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CLASSIFICATION OF RINGS WITH TOROIDAL JACOBSON GRAPH

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Abstract. Let R be a commutative ring with nonzero identity and $J(R)$ the Jacobson radical of R. The Jacobson graph of R, denoted by $\mathfrak{J}_R$, is defined as the graph with vertex set $R \setminus J(R)$ such that two distinct vertices x and y are adjacent if and only if $1 - xy$ is not a unit of R. The genus of a simple graph G is the smallest nonnegative integer n such that G can be embedded into an orientable surface S_n. In this paper, we investigate the genus number of the compact Riemann surface in which $\mathfrak{J}_R$ can be embedded and explicitly determine all finite commutative rings R (up to isomorphism) such that $\mathfrak{J}_R$ is toroidal.

Keywords: planar graph; genus of a graph; local ring; nilpotent element; Jacobson graph

MSC 2010: 05C10, 05C25, 13M05

1. Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. Beck in [10] began the study of associating a graph called the zero-divisor graph to a commutative ring being mainly interested in the coloring of the zero-divisor graph. For a commutative ring R, the *zero-divisor graph* $\Gamma(R)$ is the simple graph with R as the vertex set in which two distinct elements x and y are adjacent if and only if $xy = 0$, see [10]. In [4], Anderson and Livingston modified and studied the zero-divisor graph $\Gamma(R)$ as the graph with the nonzero zero-divisors $Z(R)^*$ of R as the vertex set. While they focus just on the zero-divisors of the rings

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(see [1], [2], [3], [4], [10]), there are many other kinds of graphs associated to rings, some of which have been extensively studied, see for example [5], [6], [13], [17], [18].

Using the notions of Jacobson radicals and semi-simplicity of rings we intend to associate a graph to a ring and investigate some of its graph theoretical properties. Throughout this paper R stands for a commutative ring with nonzero identity. Recall that the Jacobson radical of R is defined by

$$J(R) = \bigcap\{\mathfrak{m}\colon \mathfrak{m} \text{ is the maximal ideal of } R\}.$$

It is known that an element $r \in R$ belongs to $J(R)$ if and only if $1-rx$ is invertible for all $x \in R$. We recall that R is semi-simple if $J(R) = (0)$ and hence the quotient ring $R/J(R)$ is always a semi-simple ring. In [8], Azimi et al. introduced the *Jacobson graph* of R, denoted by $\mathfrak{J}_R$, as the graph with vertex set $V(\mathfrak{J}_R) = R \setminus J(R)$ in such a way that two distinct vertices $x, y \in V(\mathfrak{J}_R)$ are adjacent if and only if $1-xy \notin U(R)$, where $U(R)$ denotes the group of units of R. Also in that article, the authors classified the finite commutative rings R for which $\mathfrak{J}_R$ is planar (see [8], Theorem 4.3).

In recent years, many research articles have been published on the genera of zero-divisor graphs. The planarity of zero-divisor graphs was studied in [11], [20]. Toroidal zero-divisor graphs were classified independently by Wang [21], [14] and Wickham [23]. Genus two zero-divisor graphs of local rings are studied by Bloomfield and Wickham [12]. Also various research articles have been published on the genera of the graphs constructed out of the rings [6], [12], [19]. In [8], the authors classified the finite commutative rings R for which $\mathfrak{J}_R$ is planar (see Theorem 4.3). In this paper, we characterize all finite commutative rings whose Jacobson graph $\mathfrak{J}_R$ has genus one.

Throughout the paper, we assume that R is a finite commutative ring with identity, $Z(R)$ its set of zero-divisors, $N(R)$ its set of nilpotent elements and $U(R)$ its group of units. We denote the ring of integers modulo n by $\mathbb{Z}_n$ and the Galois field with q elements by $\mathbb{F}_q$. If X is a subset of R, we denote $X - \{0\}$ by X^*. For basic definitions on rings, one may consult [7], [16].

2. Preliminaries

In this section, we summarize notation, concepts and results related to the genus of a graph which will be needed in the subsequent sections.

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$.

