 = k(v^2 + 2uv)$, *where* $u, v \in \mathbb{Z}$ *and* $k \in \mathbb{Q}$ *are such that* $3k \in \mathbb{Z}$.

Proof. Part 1 of Corollary 2.1 yields that the necessary and sufficient condition for $K_{p_1,p_2}$ to be $D$-integral is that for some integer $r$, $p_1^2 - p_1p_2 + p_2^2 = r^2$. According to [1], page 90, all integral solutions to $p_1^2 - p_1p_2 + p_2^2 = r^2$ are given by $p_1 = k(v^2 + 2uv)$, $p_2 = k(v^2 - u^2)$, or $p_1 = k(v^2 - u^2)$, $p_2 = k(v^2 + 2uv)$, where $u, v \in \mathbb{Z}$ and $k \in \mathbb{Q}$ is such that $3k \in \mathbb{Z}$. □

## 3. Distance integral complete multipartite graphs $K_{p_1,p_2,p_3}$ and $K_{p_1,p_2,p_3,p_4}$

Using computers, we have found 292 $D$-integral complete 3-partite graphs $K_{p_1,p_2,p_3}$ for $p_1 < p_2 < p_3 \leqslant 1,000$. The primitive graphs (those, where $\mathrm{GCD}(p_1, p_2, p_3) = 1$) with less than 180 vertices are given in Table 1, rows 2–7.

Using Theorem 1.2, we can construct infinite classes of $D$-integral complete multipartite graphs for each graph from Table 1.

**Corollary 3.1.** *Let* $K_{p_1,p_2,p_3}$ *be a D-integral complete* 3*-partite graph from Table* 1*, rows 2–4. Then* $K_{b_1p_1,b_2p_2,b_3p_3}$ *is a D-integral complete multipartite graph for every* $t \in \mathbb{N}$, *where* $b_1, b_2, b_3$ *are those of Table* 1*, rows 9–11.*

| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $p_1$ | 12 | 7 | 28 | 25 | 20 | 23 | 39 | 35 |
| $p_2$ | 21 | 33 | 33 | 30 | 39 | 39 | 48 | 54 |
| $p_3$ | 28 | 81 | 60 | 81 | 84 | 81 | 56 | 75 |
| $\mu_1$ | 12 | 7 | 28 | 25 | 22 | 25 | 40 | 38 |
| $\mu_2$ | 22 | 42 | 42 | 43 | 50 | 50 | 50 | 61 |
| $\mu_3$ | 82 | 187 | 166 | 198 | 208 | 205 | 190 | 223 |
| $r$ | 504 | 9,828 | 3,780 | 5,950 | 11,970 | 11,592 | 2,448 | 8,550 |
| $b_1$ | $1+7t$ | $1+54t$ | $1+27t$ | $1+34t$ | $1+63t$ | $1+63t$ | $1+16t$ | $1+45t$ |
| $b_2$ | $1+8t$ | $1+63t$ | $1+28t$ | $1+35t$ | $1+70t$ | $1+69t$ | $1+17t$ | $1+50t$ |
| $b_3$ | $1+9t$ | $1+91t$ | $1+35t$ | $1+50t$ | $1+95t$ | $1+92t$ | $1+18t$ | $1+57t$ |

Table 1. $D$-integral complete multipartite graphs $K_{p_1,p_2,p_3}$.

P r o o f. It is sufficient to use the formulas (1.3)–(1.6) from Theorem 1.2. □

Similarly, using computers, we have found the $D$-integral complete 4-partite graph $K_{143,192,228,468}$. Using Theorem 1.2, we have the following corollary.

**Corollary 3.2.** *The graph* $K_{(1+1,368t)\cdot 143,(1+1,425t)\cdot 192,(1+1,470t)\cdot 228,(1+1,862t)\cdot 468}$ *is a $D$-integral complete multipartite graph for every $t \in \mathbb{N}$.*

P r o o f. It is sufficient to use (1.3)–(1.6) from Theorem 1.2 for $\mu_1 = 154$, $\mu_2 = 206$, $\mu_3 = 328$, $\mu_4 = 1,366$, $r = 1,675,800$. □

## 4. Distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ with $s = 5,6$

Using a computer search based on Theorem 1.1, we have found examples of $D$-integral complete multipartite graphs $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4,a_5p_5}$; they are given in Table 2, rows 2–11. Using Theorem 1.2, we can construct infinite classes of $D$-integral complete multipartite graphs for each graph from Table 2.

**Corollary 4.1.** *Let $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4,a_5p_5}$ be a $D$-integral complete multipartite graph from Table* 2, *rows* 2–11. *Then $K_{b_1p_1,b_2p_2,b_3p_3,b_4p_4,b_5p_5}$ is a $D$-integral complete multipartite graph for every $t \in \mathbb{N}$, where $b_1, b_2, b_3, b_4, b_5$ are those of Table* 2, *rows* 18–22.

