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ON THE FRACTIONAL DIFFERENTIABILITY OF THE SPATIAL DERIVATIVES OF WEAK SOLUTIONS TO NONLINEAR PARABOLIC SYSTEMS OF HIGHER ORDER

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Abstract. We are concerned with the problem of differentiability of the derivatives of order $m+1$ of solutions to the "nonlinear basic systems" of the type

$$(-1)^m \sum_{|\alpha|=m} D^\alpha A^\alpha\bigl(D^{(m)}u\bigr) + \frac{\partial u}{\partial t} = 0 \quad \text{in } Q.$$

We are able to show that

$$D^\alpha u \in L^2\bigl(-a, 0, H^\vartheta\bigl(B(\sigma), \mathbb{R}^N\bigr)\bigr), \quad |\alpha| = m+1,$$

for $\vartheta \in (0, 1/2)$ and this result suggests that more regularity is not expectable.

Keywords: nonlinear parabolic system; fractional differentiability; spatial derivative; weak solution

MSC 2010: 35R11, 35K41

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geqslant 1$, let x be a point of $\mathbb{R}^n$, $t \in \mathbb{R}$ and $X = (x, t)$ a point of $\mathbb{R}^n \times \mathbb{R}$. Let N be an integer $N \geqslant 1$, and $(\ ,\ )_k$ and $\|\cdot\|_k$ the scalar product and the norm in $\mathbb{R}^k$, respectively.

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[1] This paper was accepted for publication and scheduled to appear in issue 3 of volume 43 (1993) of this journal. For reasons unknown, however, it seems to have disappeared from the processing queue, which moreover remained unnoticed until today. With an apology to the author, the paper is finally printed here.

If T is a positive number we set $Q = \Omega \times (-T, 0)$ and we denote by $B(x^0, \sigma)$ the cube of $\mathbb{R}^n$:

$$B(x^0, \sigma) = \{x \in \mathbb{R}^n \colon\ |x_i - x_i^0| < \sigma,\ i = 1, 2, \ldots, n\},$$

and we set

$$Q(X_0, \sigma) = B(x^0, \sigma) \times (t_0 - \sigma^{2m}, t_0)$$

where m is a positive integer and $X_0 = (x^0, t_0)$, with $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in \mathbb{R}^n$ and $\sigma > 0$. For the sake of brevity we set $B(x^0, \sigma) = B(\sigma)$.

Moreover, we say that $Q(X_0, \sigma) \subset\subset Q$ if

$$B(x^0, \sigma) \subset\subset \Omega \quad \text{and} \quad \sigma^{2m} > t_0 + T \geqslant T.$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multindex and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$; we denote by $\mathscr{R}$, $\mathscr{R}^*$ and $\mathscr{R}'$ respectively the Cartesian products $\prod\limits_{|\alpha| \leqslant m} \mathbb{R}_\alpha^N$, $\prod\limits_{|\alpha| \leqslant m-1} \mathbb{R}_\alpha^N$ and $\prod\limits_{|\alpha| = m} \mathbb{R}_\alpha^N$, while $p = \{p^\alpha\}_{|\alpha| \leqslant m}$, $p^* = \{p^\alpha\}_{|\alpha| \leqslant m-1}$ and $p' = \{p^\alpha\}_{|\alpha| = m}$, $p^\alpha \in \mathbb{R}^N$, are respectively points of $\mathscr{R}$, $\mathscr{R}^*$ and $\mathscr{R}'$.

If $u \colon Q \to \mathbb{R}^n$, we set

$$Du = \{D^\alpha u\}_{|\alpha| \leqslant m}, \quad \delta u = \{D^\alpha u\}_{|\alpha| \leqslant m-1},$$
$$D^{(k)} u = \{D^\alpha u\}_{|\alpha| = k}, \quad k = 1, 2, \ldots, m,$$

while $D_s\, u$, $s = 1, 2, \ldots, n$, denotes the derivative of u with respect to the variable x_s.

