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EXTREMELY PRIMITIVE GROUPS AND LINEAR SPACES

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Abstract. A non-regular primitive permutation group is called extremely primitive if a point stabilizer acts primitively on each of its nontrivial orbits. Let $\mathcal{S}$ be a nontrivial finite regular linear space and $G \leqslant \mathrm{Aut}(\mathcal{S})$. Suppose that G is extremely primitive on points and let $\mathrm{rank}(G)$ be the rank of G on points. We prove that $\mathrm{rank}(G) \geqslant 4$ with few exceptions. Moreover, we show that $\mathrm{Soc}(G)$ is neither a sporadic group nor an alternating group, and $G = \mathrm{PSL}(2, q)$ with $q+1$ a Fermat prime if $\mathrm{Soc}(G)$ is a finite classical simple group.

Keywords: linear space; automorphism; point-primitive automorphism group; extremely primitive permutation group

MSC 2010: 05B05, 05B25, 20B15, 20B25

1. Introduction

For positive integers v and b satisfying $b \geqslant v$, a *finite linear space* $\mathcal{S}$ is an incidence structure $(\mathcal{P}, \mathcal{L})$ consisting of a set $\mathcal{P}$ of v points and a collection $\mathcal{L}$ of b distinguished subsets of $\mathcal{P}$ called lines, such that any two points are incident with exactly one line. The linear space is said to be nontrivial if every line is incident with at least three points and there are at least two lines. If every line has the same number, say k, of points, then $\mathcal{S}$ is called a *regular linear space*, or a 2-$(v, k, 1)$ *design*. Further, in a regular linear space, the set of r lines through a given point, say α, is named the *pencil* through α, denoted by $\mathcal{L}(\alpha)$.

An *automorphism* of $\mathcal{S}$ is a permutation of $\mathcal{P}$ which leaves $\mathcal{L}$ invariant. The full automorphism group of $\mathcal{S}$ is denoted by $\mathrm{Aut}(\mathcal{S})$ and any subgroup $G \leqslant \mathrm{Aut}(\mathcal{S})$ is called an *automorphism group* of $\mathcal{S}$. Such a group G is called *point-primitive* if it

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is primitive on $\mathcal{P}$, and *flag-transitive* if it acts transitively on the set of flags of $\mathcal{S}$, where a *flag* is an incident point-line pair (α, λ). A flag-transitive pair $(\mathcal{S}, G)$ is said to be *locally primitive* if every point stabilizer G_α acts primitively on the pencil $\mathcal{L}(\alpha)$ through α (see [8]).

Several papers have already been devoted to the study of linear spaces having a point-primitive automorphism group. First observe that any flag-transitive group of a nontrivial linear space must be point-primitive (see [11]). In 1990, Buekenhout et al. classified the linear spaces admitting a flag-transitive automorphism group [3], apart from the unresolved case where the group is a one-dimensional affine group, which was settled in a later work of Liebeck [14]. The linear spaces admitting an automorphism group which is 2-transitive on points have been classified by Kantor [13]. A natural generalization is the classification of linear spaces admitting a primitive rank 3 automorphism group. Devillers classified these linear spaces when the automorphism groups are of almost simple type and grid type [10], [9]. In recent works [1], [16], Biliotti, Francot and Montinaro have completed the classification in the case when the automorphism groups are of affine type.

Our purpose is to explore the theory of point-primitive linear spaces, with the focus on the case when an automorphism group acts extremely primitively on points. Here, a non-regular primitive permutation group is *extremely primitive* if the point stabilizer acts primitively on each of its nontrivial orbits. According to [15], Theorem 1.1, a finite extremely primitive group is either of affine type or almost simple. The almost simple extremely primitive groups whose socle is sporadic, alternating or classical are classified by Burness, Praeger and Seress [5], [4]. In this paper, we use these works to investigate the finite linear spaces admitting an extremely primitive automorphism group. The main results of this paper are as follows.

Theorem 1.1. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ *be a nontrivial finite regular linear space and let* $G \leqslant \mathrm{Aut}(\mathcal{S})$ *be extremely primitive on* $\mathcal{P}$. *Then* $\mathrm{rank}(G) \geqslant 3$. *In particular, if* $\mathrm{rank}(G) = 3$, *then* $\mathcal{S}$ *is the affine space* $AG(m, 3)$ *and* $G \leqslant A\Gamma L(1, 3^m)$.

Theorem 1.2. *Let* $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ *be a nontrivial finite regular linear space and let* $G \leqslant \mathrm{Aut}(\mathcal{S})$ *be almost simple and extremely primitive on* $\mathcal{P}$. *Then* $\mathrm{Soc}(G)$ *is a group of Lie type, with* $G = \mathrm{PSL}(2, 2^{2^n})$ *if* $\mathrm{Soc}(G)$ *is classical.*

Remark 1.3. (i) For the case when $\mathrm{rank}(G) = 3$, the pair $(\mathcal{S}, G)$ in Theorem 1.1 is locally primitive, $m \geqslant 3$ is an odd integer and $r = (3^m - 1)/2$ is a prime.