The main objective of topological graph theory is to embed a graph into a surface. Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The genus of a graph G, denoted by $g(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A graph G with genus 0 is called a *planar graph* while a graph G with genus 1 is called a *toroidal graph*. Further note that if H is a subgraph of a graph G, then $g(H) \leqslant g(G)$. A result of Battle, Harary, Kodama, and Youngs states that the genus of a graph is the sum of the genera of its blocks, see [9]. For example, the graph $\mathbb{H}$ in Figure 2.1 has two blocks, both isomorphic to $K_{3,3}$, and so has genus 2, see Wickham [23]. For details on the notion of embedding a graph in a surface, see [22].

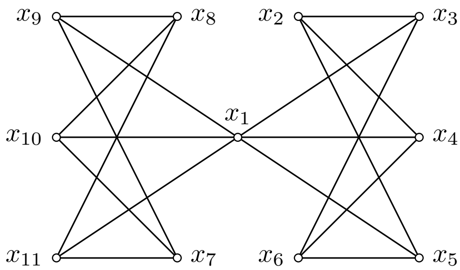


Figure 2.1. Graph $\mathbb{H}$.

Now we summarize some results and bounds for the genus of a graph.

Lemma 2.1 ([22]). *$g(K_n) = \lceil (n-3)(n-4)/12 \rceil$ if $n \geqslant 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 2.2 ([22]). *$g(K_{m,n}) = \lceil (m-2)(n-2)/4 \rceil$ if $m, n \geqslant 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{3,n}) = 2$ if $n = 7, 8, 9, 10$ and $g(K_{5,4}) = g(K_{6,4}) = 2$.*

3. Genus of Jacobson graph

The main goal of this section is to determine all finite rings R whose Jacobson graph has genus one. Azimi et al. [8] determined the finite commutative rings R for which $\mathfrak{J}_R$ is planar. The following observation proved by Azimi et al. [8] is used frequently in this article.

Theorem 3.1 ([8], Theorem 2.2)**.** *Let $(R,\mathfrak{m})$ be a finite local ring with associated field F. Then the connected components of $\mathfrak{J}_R$ are either complete graphs of size $|\mathfrak{m}|$ or complete bipartite graphs $K_{|\mathfrak{m}|,|\mathfrak{m}|}$. Moreover,*

(1) *if $|F|$ is odd, then $\mathfrak{J}_R$ has two complete components and $(|F|-3)/2$ complete bipartite components, and*
(2) *if $|F|$ is even, then $\mathfrak{J}_R$ has one complete component and $(|F|-2)/2$ complete bipartite components.*

Theorem 3.2 ([8], Theorem 4.3)**.** *Let R be a commutative finite ring. Then $\mathfrak{J}_R$ is planar if and only if either R is a field, or R is isomorphic to one of the following rings:*

$$\mathbb{Z}_4,\ \mathbb{F}_2\times\mathbb{F}_2,\ \frac{\mathbb{F}_2[x]}{\langle x^2\rangle},\ \mathbb{F}_2\times\mathbb{F}_3,\ \mathbb{Z}_8,\ \mathbb{F}_2\times\mathbb{Z}_4,\ \mathbb{F}_2\times\frac{\mathbb{F}_2[x]}{\langle x^2\rangle},\ \mathbb{F}_2\times\mathbb{F}_2\times\mathbb{F}_2,\ \frac{\mathbb{F}_2[x]}{\langle x^3\rangle},$$
$$\frac{\mathbb{Z}_4[x]}{\langle 2x,x^2\rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x,x^2-2\rangle},\ \mathbb{F}_2\times\mathbb{F}_4,\ \frac{\mathbb{F}_2[x,y]}{\langle x,y\rangle^2},\ \mathbb{Z}_9,\ \mathbb{F}_3\times\mathbb{F}_3,\ \frac{\mathbb{F}_3[x]}{\langle x^2\rangle}.$$

The next theorem gives the genus of the Jacobson graph of a finite commutative local ring.