Let $A^\alpha(p')$, $|\alpha| = m$, be vectors of $\mathbb{R}^N$ defined in $\mathscr{R}'$, continuous in p' and such that

$$A^\alpha(0) = 0, \quad |\alpha| = m;$$

we shall call "basic system" the nonlinear differential system

$$E_0 u = (-1)^m \sum_{|\alpha| = m}^{n} D^\alpha A^\alpha(D^{(m)} u) + \frac{\partial u}{\partial t} = 0. \tag{1.1}$$

We suppose that the vectors $p' \to A^\alpha(p')$ are twice differentiable with derivatives $\partial A_h^\alpha / \partial p_k^\beta$, $\partial^2 A_h^\alpha / \partial p_l^\gamma \partial p_k^\beta$, $|\alpha| = |\beta| = |\gamma| = m$, with $h, k, l = 1, 2, \ldots, N$, continuous and bounded in $\mathscr{R}'$:

$$\left\{ \sum_{h,k=1}^{N} \sum_{|\alpha| = m} \sum_{|\beta| = m} \left| \frac{\partial A_h^\alpha(p')}{\partial p_k^\beta} \right|^2 \right\}^{1/2} \leqslant M, \quad p' \in \mathscr{R}' \tag{1.2}$$

$$\left\{ \sum_{h,k,l=1}^{N} \sum_{|\alpha| = m} \sum_{|\beta| = m} \sum_{|\gamma| = m} \left| \frac{\partial^2 A_h^\alpha(p')}{\partial p_l^\gamma \partial p_k^\beta} \right|^2 \right\}^{1/2} \leqslant M', \quad p' \in \mathscr{R}'$$

where M, M' are positive constants.

We also suppose that the operator E_0 is strongly parabolic, that is there exists $\nu > 0$ such that

$$\sum_{h,k=1}^{n} \sum_{|\alpha|=m} \sum_{|\beta|=m} \frac{\partial A_h^\alpha(p')}{\partial p_k^\beta} \xi_h^\alpha \xi_\kappa^\beta \geqslant \nu \sum_{|\alpha|=m} \|\xi^\alpha\|^2 \tag{1.3}$$

for every $p' \in \mathscr{R}'$ and for any system $\{\xi^\alpha\}_{|\alpha|=m}$ of vectors of $\mathbb{R}^N$. If we set $A_{\alpha\beta} = \{A_{\alpha\beta}^{hk}\}$ with $A_{\alpha\beta}^{hk}(p') = \int_0^1 (\partial A_h^\alpha(\tau p')/\partial p_k^\beta)\,\mathrm{d}\tau$, then thanks to the fact that $A^\alpha(0) = 0$, we get:

$$\begin{aligned} \sum_{k=1}^{N} \sum_{|\beta|=m} A_{\alpha\beta}^{hk}(p') p_k^\beta &= \int_0^1 \sum_{k=1}^{N} \sum_{|\beta|=m} \frac{\partial A_h^\alpha(\tau p')}{\partial p_k^\beta} p_k^\beta \,\mathrm{d}\tau = \int_0^1 \frac{\partial A_h^\alpha(\tau p')}{\partial \tau}\,\mathrm{d}\tau \\ &= A_h^\alpha(p') \end{aligned} \tag{1.4}$$

and therefore

$$A^\alpha(p') = \sum_{|\beta|=m} A_{\alpha\beta}(p') p^\beta.$$

Furthermore setting

$$A_{\alpha\beta\gamma} = \{\mathscr{A}_{\alpha\beta\gamma}^{hkl}\} \tag{1.5}$$

and

$$\mathscr{A}_{\alpha\beta\gamma}^{hkl} = \int_0^1 \frac{\partial A_{\alpha\beta}^{hk}(p_1' + \eta p_2')}{\partial p_l^\gamma}\,\mathrm{d}\eta$$

with

$$p_1' = \{p^{(1)\alpha}\}_{|\alpha|=m}, \quad p_2' = \{p^{(2)\alpha}\}_{|\alpha|=m}, \quad p^{(1)\alpha}, p^{(2)\alpha} \in \mathbb{R}^N,$$
$$|\alpha|, |\beta|, |\gamma| = m \quad \text{and} \quad h, k, l = 1, 2, \ldots, N,$$

we obtain

$$\begin{aligned} \sum_{l=1}^{N} \sum_{|\gamma|=m} \mathscr{A}_{\alpha\beta\gamma}^{hkl} p_1^{(2)\gamma} &= \sum_{l=1}^{N} \sum_{|\gamma|=m} \int_0^1 \frac{\partial A_{\alpha\beta}^{hk}(p_1' + \eta p_2')}{\partial p_l^\gamma} p_l^{(2)\gamma}\,\mathrm{d}\eta \\ &= \int_0^1 \frac{\partial A_{\alpha\beta}^{hk}(p_1' + \eta p_2')}{\partial \eta}\,\mathrm{d}\eta = A_{\alpha\beta}^{hk}(p_1' + p_2') - A_{\alpha\beta}^{hk}(p_1'). \end{aligned} \tag{1.6}$$