(ii) For the case when $G = \mathrm{PSL}(2, 2^{2^n})$ in Theorem 1.2, the action of G on $\mathcal{P}$ is permutationally isomorphic to the action of G on the cosets of a dihedral subgroup $D_{2(2^{2^n}+1)}$, where $2^{2^n} + 1$ is a Fermat prime.

Combining Theorems 1.1 and 1.2 with [5], Theorem 1, and [4], Theorem 1.1, we have the following corollary.

Corollary 1.4. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a nontrivial finite regular linear space and let $G \leqslant \mathrm{Aut}(\mathcal{S})$ be extremely primitive on $\mathcal{P}$. Then one of the following statements holds:*

(i) $\mathrm{Soc}(G)$ *is an elementary abelian group.*

(ii) $\mathrm{Soc}(G)$ *is an exceptional group of Lie type.*

(iii) $G = \mathrm{PSL}(2, 2^{2^n})$, *the group G acting on $\mathcal{P}$ is permutationally isomorphic to G acting on the cosets of a dihedral subgroup $D_{2(2^{2^n}+1)}$, where $2^{2^n}+1$ is a Fermat prime.*

2. Preliminary results

Our discussion is based on a partial classification of the finite extremely primitive groups. The extremely primitive permutation groups of almost simple type are classified in [5], [4], except the case of exceptional groups.

Lemma 2.1 ([5], Theorem 1)**.** *Let G be a finite almost simple primitive permutation group, with stabilizer H and socle G_0. Assume that G_0 is a sporadic or alternating group. Then G is extremely primitive if and only if (G_0, H) is one of the cases listed in Table 1, where $a = |G : G_0|$.*

Case	G_0	H	Rank	Subdegrees	Conditions
1	A_{2m}	$N_G((S_m \wr S_2) \cap G)$	$(m+1)/2$	$\binom{m}{i}^2$, $0 \leqslant i \leqslant (m-1)/2$	m odd
2	A_n	$N_G(A_{n-1})$	2	$1, n-1$	$G \leqslant S_n$
3	A_n	$N_G(D_{10})$	2	$1, 5$	$n = 5$
4	M_{11}	S_6	2	$1, 10$	
5	M_{11}	$\mathrm{PSL}(2, 11)$	2	$1, 11$	
6	M_{12}	M_{11}	2	$1, 11$	$G = G_0$
7	M_{22}	$\mathrm{PSL}(3, 4).a$	2	$1, 21$	
8	M_{23}	M_{22}	2	$1, 22$	
9	M_{24}	M_{23}	2	$1, 23$	
10	J_2	$\mathrm{PSU}(3, 3).a$	3	$1, 36, 63$	
11	HS	$M_{22}.a$	3	$1, 22, 77$	
12	HS	$\mathrm{PSU}(3, 5)$	2	$1, 175$	$G = G_0$
13	Suz	$G_2(4).a$	3	$1, 416, 1365$	
14	McL	$\mathrm{PSU}(4, 3).a$	3	$1, 112, 162$	
15	Ru	${}^2F_4(2)$	3	$1, 1755, 2304$	
16	Co_2	$\mathrm{PSU}(6, 2).2$	3	$1, 891, 1408$	
17	Co_2	McL	6	1, 275, 2025, 7128, 15400, 22275	
18	Co_3	McL.2	2	$1, 275$	

Table 1. The extremely primitive sporadic and alternating groups.

Lemma 2.2 ([4], Theorem 1.1). *Let G be a finite almost simple classical primitive permutation group, with stabilizer H and socle G_0. Then G is extremely primitive if and only if (G_0, H) is one of the cases listed in Table 2.*

Case	G_0	Type of H	v	Rank	Conditions
1	$\mathrm{PSL}(2,q)$	P_1	$q+1$	2	$q \geqslant 4$
2	$\mathrm{PSp}(2m,2)'$	$O^{\pm}(2m,2)$	$2^{2m-1} \mp 2^{m-1}$	2	$n \geqslant 4$
3	$\mathrm{PSL}(2,q)$	$D_{2(q+1)}$	$q(q-1)/2$	$q/2$	$G = G_0$, $q > 2$, $q+1$ Fermat prime
4	$\mathrm{PSL}(4,2)$	A_7	8	2	
5	$\mathrm{PSU}(4,3)$	$\mathrm{PSL}(3,4)$	162	3	$G = G_0.2^2$ or $G_0.2$
6	$\mathrm{PSL}(3,4)$	A_6	56	3	$G = G_0.2^2$ or $G_0.2$
7	$\mathrm{PSL}(2,11)$	A_5	11	2	$G = G_0$

Table 2. The extremely primitive classical groups.