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a finite commutative local ring with associated field F, $|F|=\alpha$ and $|\mathfrak{m}|=\beta$. Then the following formulas are true:*

$$g(\mathfrak{J}_R)=\begin{cases}\left\lceil\frac{(\beta-3)(\beta-4)}{12}\right\rceil+\frac{(\alpha-2)}{2}\left\lceil\frac{(\beta-2)^2}{4}\right\rceil & \text{if } \alpha \text{ is even;}\\ 2\left\lceil\frac{(\beta-3)(\beta-4)}{12}\right\rceil+\frac{(\alpha-3)}{2}\left\lceil\frac{(\beta-2)^2}{4}\right\rceil & \text{if } \alpha \text{ is odd.}\end{cases}$$

P r o o f. By Theorem 3.1,

$$\mathfrak{J}_R=\begin{cases}K_\beta\cup\underbrace{K_{\beta,\beta}\cup\ldots\cup K_{\beta,\beta}\cup\ldots\cup K_{\beta,\beta}}_{(\alpha-2)/2\text{ copies}} & \text{if } \alpha \text{ is even;}\\ K_\beta\cup K_\beta\cup\underbrace{K_{\beta,\beta}\cup\ldots\cup K_{\beta,\beta}\cup\ldots\cup K_{\beta,\beta}}_{(\alpha-3)/2\text{ copies}} & \text{if } \alpha \text{ is odd.}\end{cases}$$

By Lemmas 2.1 and 2.2, we have

$$g(\mathfrak{J}_R)=\begin{cases}\left\lceil\frac{(\beta-3)(\beta-4)}{12}\right\rceil+\frac{(\alpha-2)}{2}\left\lceil\frac{(\beta-2)^2}{4}\right\rceil & \text{if } \alpha \text{ is even;}\\ 2\left\lceil\frac{(\beta-3)(\beta-4)}{12}\right\rceil+\frac{(\alpha-3)}{2}\left\lceil\frac{(\beta-2)^2}{4}\right\rceil & \text{if } \alpha \text{ is odd.}\end{cases}$$

□

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a finite commutative local ring. Then $g(\mathfrak{J}_R) = 1$ if and only if R is isomorphic to $\mathbb{F}_4[x]/\langle x^2\rangle$ or $\mathbb{Z}_4[x]/\langle x^2 + x + 1\rangle$.*

Proof. The proof follows from Theorem 3.3. □

The fact given in the following Lemma 3.5 will be used in this paper on many occasions.

Lemma 3.5. *Let R be a finite commutative ring. For any maximal ideal M in R, the subgraph induced by $1 + M$ in $\mathfrak{J}_R$ is complete.*

Proof. Let M be any maximal ideal in R. Let $x, y \in 1 + M$. Then $x = 1 + a$, $y = 1 + b$ for some $a, b \in M$. Also $1 - xy = 1 - (1 + a)(1 + b) = a + b + ab \in M$ and so x and y are adjacent in $\mathfrak{J}_R$. Hence the subgraph induced by $1 + M$ is complete. □

If R is a finite commutative ring with identity, then $R = R_1 \times R_2 \times \ldots \times R_n$ where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geqslant 2$. Hence $\mathrm{Max}(R) = \{R_1 \times \ldots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \ldots \times R_n \colon 1 \leqslant i \leqslant n\}$ is the set of maximal ideals of R.

In the following theorem, we characterize all finite commutative nonlocal rings whose $\mathfrak{J}_R$ is toroidal.

Theorem 3.6. *Let R be a finite commutative nonlocal ring. Then $g(\mathfrak{J}_R) = 1$ if and only if R is isomorphic to one of the following rings:*

$$\mathbb{F}_2 \times \mathbb{F}_5, \quad \mathbb{F}_2 \times \mathbb{F}_7, \quad \mathbb{F}_3 \times \mathbb{F}_4, \quad \mathbb{F}_3 \times \mathbb{Z}_4, \quad \mathbb{F}_3 \times \frac{\mathbb{F}_2[x]}{\langle x^2\rangle}.$$

Proof. Let us assume that $g(\mathfrak{J}_R) = 1$. It is well known that $R = R_1 \times R_2 \times \ldots \times R_n$ where each $(R_i, \mathfrak{m}_i)$ is a local ring and $n \geqslant 2$. Note that $|R_i| \geqslant 2$ for all i.

Suppose that $n \geqslant 4$. Then there exists $M = R_1 \times \ldots \times R_{i-1} \times \mathfrak{m}_i \times R_{i+1} \times \ldots \times R_n \in \mathrm{Max}(R)$ such that $|M| \geqslant 8$ for some i. By Lemma 3.5, the subgraph induced by $1 + M$ in $\mathfrak{J}_R$ is complete and hence $\mathfrak{J}_R$ contains K_8 as a subgraph. By Lemma 2.1, $g(\mathfrak{J}_R) \geqslant 2$, a contradiction. Hence $n \leqslant 3$.