P r o o f. It is sufficient to use (1.3)–(1.6) from Theorem 1.2. □

Similarly, using a computer search based on Theorem 1.1, we have found an example of $D$-integral complete multipartite graph $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4,a_5p_5,a_6p_6}$.

| No. | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $a_1$ | 11 | 31 | 44 | 56 | 23 | 39 | 44 |
| $p_1$ | 3 | 11 | 4 | 10 | 10 | 7 | 8 |
| $a_2$ | 1 | 9 | 52 | 2 | 39 | 37 | 52 |
| $p_2$ | 12 | 35 | 8 | 22 | 14 | 10 | 16 |
| $a_3$ | 2 | 2 | 12 | 13 | 6 | 23 | 12 |
| $p_3$ | 18 | 45 | 23 | 37 | 22 | 23 | 46 |
| $a_4$ | 3 | 3 | 11 | 9 | 6 | 31 | 11 |
| $p_4$ | 28 | 49 | 25 | 46 | 35 | 28 | 50 |
| $a_5$ | 1 | 1 | 6 | 3 | 21 | 7 | 6 |
| $p_5$ | 39 | 56 | 29 | 57 | 55 | 50 | 58 |
| $\mu_1$ | 4 | 19 | 3 | 17 | 9 | 6 | 8 |
| $\mu_2$ | 11 | 40 | 13 | 22 | 18 | 12 | 28 |
| $\mu_3$ | 19 | 45 | 22 | 40 | 26 | 23 | 46 |
| $\mu_4$ | 34 | 53 | 26 | 53 | 38 | 44 | 54 |
| $\mu_5$ | 226 | 978 | 1,332 | 1,700 | 2,308 | 2,413 | 2,666 |
| $r$ | 37,800 | 10,445,820 | 22,621,305 | 100,792,440 | 8,208,200 | 1,721,720 | 45,242,610 |
| $b_1$ | $11 + 1,848t$ | $31 + 334,180t$ | $44 + 748,374t$ | $56 + 3,335,920t$ | $23 + 82,082t$ | $39 + 27,885t$ | $44 + 748,374t$ |
| $b_2$ | $1 + 175t$ | $9 + 99,484t$ | $52 + 887,110t$ | $2 + 119,991t$ | $39 + 139,425t$ | $37 + 26,488t$ | $52 + 887,110t$ |
| $b_3$ | $2 + 360t$ | $2 + 22,344t$ | $12 + 207,060t$ | $13 + 786,968t$ | $6 + 21,525t$ | $23 + 16,555t$ | $12 + 207,060t$ |
| $b_4$ | $3 + 567t$ | $3 + 33,660t$ | $11 + 190,095t$ | $9 + 547,785t$ | $6 + 21,648t$ | $31 + 22,360t$ | $11 + 190,095t$ |
| $b_5$ | $1 + 200t$ | $1 + 11,305t$ | $6 + 104,006t$ | $3 + 183,816t$ | $21 + 76,440t$ | $7 + 5,096t$ | $6 + 104,006t$ |

Table 2. $D$-integral complete multipartite graphs $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4,a_5p_5}$.

**Corollary 4.2.** 1. *The graph* $K_{722,608\cdot 4,706,668\cdot 8,364,041\cdot 14,73,308\cdot 23,73,420\cdot 25,214,524\cdot 32}$ *is a $D$-integral complete multipartite graph and* $\mu_1 = 3$, $\mu_2 = 9$, $\mu_3 = 18$, $\mu_4 = 22$, $\mu_5 = 26$, $\mu_6 = 24,026,718$.

2. *Let* $b_1 = 722,608 + 825,792t$, $b_2 = 706,668 + 807,576t$, $b_3 = 364,041 + 416,024t$, $b_4 = 73,308 + 83,776t$, $b_5 = 73,420 + 83,904t$, $b_6 = 214,524 + 245,157t$. *The graph* $K_{b_1\cdot 4,b_2\cdot 8,b_3\cdot 14,b_4\cdot 23,b_5\cdot 25,b_6\cdot 32}$ *is a $D$-integral complete multipartite graph for every* $t \in \mathbb{N}$.

P r o o f. For case 1 it is sufficient to use Theorem 1.1. For Case 2 it is sufficient to use (1.3)–(1.6) from Theorem 1.2 ($r = 27,457,584$). □

## 5. Conclusion

In the paper, we give new results for $D$-integrality of complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$, where $s = 1, 2, 3, 4, 5, 6$, and answer affirmatively questions 4.1 and 4.2 of Yang and Wang (see [15]). However, when $s > 6$, we have not found such $D$-integral graphs. Thus, we raise the following questions.

**Question 5.1.** Are there any distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ for $s \geqslant 7$?

**Question 5.2.** Are there any distance integral complete multipartite graphs $K_{a_1p_1,a_2p_2,\ldots,a_sp_s}$ with $a_1 = a_2 = \ldots = a_s = 1$ for $s \geqslant 5$?

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