Remark 2.3. In Table 2, 'Type of H' describes the approximate group-theoretic structure of H, v is the degree of G, 'P_1' denotes a Borel subgroup of G which is the stabilizer of a 1-dimensional subspace of the natural G_0-module. The entries in the column 'Subdegrees' in Table 1 can be found in the on-line Atlas of Finite Group Representations [18], and the information in the columns 'v' and 'Rank' in Table 2 can be found in [4], Table 1 (or computed directly, using MAGMA [2]).

In the following, we always assume that $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a nontrivial finite regular linear space with parameters v, b, k and r as described in Section 1, and $G \leqslant \mathrm{Aut}(\mathcal{S})$. The first lemma is well-known.

Lemma 2.4 ([6], Lemma 2.1). *For parameters v, b, k and r of $\mathcal{S}$, we have*

$$r = \frac{v-1}{k-1}, \quad b = \frac{v(v-1)}{k(k-1)},$$

and $k(k-1)+1 \leqslant v$.

Lemma 2.5. *Let G be point-transitive and let $\mathcal{L}_0$ be an orbit of G on $\mathcal{L}$ of length b_0. Then every point occurs in exactly r_0 lines in $\mathcal{L}_0$, where $r_0 = b_0 k/v$.*

Proof. Let (α, λ) be a flag of $\mathcal{S}$, let $\mathcal{L}_0(\alpha) = \mathcal{L}(\alpha) \cap \mathcal{L}_0$ and let $\beta \in \mathcal{P}$ be another point. Since $G \leqslant \mathrm{Aut}(\mathcal{S})$ is point-transitive, there exists an element $g \in G$ such that $\alpha^g = \beta$. Thus

$$\mathcal{L}_0(\alpha)^g = \mathcal{L}(\alpha)^g \cap \mathcal{L}_0 = \mathcal{L}(\beta) \cap \mathcal{L}_0 = \mathcal{L}_0(\beta).$$

It follows that the number of lines in $\mathcal{L}_0(\alpha)$ is independent of the choice of α, denoted by r_0. Let $X = \{(\alpha,\lambda)\colon\ \alpha\in\lambda,\ \lambda\in\mathcal{L}_0\}$; counting it in two ways, we get $b_0k = vr_0$. $\square$

Let H be a group acting transitively on a set Ω. Recall that a *block* Δ for H is a subset of Ω such that for every $g\in H$ either $\Delta^g=\Delta$ or $\Delta^g\cap\Delta=\emptyset$. Clearly, the empty set $\emptyset$, the singletons $\{\alpha\}$ ($\alpha\in\Omega$), and Ω are blocks; they are called the trivial blocks. Any other block is called nontrivial.

Lemma 2.6. *Let G be point-transitive and let (α,λ) be a flag. Assume that Δ is a nontrivial orbit of G_α on $\mathcal{P}$ that meets λ. Then $\lambda\cap\Delta$ is a block for G_α on Δ. Moreover, if $\Delta_1,\Delta_2,\ldots,\Delta_s$ are the nontrivial orbits of G_α that meet λ, that is,*

$$\lambda=\{\alpha\}\cup(\lambda\cap\Delta_1)\cup(\lambda\cap\Delta_2)\cup\ldots\cup(\lambda\cap\Delta_s),$$

where $|\lambda\cap\Delta_i|>0$ and $\Delta_i\neq\Delta_j$ $(i\neq j)$ for all $i,j=1,2,\ldots,s$, then

$$k=1+\sum_{i=1}^{s}\frac{|\Delta_i|}{t}, \tag{2.1}$$

where

$$t=|\lambda^{G_\alpha}|=\frac{|\Delta_i|}{|\lambda\cap\Delta_i|},\quad i=1,2,\ldots,s.$$

Proof. Assume that $(\lambda\cap\Delta)\cap(\lambda\cap\Delta)^g\neq\emptyset$ for some $g\in G_\alpha$. Let $\beta\in(\lambda\cap\Delta)\cap(\lambda\cap\Delta)^g$. Then α and β are incident with both λ and λ^g. It follows that $\lambda=\lambda^g$, so $\lambda\cap\Delta=(\lambda\cap\Delta)^g$. Therefore, $\lambda\cap\Delta$ is a block for G_α, and $(\lambda\cap\Delta)^{G_\alpha}$ is a partition of Δ with t parts of length $|\lambda\cap\Delta|$, where $t=|\Delta|/|\lambda\cap\Delta|$. Since

$$\frac{|\Delta|}{|\lambda\cap\Delta|}=|(\lambda\cap\Delta)^{G_\alpha}|=|G_\alpha:(G_\alpha)_{\lambda\cap\Delta}|=|G_\alpha:G_{\alpha\lambda}|=|\lambda^{G_\alpha}|,$$

then $t=|\lambda^{G_\alpha}|$, and hence $k=1+\sum\limits_{i=1}^{s}|\lambda\cap\Delta_i|=1+\sum\limits_{i=1}^{s}|\Delta_i|/t$. $\square$

The next lemma is a slight modification of [1], Proposition 2, that will be useful in the proofs of our main results.