Suppose that $n = 3$. If $|R_1| \geqslant 3$ and $|R_2| \geqslant 3$, then there exists a maximal ideal $M = R_1 \times R_2 \times \mathfrak{m}_3$ such that $|M| \geqslant 9$. By Lemma 3.5, $\mathfrak{J}_R$ contains K_9 as a subgraph and hence by Lemma 2.1, $g(\mathfrak{J}_R) \geqslant 3$, a contradiction. Hence $|R_1| = 2$ and $|R_2| = 2$ and so $R_1 \cong \mathbb{F}_2$, $R_2 \cong \mathbb{F}_2$.

If $|R_3| \geqslant 4$, then there exists $M = (0) \times R_2 \times R_3 \in \mathrm{Max}(R)$ such that $|M| \geqslant 8$. By virtue of Lemmas 3.5 and 2.1, K_8 is a subgraph of $\mathfrak{J}_R$ and $g(\mathfrak{J}_R) > 1$, a contradiction. From this, we get $|R_3| = 2$ or 3. By Theorem 3.2, $R_3 \ncong \mathbb{F}_2$ and so $R_3 \cong \mathbb{F}_3$.

Consider the case that $R = \mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_3$. Note that $\mathbb{G}$ is a subgraph of $\mathfrak{J}_R$. Then $K_{3,6}$ is a subgraph of $\mathbb{G}$ (see Figure 3.1). Recall that the genus of $K_{3,6}$ is one and hence one can fix an embedding of $K_{3,6}$ on the surface of torus. By Euler's formula, there are 9 faces in the embedding of $K_{3,6}$, say $\{S_1, \ldots, S_9\}$. Let n_i be the length of the face S_i. Note that $\sum_{i=1}^{9} n_i = 36$ and $n_i \geqslant 4$ for every i. Thus $n_i = 4$ for every i. Let $U = \{(0,1,2), (0,1,0), (0,1,1)\} \subset V(K_{3,6})$. Further, the subgraph G' of $\mathbb{G}$ induced by the vertices in U is K_3, $E(G') \cap E(K_{3,6}) = \emptyset$. Since K_3 cannot be embedded in the torus along with an embedding with only rectangles as faces, one cannot have an embedding of G' and $K_{3,6}$ together in the torus. This implies that $g(\mathbb{G}) \geqslant 2$. Since $g(\mathbb{G}) \leqslant g(\mathfrak{J}_R)$, $g(\mathfrak{J}_R) \geqslant 2$, a contradiction.

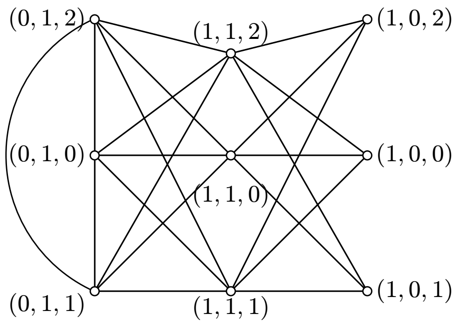


Figure 3.1. $\mathbb{G}$.

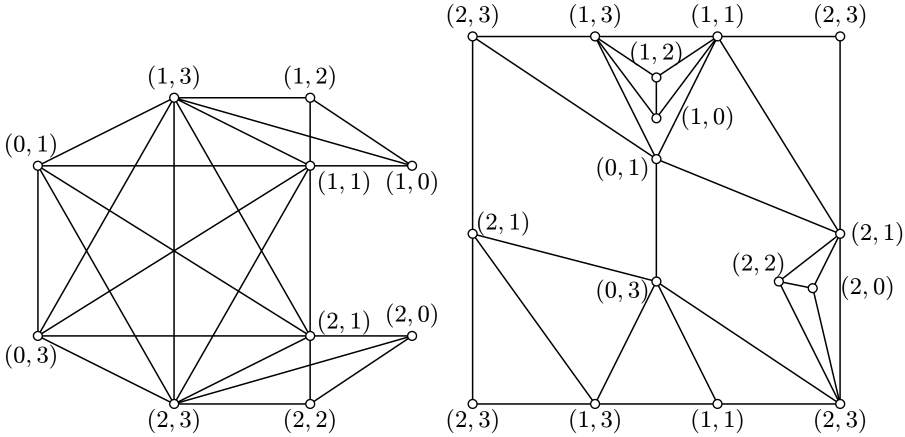
Suppose $n = 2$. If $\mathfrak{m}_i \neq \{0\}$ for all i, then $|R_i| \geqslant 4$ for all i, $M_1 = \mathfrak{m}_1 \times R_2$ and so $|M_1| \geqslant 8$. From this we get $\mathfrak{J}_R$ would contain a copy of K_8, it follows that $g(\mathfrak{J}_R) \geqslant 2$. Hence $\mathfrak{m}_i = \{0\}$ for some i.