The object of the present work is to show a result on fractional differentiability for the spatial derivatives of order $m+1$ of the solutions to basic systems of order $2m$ of the type (1.1). The type of solution that we consider is the following: a vector $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N))$ is a solution of the system (1.1) if it satisfies

$$\int_Q \Big\{ \sum_{|\alpha|=m} (A^\alpha(D^{(m)}u), D^\alpha\varphi) - \Big(u, \frac{\partial\varphi}{\partial t}\Big) \Big\}\,\mathrm{d}X = 0 \tag{1.7}$$

for all $\varphi \in L^2\bigl(-T, 0, H_0^m(\Omega, \mathbb{R}^N)\bigr) \cap H^1\bigl(-T, 0, L^2(\Omega, \mathbb{R})^N\bigr)$ such that $\varphi(x, -T) = \varphi(x, 0) = 0$ in Ω.

We recall that $H^m(\Omega, \mathbb{R}^N)$ and $H_0^m(\Omega, \mathbb{R}^N)$ are the usual Sobolev spaces. Afterwards we shall set

$$|u|_{\vartheta,\Omega} = \left(\int_\Omega \mathrm{d}y \int_\Omega \frac{\|u(x) - u(y)\|^2}{\|x - y\|^{n+2\vartheta}} \,\mathrm{d}x\right)^{1/2}, \quad 0 < \vartheta < 1.$$

In [1] it has been shown that if $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N))$ is a solution to the system (1.1) then there exist $D^\alpha u \in L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ for all α, $|\alpha| = m + 1$, and the derivative $D_s u$, $s = 1, 2, \ldots, n$, satisfies the system

$$(1.8)\quad \int_{Q(X_0,\sigma')} \left\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(D^{(m)}u) D^\beta D_s u, D^\alpha \varphi) - \Bigl(D_s u, \frac{\partial \varphi}{\partial t}\Bigr) \right\} \mathrm{d}X = 0$$

for all $\varphi \in C_0^\infty\bigl(Q(X_0, \sigma'), \mathbb{R}^N\bigr)$ and for all $Q(X_0, \sigma') \subset\subset Q$.

Under the above assumptions (1.2) and (1.3), we are able to show that, if $D^\alpha u \in L^3_{\mathrm{loc}}(Q, \mathbb{R}^N)$, $|\alpha| \leqslant m + 1$, then $D^\alpha u \in L^2(-a, 0, H^\vartheta(B(\sigma), \mathbb{R}^N))$ for $|\alpha| = m + 1$, $a \in (0, T)$ and for all $\vartheta \in \bigl(0, 1/2\bigr)$; such result suggests that this is the sharpest regularity for the solutions of a nonlinear parabolic system not only of basic type but also of general type (see [2] and [3] for such type of systems; for the case $m = 1$ see also [4], [5], [8], [9] and [10]).

2. Local differentiability of the derivatives of order $m + 1$

In this section we show the result on fractional local differentiability for the derivatives of order $m+1$ of the solutions to basic systems that is the purpose of this paper.

In fact the following theorem holds:

Theorem 2.1. *Let $u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N))$ be a solution of the system*

$$(2.1)\qquad (-1)^m \sum_{|\alpha|=m}^{n} D^\alpha A^\alpha(D^{(m)}u) + \frac{\partial u}{\partial t} = 0$$

and assume that $D^\alpha u \in L^3_{\mathrm{loc}}(Q, \mathbb{R}^N)$, $|\alpha| \leqslant m + 1$. Then for $\vartheta \in \bigl(0, 1/2\bigr)$, for all $2a \in (0, T)$, and for all σ, σ' such that $B(x^0, 3\sigma) \subset\subset B(x^0, \sigma') \subset\subset \Omega$ it results

$$D^\alpha u \in L^2(-a, 0, H^\vartheta(B(\sigma), \mathbb{R}^N)), \qquad |\alpha| = m + 1,$$

and the following inequality holds

$$\int_{-a}^{0} |D^\alpha u|^2_{\vartheta,B(\sigma)}\,\mathrm{d}t \leqslant c(\nu, M, M', m, n, a)\mathscr{A}$$

where $\mathscr{A}$ *is defined by* (2.5).