Lemma 2.7. *Let G be point-transitive of rank 3. If the subdegrees of G on $\mathcal{P}$ are 1, r_1 and r_2, then one of the following cases occurs:*

(i) *G acts flag-transitively on $\mathcal{S}$, and for each $\lambda \in \mathcal{L}$, the group G_λ induces a rank 3 group on λ.*

(ii) *$k-1 \mid \gcd(r_1, r_2)$. If $|G|$ is odd then G is line-transitive on $\mathcal{S}$. If $|G|$ is even then G_λ induces a 2-transitive group on λ for each $\lambda \in \mathcal{L}$, and G has precisely two orbits on $\mathcal{L}$.*

Proof. Let (α, λ) be a flag, let Δ_1 and Δ_2 be the two nontrivial orbits of G_α on $\mathcal{P}$ of length r_1 and r_2, respectively. Write $\lambda = \{\alpha\} \cup (\Delta_1 \cap \lambda) \cup (\Delta_2 \cap \lambda)$.

(i) First assume that $\Delta_1 \cap \lambda \neq \emptyset$ and $\Delta_2 \cap \lambda \neq \emptyset$. Then every line in $\mathcal{L}(\alpha)$ meets Δ_1 and Δ_2. Therefore, G_α is transitive on $\mathcal{L}(\alpha)$ and it follows that G is flag-transitive. Furthermore, G_λ is transitive on λ, with $\Delta_1 \cap \lambda$ and $\Delta_2 \cap \lambda$ being the two nontrivial orbits of $(G_\lambda)_\alpha$ on λ. That is, G_λ induces a rank 3 group on λ.

(ii) For the rest we may assume that there is one suborbit which does not meet λ. Without loss of generality we assume that $\Delta_1 \cap \lambda = \emptyset$, so either $\Delta_1 \cap \lambda' = \emptyset$ or $\Delta_2 \cap \lambda' = \emptyset$ for any other line $\lambda' \in \mathcal{L}(\alpha)$. Let $\mathcal{L}_i(\alpha) = \{\lambda\colon\ \lambda \cap \Delta_i \neq \emptyset,\ \lambda \in \mathcal{L}(\alpha)\}$ for $i = 1, 2$. It is obvious that the pencil $\mathcal{L}(\alpha)$ is the union of $\mathcal{L}_1(\alpha)$ and $\mathcal{L}_2(\alpha)$. Write $\lambda_1 = \{\alpha\} \cup (\Delta_1 \cap \lambda_1)$ and $\lambda_2 = \{\alpha\} \cup (\Delta_2 \cap \lambda_2)$. Then by Lemma 2.6, $k-1 = |\Delta_i \cap \lambda_i| \mid r_i$ for $i = 1, 2$.

If $|G|$ is odd, then G is 2-homogeneous. Therefore G is transitive on lines, for it is transitive on unordered pairs of points.

Now assume $|G|$ is even. The orbits of G are self-paired, hence for every ordered pair of distinct points $(\beta, \gamma) \in \mathcal{P} \times \mathcal{P}$, we have $(\beta, \gamma)^g = (\gamma, \beta)$ for some $g \in G$. Thus G_λ is transitive on λ. Let $(\beta_1, \gamma_1), (\beta_2, \gamma_2)$ be two ordered pairs of distinct points of λ. Then there exist two elements $g_1, g_2 \in G_\lambda$ such that

$$(\beta_1, \gamma_1)^{g_1} = (\alpha, \gamma_1^{g_1}),\ (\beta_2, \gamma_2)^{g_2} = (\alpha, \gamma_2^{g_2}).$$

Hence $\{\gamma_1^{g_1}, \gamma_2^{g_2}\} \subseteq \Delta_1 \cap \lambda \subseteq \Delta_1$ and thus $\gamma_1^{g_1 g} = \gamma_2^{g_2}$ for some $g \in G_\alpha$. Therefore $(\beta_1, \gamma_1)^{g_1 g g_2^{-1}} = (\beta_2, \gamma_2)$ and we conclude that G_λ induces a 2-transitive group on λ.