Suppose $\mathfrak{m}_1 \neq \{0\}$. Then R_2 is a field. If $|\mathfrak{m}_1^*| \geqslant 2$, then $|R_1| \geqslant 8$, $|R_1 \times (0)| \geqslant 8$ and so $\mathfrak{J}_R$ contains K_8 as a subgraph, a contradiction. Thus $|\mathfrak{m}_1^*| = 1$ and so R_1 is isomorphic to one of the following rings:

$$\mathbb{Z}_4 \quad \text{or} \quad \frac{\mathbb{F}_2[x]}{\langle x^2 \rangle}.$$

If $|R_2| \geqslant 4$, then $|\mathfrak{m}_1 \times R_2| \geqslant 8$ and so K_8 is a subgraph of $\mathfrak{J}_R$, a contradiction. Hence $|R_2| \leqslant 3$. By Theorem 3.2, $R_2 \cong \mathbb{F}_3$ and hence R is isomorphic to one of the following rings: $\mathbb{F}_3 \times \mathbb{Z}_4$, $\mathbb{F}_3 \times \mathbb{F}_2[x]/\langle x^2 \rangle$.

Consider the ring $\mathbb{F}_3 \times \mathbb{F}_2[x]/\langle x^2 \rangle$. Define a mapping $f\colon V(\mathfrak{J}_{\mathbb{F}_3 \times \mathbb{F}_2[x]/\langle x^2 \rangle}) \to V(\mathfrak{J}_{\mathbb{F}_3 \times \mathbb{Z}_4})$ by $f((1, \langle x^2 \rangle)) = (1,0)$, $f((2, \langle x^2 \rangle)) = (2,0)$, $f((0, 1 + \langle x^2 \rangle)) = (0,1)$, $f((1, 1 + \langle x^2 \rangle)) = (1,1)$, $f((2, 1 + \langle x^2 \rangle)) = (2,1)$, $f((1, x + \langle x^2 \rangle)) = (1,2)$, $f((2, x + \langle x^2 \rangle)) = (2,2)$, $f((0, 1 + x + \langle x^2 \rangle)) = (0,3)$, $f((1, 1 + x + \langle x^2 \rangle)) =$



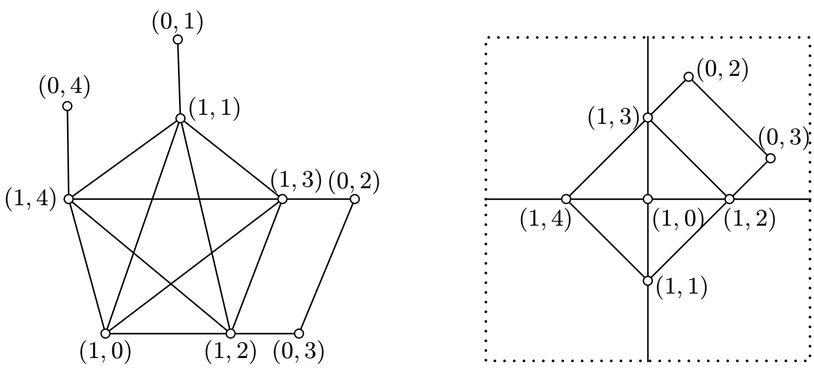
(a) (b)

Figure 3.2. (a) $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{Z}_4}$, (b) embedding of $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{Z}_4}$.

$(1,3)$, $f((2, 1 + x + \langle x^2\rangle)) = (2,3)$. Then f is a graph isomorphism and hence $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_2[x]/\langle x^2\rangle} \cong \mathfrak{J}_{\mathbb{F}_3\times\mathbb{Z}_4}$.