To complete the proof, it remains to show that G has precisely two orbits on $\mathcal{L}$. To see this, let $\lambda_1 \in \mathcal{L}_1(\alpha)$ and $\lambda_2 \in \mathcal{L}_2(\alpha)$. If there is an element $g \in G$ such that $\lambda_1^g = \lambda_2$, then we have two points $\beta \in \lambda_1, \gamma \in \lambda_2$ such that $(\alpha, \beta)^g = (\gamma, \alpha)$, which implies that (α, β) and (α, γ) are in the same orbit of G. Therefore, β and γ are in the same orbit of G_α on $\mathcal{P} \setminus \{\alpha\}$, a contradiction. Thus λ_1 and λ_2 are in different orbits of G on $\mathcal{L}$. Moreover, G is point-transitive and G_α is transitive on $\mathcal{L}_1(\alpha)$ and $\mathcal{L}_2(\alpha)$, respectively, so G has precisely two orbits on $\mathcal{L}$. $\square$

The following lemma gives the classification of 2-$(v,3,1)$ designs admitting a line-transitive automorphism group (see [7], Theorem 3.2).

Lemma 2.8. *Let $\mathcal{S}=(\mathcal{P},\mathcal{L})$ be a 2-$(v,3,1)$ design admitting a line-transitive automorphism group G, then one of the following conclusions holds:*

(i) *G is 2-transitive on $\mathcal{P}$.*

(ii) *$|G|$ is odd, $G\leqslant A\Gamma L(1,p^m)$ contains the translation subgroup, where p is a prime, m is a natural number and one of the following conditions holds:*

  (a) *$p=3$, m is odd, $\mathcal{S}$ is an affine geometry $AG(m,3)$ and G has rank 3 on $\mathcal{P}$.*

  (b) *$p^m\equiv 7 \pmod{12}$ and $\mathcal{S}$ is a Netto design.*

  (c) *$p^m\equiv 7 \pmod{12}$ and G has rank 7 on $\mathcal{P}$.*

3. Proofs of the main results

Now we assume that $\mathcal{S}$ is a nontrivial finite regular linear space and $G\leqslant \mathrm{Aut}(\mathcal{S})$ is extremely primitive on points. Let (α,λ) be a flag and let Δ be an orbit of G_α on $\mathcal{P}\setminus\{\alpha\}$ that meets λ. Then we have $|\lambda\cap\Delta|=1$ or $|\Delta|$ by Lemma 2.6.

Proof of Theorem 1.1. First we suppose that $\mathrm{rank}(G)=2$. Then G_α acts primitively on $\mathcal{P}\setminus\{\alpha\}$, thus G is 2-primitive and $k=2$ or $k=v$. This implies that $\mathcal{S}$ is trivial.

Now suppose that $\mathrm{rank}(G)=3$, and Δ_1, Δ_2 are the two orbits of G_α on $\mathcal{P}\setminus\{\alpha\}$ of length r_1,r_2, respectively. Let (α,λ) be a flag.

If G is not flag-transitive, then we have two distinct lines $\lambda_1,\lambda_2\in\mathcal{L}(\alpha)$ such that $\Delta_i\cap\lambda_i=\emptyset$ for $i=1,2$ by Lemma 2.6. Then

$$k=1+|\Delta_2\cap\lambda_1|=1+|\Delta_1\cap\lambda_2|.$$

Thus

$$|\Delta_2|=|\Delta_2\cap\lambda_1|=|\Delta_1\cap\lambda_2|=|\Delta_1|=k-1,$$

otherwise, $k=2$. Therefore,

$$|\Delta_1|=|\Delta_2|=\frac{v-1}{2}=k-1.$$

This implies that $r=(v-1)/(k-1)=2$, which is impossible. Therefore, G is flag-transitive.

Since G is flag-transitive, $\Delta_i\cap\lambda\neq\emptyset$ for $i=1$ and 2. In view of equation (2.1), we obtain

$$k\in\{1+1+1,1+r_1+1,1+1+r_2,1+r_1+r_2\}.$$

If $k = 1 + r_1 + 1$, that is, $|\lambda \cap \Delta_1| = r_1$ and $|\lambda \cap \Delta_2| = 1$, then $|\lambda^{G_\alpha}| = 1 = r_2$ and thus $k = v$, a contradiction. Similarly, we get $k = v$ if $k \in \{1 + 1 + r_2, 1 + r_1 + r_2\}$. So $k = 1 + 1 + 1 = 3$, and then $G \leqslant A\Gamma L(1, p^m)$ according to Lemma 2.8, where p is a prime and m is a natural number.

Let $\beta \in \lambda$, then $G_{\alpha\beta} \leqslant G_{\alpha\lambda} \leqslant G_\alpha$. Since $r \neq 1$, we have $G_{\alpha\lambda} \neq G_\alpha$. Thus $G_{\alpha\beta} = G_{\alpha\lambda}$ is maximal in G_α since G is extremely primitive. This implies that $(\mathcal{S}, G)$ is locally primitive, and then $r = (v-1)/(k-1) = (p^m - 1)/2$ is prime by [8], Theorem 4. It follows that p is an odd prime and we have the factorization

$$\frac{p^m - 1}{2} = \frac{p-1}{2}(p^{m-1} + p^{m-2} + \ldots + p + 1).$$

Since $p^{m-1} + \ldots + p + 1 > 1$, we have $(p-1)/2 = 1$ and thus $p = 3$. Since $3^m \not\equiv 7 \pmod{12}$, Lemma 2.8 implies that $\mathcal{S}$ is the affine space $AG(m, 3)$ with m odd. $\square$

Proof of Theorem 1.2. We will prove Theorem 1.2 in two cases: (1) $\mathrm{Soc}(G)$ is a sporadic or an alternating group; (2) $\mathrm{Soc}(G)$ is a finite classical simple group.

Lemma 3.1. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a nontrivial finite regular linear space and assume that $G \leqslant \mathrm{Aut}(\mathcal{S})$ is extremely primitive on points. Then $\mathrm{Soc}(G)$ is neither a sporadic group nor an alternating group.*

Proof. Suppose that $\mathrm{Soc}(G) = G_0$ is a sporadic group or an alternating group. Let $H = G_\alpha$ be the stabilizer of a point $\alpha \in \mathcal{P}$. Then (G_0, H) is one of the cases listed in Table 1. According to Theorem 1.1, we know that the only possible cases are the ones numbered 1 and 17 in the table.

First consider Case 1 in Table 1, so $G_0 = A_{2m}$ and $H = N_G((S_m \wr S_2) \cap G)$. Here $\mathcal{P}$ is the set of all partitions of $\{1, 2, \ldots, 2m\}$ into two parts of size m, where m is odd, so $v = \binom{2m}{m}/2$ and the nontrivial subdegrees are $\binom{m}{i}^2$ for $1 \leqslant i \leqslant (m-1)/2$. By Theorem 1.1, we have $m \geqslant 5$. Let $\alpha \in \mathcal{P}$ and let Δ be the orbit of G_α on $\mathcal{P}$ of length $\binom{m}{(m-1)/2}^2$. Choose one point $\beta \in \Delta$ and let λ be the unique line through α and β, so $|\lambda \cap \Delta| = 1$ or $|\Delta|$ by Lemma 2.6. If $|\lambda \cap \Delta| = |\Delta|$, then $k \geqslant 1 + \binom{m}{(m-1)/2}^2$. We have

$$v - 1 = \sum_{i=1}^{(m-1)/2} \binom{m}{i}^2 < \frac{m-1}{2}(k-1) < k(k-1),$$

a contradiction (see Lemma 2.4). Thus $|\lambda \cap \Delta| = 1$, and there must be another orbit Δ_1 of G_α meeting λ since $k > 2$. Since $|\Delta| = |\Delta|/|\lambda \cap \Delta| = |\Delta_1|/|\lambda \cap \Delta_1|$ we have $\binom{m}{(m-1)/2}^2 \mid |\Delta_1|$, which is impossible. Therefore, Case 1 in Table 1 is ruled out.

Finally, let us turn to Case 17 in Table 1, where $G = Co_2$ and $H = McL$. Here $v = 47104$ and the subdegrees are 1, 275, 2025, 7128, 15400 and 22275. Let $\alpha \in \mathcal{P}$

and let Δ be the orbit of G_α on $\mathcal{P}$ of length 22275. Choose one point $\beta \in \Delta$ and let λ be the unique line through α and β, so $|\lambda \cap \Delta| = 1$ or $|\Delta|$ by Lemma 2.6. If $|\lambda \cap \Delta| = |\Delta|$, then $k \geqslant 1 + 22275$, hence $v - 1 < k(k-1)$, a contradiction. Thus $|\lambda \cap \Delta| = 1$, and there must be another nontrivial orbit Δ_1 of G_α that meets λ with $|\Delta| = |\Delta|/|\lambda \cap \Delta| = |\Delta_1|/|\lambda \cap \Delta_1|$. So $22275 \mid |\Delta_1|$, which is impossible. Therefore, Case 17 in Table 1 is also ruled out. $\square$

Lemma 3.2. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be a nontrivial finite regular linear space and suppose $G \leqslant \mathrm{Aut}(\mathcal{S})$ is extremely primitive on points. If $\mathrm{Soc}(G)$ is a finite classical simple group, then $G = \mathrm{PSL}(2, q)$ and $\mathcal{S}$ has parameters*

$$(b, v, k, r) = \Big(\frac{t^2 q(q-1)(q+1)}{q-2+2t}, \frac{q(q-1)}{2}, \frac{q-2}{2t} + 1, t(q+1)\Big),$$

where $q = 2^{2^n}$, t is an odd positive integer and G acting on $\mathcal{P}$ is permutationally isomorphic to G acting on the cosets of a dihedral subgroup $D_{2(q+1)}$. Moreover, either $t = 1$ and $\mathcal{S}$ is a Witt-Bose-Shrikhande space, or $t \geqslant 73$ and $n \geqslant 4$.