Suppose R_1 and R_2 are fields. Then $|R_i| \leqslant 7$ for all i. Otherwise if $|R_i| \geqslant 8$ for some i, then there is a maximal ideal containing at least 8 elements, so that $\mathfrak{J}_R$ would contain a copy of K_8 and so $g(\mathfrak{J}_R) \geqslant 2$. By Theorem 3.2, R is not isomorphic to the rings $\mathbb{F}_2\times\mathbb{F}_2$, $\mathbb{F}_2\times\mathbb{F}_3$, $\mathbb{F}_3\times\mathbb{F}_3$ and $\mathbb{F}_2\times\mathbb{F}_4$. For further use in the proof, we list below all finite commutative rings R with $|R_i| \leqslant 7$ for $i = 1, 2$:

$$\mathbb{F}_2\times\mathbb{F}_5,\quad \mathbb{F}_2\times\mathbb{F}_7,\quad \mathbb{F}_3\times\mathbb{F}_4,\quad \mathbb{F}_3\times\mathbb{F}_5,\quad \mathbb{F}_3\times\mathbb{F}_7,$$
$$\mathbb{F}_4\times\mathbb{F}_4,\quad \mathbb{F}_4\times\mathbb{F}_5,\quad \mathbb{F}_4\times\mathbb{F}_7,\quad \mathbb{F}_5\times\mathbb{F}_5,\quad \mathbb{F}_5\times\mathbb{F}_7,\quad \mathbb{F}_7\times\mathbb{F}_7.$$



(a) (b)

Figure 3.3. (a) $\mathfrak{J}_{\mathbb{F}_2\times\mathbb{F}_5}$, (b) embedding of $\mathfrak{J}_{\mathbb{F}_2\times\mathbb{F}_5}$.

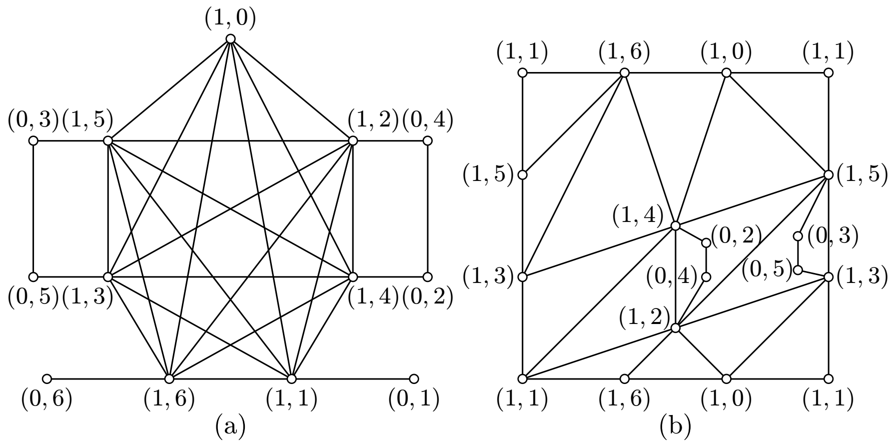


Figure 3.4. (a) $\mathfrak{J}_{\mathbb{F}_2\times\mathbb{F}_7}$, (b) embedding of $\mathfrak{J}_{\mathbb{F}_2\times\mathbb{F}_7}$.