Proof. Suppose $\mathrm{Soc}(G) = G_0$ is a simple classical group and let H be the stabilizer of a point. Then (G_0, H) is one of the cases listed in Table 2.

Cases 1, 2 and 4–7 are ruled out by Theorem 1.1. Therefore, it remains to consider Case 3 in Table 2.

Here $G = \mathrm{PSL}(2, q)$, $G_\alpha = D_{2(q+1)}$ is a dihedral subgroup and $q + 1$ is a Fermat prime, so $q = 2^{2^n}$ for a positive integer n. In view of Proposition 5.3 of [4], $\mathrm{rank}(G) = q/2$ and all the nontrivial subdegrees are $q + 1$. Let (α, λ) be a flag and let Δ be a nontrivial orbit of G_α meeting λ. Then $|\lambda \cap \Delta| = 1$ or $q + 1$ by Lemma 2.6. If $|\lambda \cap \Delta| = q+1$, then $k \geqslant 1+q+1$, which contradicts Lemma 2.4 since $v = q(q-1)/2$. Thus $|\lambda \cap \Delta| = 1$, and it follows that the orbit of G_α on λ is of length $q + 1$. If G_α has t orbits on $\mathcal{L}(\alpha)$, then $t = r/(q + 1)$, which implies that $k = (q - 2)/(2t) + 1$. From the equality $bk = vr$ we get

$$\frac{q(q-1)}{2} t(q+1) = b\Big(\frac{q-2}{2t} + 1\Big),$$

thus $b = t^2 q(q-1)(q+1)/(q-2+2t)$. In particular, if $t = 1$ then G is flag-transitive and it follows from [3] that $\mathcal{S}$ is the Witt-Bose-Shrikhande space with parameters $(v, b, k, r) = (q(q-1)/2, q^2 - 1, q/2, q + 1)$.

From now on, we suppose that $t > 1$. Let τ be an involution in G. Then there are two points $\alpha, \beta \in \mathcal{P}$ such that $\langle\tau\rangle \leqslant G_{\alpha\beta}$. Recall that the length of nontrivial orbits of G_α through β is $q + 1$, and hence $|G_{\alpha\beta}| = |G_\alpha|/|\beta^{G_\alpha}| = 2(q+1)/(q+1) = 2$. Thus $G_{\alpha\beta} = \langle\tau\rangle$. Let $\mathcal{P}_\tau = \mathrm{Fix}_{\mathcal{P}}(\tau)$ and

$$\mathcal{L}_\tau = \{\lambda \cap \mathrm{Fix}_{\mathcal{P}}(\tau)\colon\ \lambda \in \mathcal{L},\ |\lambda \cap \mathrm{Fix}_{\mathcal{P}}(\tau)| \geqslant 2\}.$$

Then $\mathcal{S}_\tau = (\mathcal{P}_\tau, \mathcal{L}_\tau)$ is an induced linear space with v_0 points. Both G and $G_\alpha \cong Z_{q+1} : Z_2$ have unique conjugacy classes of involutions and it is easy to see that $|N_G(\langle\tau\rangle)| = q$ (see [12]) and $|N_{G_\alpha}(\langle\tau\rangle)| = 2$. Therefore by [17], Lemma 2.1, we have

$$v_0 = |N_G(\langle\tau\rangle) : N_{G_\alpha}(\langle\tau\rangle)| = \frac{q}{2}.$$

Let λ be the line through α and γ, where $\gamma \in \mathcal{P}_\tau$. Note that $G_{\alpha\gamma} = \langle\tau\rangle \leqslant G_\lambda$. If β_0 is another point in λ, then $\beta_0^{\langle\tau\rangle} \subseteq \beta_0^{\langle\tau\rangle} \cap \lambda$. Since $|\beta_0^{G_\alpha} \cap \lambda| = 1$, it follows that $|\beta_0^{\langle\tau\rangle} \cap \lambda| = 1$. Thus $\beta_0 \subseteq \mathrm{Fix}_{\mathcal{P}}(\tau)$. This implies that the induced linear space is regular with line size $k_0 = k$.