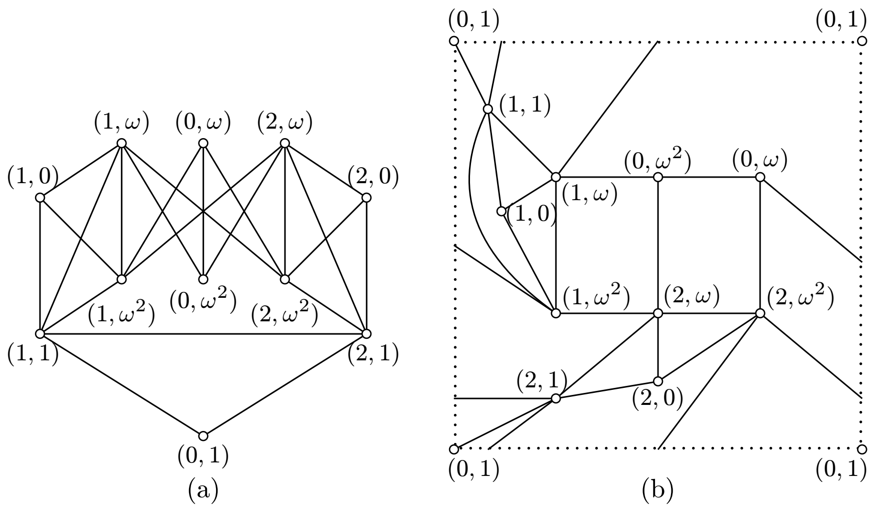


Figure 3.5. (a) $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_4}$, (b) embedding of $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_4}$.

Consider the ring $\mathbb{F}_3 \times \mathbb{F}_5$. Note that $2K_5$ is a subgraph of $\mathfrak{J}_R$ (see Figure 3.6). By Lemma 4.4 [15], $g(2K_5) > 1$. This yields $g(\mathfrak{J}_R) \geqslant 2$.

Note that $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_5}$ is a subgraph of $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_7}$. Hence $g(\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_7}) \geqslant 2$.

Now consider the ring $\mathbb{F}_4\times\mathbb{F}_4$. Let $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$. Then the graph in Figure 3.7 is a subgraph of $\mathfrak{J}_{\mathbb{F}_4\times\mathbb{F}_4}$. Note that the graph in Figure 3.7 is not toroidal. Therefore $g(\mathfrak{J}_{\mathbb{F}_4\times\mathbb{F}_4}) \geqslant 2$.

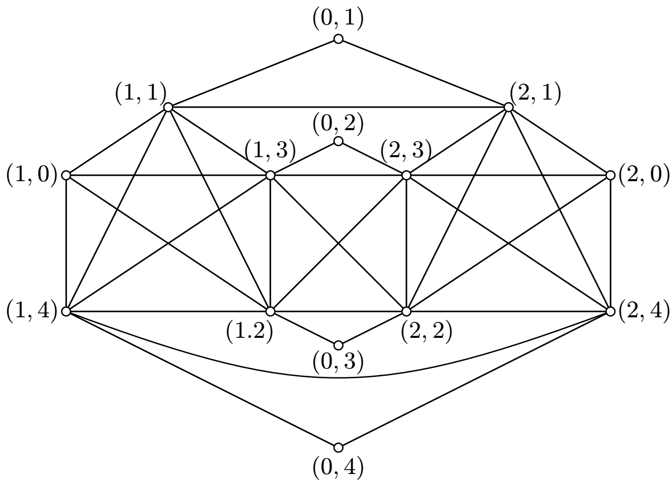


Figure 3.6. $\mathfrak{J}_{\mathbb{F}_3\times\mathbb{F}_5}$.

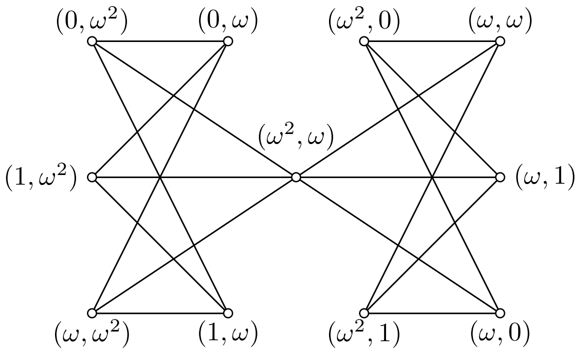


Figure 3.7.

Note that $\mathfrak{J}_{\mathbb{F}_4\times\mathbb{F}_4}$ is a subgraph of $\mathfrak{J}_{R_1\times R_2}$ and $g(\mathfrak{J}_{R_1\times R_2})\geqslant 2$, where each R_i is a field with $|R_i|\geqslant 4$. Hence R is isomorphic to one of the following rings:

$$\mathbb{F}_2\times\mathbb{F}_5,\quad \mathbb{F}_2\times\mathbb{F}_7,\quad \mathbb{F}_3\times\mathbb{F}_4,\quad \mathbb{F}_3\times\mathbb{Z}_4,\quad \mathbb{F}_3\times\frac{\mathbb{F}_2[x]}{\langle x^2\rangle}.$$

$\square$

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