Let $\sigma \in G$ be the element interchanging α and β, and let f be the number of lines in $\mathcal{S}_\tau$ fixed by σ. Since $\langle\tau\rangle^\sigma = G_{\alpha\gamma}^\sigma = G_{\alpha\gamma} = \langle\tau\rangle$, we have $\sigma \in N_G(\langle\tau\rangle) \cong Z_2^{2^n}$ and then σ is an involution. If $\mathrm{Fix}_{\mathcal{P}_\tau}(\sigma) \neq \emptyset$, then there are at least two points $\beta_1, \beta_2 \in \mathrm{Fix}_{\mathcal{P}}(\tau)$ fixed by σ. This implies that $|G_{\beta_1\beta_2}| > 2$, a contradiction. So $|\mathrm{Fix}_{\mathcal{P}_\tau}(\sigma)| = 0$. Hence k is even and

$$f \times \frac{k}{2} = \frac{v_0}{2},$$

therefore $k \mid q/2$. Let $k = 2^s$. Then $s \mid 2^n - 1$ since $k - 1 \mid q/2 - 1$. Note that $2 < k < 2^{2^n-1}$, $1 < s < 2^n - 1$ is odd and $n \geqslant 4$. Since $k = (q-2)/(2t) + 1$, it follows that $2^s t = 2^{2^n-1} + (t-1)$. Hence $2^s \mid t-1$. Let $t - 1 = 2^s t_0$. Substituting this in $2^s t = 2^{2^n-1} + (t-1)$ we get $2^s t = 2^{2^n-1} + 2^s t_0$. Thus $t = 2^{2^n-1-s} + t_0 = 2^s t_0 + 1$. This implies that $2^s \mid t_0 - 1$ since $2s \leqslant 2^n - 1$. If $t_0 = 1$, then $t - 1 = 2^s = 2^{2^n-1-s}$, which is impossible. Thus $t_0 - 1 \geqslant 2^s$, and then $t \geqslant 2^s(2^s + 1) + 1 \geqslant 73$ since $s \geqslant 3$. □

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References

[1] *M. Biliotti, A. Montinaro, E. Francot*: 2-$(v,k,1)$ designs with a point-primitive rank 3 automorphism group of affine type. Des. Codes Cryptography *76* (2015), 135–171.
[2] *W. Bosma, J. Cannon, C. Playoust*: The Magma algebra system I: The user language. J. Symb. Comput. *24* (1997), 235–265.
[3] *F. Buekenhout, A. Delandtsheer, J. Doyen, P. B. Kleidman, M. W. Liebeck, J. Saxl*: Linear spaces with flag-transitive automorphism groups. Geom. Dedicata *36* (1990), 89–94.
[4] *T. C. Burness, C. E. Praeger, Á. Seress*: Extremely primitive classical groups. J. Pure Appl. Algebra *216* (2012), 1580–1610.
[5] *T. C. Burness, C. E. Praeger, Á. Seress*: Extremely primitive sporadic and alternating groups. Bull. Lond. Math. Soc. *44* (2012), 1147–1154.
[6] *A. R. Camina, P. M. Neumann, C. E. Praeger*: Alternating groups acting on finite linear spaces. Proc. Lond. Math. Soc. (3) *87* (2003), 29–53.

[7] *P. C. Clapham*: Steiner triple systems with block-transitive automorphism groups. Discrete Math. *14* (1976), 121–131.

[8] *A. Delandtsheer*: 2-designs with a group transitive on the pairs of intersecting lines. Simon Stevin *66* (1992), 107–112.

[9] *A. Devillers*: A classification of finite partial linear spaces with a primitive rank 3 automorphism group of grid type. Eur. J. Comb. *29* (2008), 268–272.

[10] *A. Devillers*: A classification of finite partial linear spaces with a primitive rank 3 automorphism group of almost simple type. Innov. Incidence Geom. *2* (2005), 129–175.

[11] *D. G. Higman, J. E. McLaughlin*: Geometric ABA-groups. Ill. J. Math. *5* (1961), 382–397.

[12] *B. Huppert*: Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften 134, Springer, Berlin, 1967. (In German.)

[13] *W. M. Kantor*: Homogeneous designs and geometric lattices. J. Comb. Theory, Ser. A *38* (1985), 66–74.

[14] *M. W. Liebeck*: The classification of finite linear spaces with flag-transitive automorphism groups of affine type. J. Comb. Theory, Ser. A *84* (1998), 196–235.

[15] *A. Mann, C. E. Praeger, Á. Seress*: Extremely primitive groups. Groups Geom. Dyn. *1* (2007), 623–660.

[16] *A. Montinaro*: 2-$(v, k, 1)$ designs admitting a primitive rank 3 automorphism group of affine type: the extraspecial and the exceptional classes. J. Comb. Des. *23* (2015), 481–498.

[17] *C. E. Praeger, M. Y. Xu*: Vertex-primitive graphs of order a product of two distinct primes. J. Comb. Theory, Ser. B *59* (1993), 245–266.

[18] *R. Wilson* et al.: Atlas of Finite Group Representations. Version 3. Available at `http://brauer.maths.qmul.ac.uk/Atlas/v3/`.

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