P r o o f. To achieve this result let us take in (1.8), as test function, the function φ constructed in the following way: let $\vartheta(x) \in C_0^\infty(\mathbb{R}^N)$ be a function with the following properties

$$0 \leqslant \vartheta \leqslant 1, \quad \vartheta = 1 \quad \text{in } B(\sigma), \quad \vartheta = 0 \quad \text{in } \mathbb{R}^n \setminus B(\tfrac{3}{2}\sigma),$$

$$|D^\gamma \vartheta| \leqslant c\sigma^{-|\gamma|}, \quad |\gamma| \leqslant m.$$

Let $\varrho_p(t)$, with p an integer greater than $2/a$, be a function defined in $\mathbb{R}$ as follows

$$\begin{cases} \varrho_p(t) = 1 & \text{if } -a \leqslant t \leqslant -2/p, \\ \varrho_p(t) = 0 & \text{if } t > -1/p \text{ or } t < -2a, \\ \varrho_p(t) = t/a + 2 & \text{if } -2a \leqslant t \leqslant -a, \\ \varrho_p(t) = -(pt+1) & \text{if } -2/p \leqslant t \leqslant -1/p. \end{cases}$$

Let $\{g_s(t)\}$ be a sequence of symmetric mollifying functions such that

$$\begin{cases} g_s(t) \in C_0^\infty(\mathbb{R}),\ g_s(t) \geqslant 0,\ g_s(t) = g_s(-t), \\ \operatorname{supp} g_s(t) \subseteq [-1/s, 1/s], \\ \int_{\mathbb{R}} g_s(t)\,\mathrm{d}t = 1. \end{cases}$$

Then for every $p > 2/a$, for all $s > \max\{p, 1/(T-2a)\}$, and for $|h| < \sigma$ we consider the function

$$\varphi = \tau_{r,-h}\{\vartheta^{2m}\varrho_p[(\varrho_p\tau_{r,h}D_s u) * g_s]\} = \tau_{r,-h}\psi(x,t)$$

where, for the sake of brevity, we set

$$\psi(x,t) = \vartheta^{2m}\varrho_p[(\varrho_p\tau_{r,h}D_s u) * g_s]$$

and furthermore

$$\tau_{r,h}u = u(x + he^r, t) - u(x,t),$$

r integer, $1 \leqslant r \leqslant n$, with $e = (e^1, e^2, \ldots, e^n)$ the canonical basis.

From (1.8) we then obtain

$$\int_Q \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} \big(A_{\alpha\beta}(D^{(m)}u) D^\beta D_s u, D^\alpha \tau_{r,-h}\psi(x,t)\big) - \Big(D_s u, \frac{\partial \tau_{r,-h}\psi(x,t)}{\partial t}\Big)\Big\} \, \mathrm{d}X = 0,$$

that is,

$$\begin{aligned} &\int_Q \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha,\beta}(D^{(m)}u(x,t)) D^\beta D_s u, D^\alpha \psi(x-he^r,t)) \Big\} \, \mathrm{d}X \\ &\quad - \int_Q \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha,\beta}(D^{(m)}u(x,t)) D^\beta D_s u, D^\alpha \psi(x,t)) \Big\} \, \mathrm{d}X \\ &\quad - \int_Q \vartheta^{2m}(x-he^r)(D_s u, \varrho_p'(t))(\varrho_p \tau_{r,h} D_s u(x-he^r,t) * g_s) \, \mathrm{d}X \\ &\quad - \int_Q \vartheta^{2m}(x)(D_s u, \varrho_p'(t))(\varrho_p \tau_{r,h} D_s u * g_s) \, \mathrm{d}X = 0, \end{aligned}$$

having taken into account that by virtue of the symmetry of $g_s(t)$ the integral term vanishes. By a simple calculation we obtain

$$\begin{aligned} &\int_Q \Big\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} [(A_{\alpha\beta}(D^{(m)}u(x+he^r,t)) D^\beta D_s u(x+he^r,t), D^\alpha \psi(x,t)) \\ &\quad - (A_{\alpha\beta}(D^{(m)}u(x,t)) D^\beta D_s u(x,t), D^\alpha \psi(x,t))] \Big\} \, \mathrm{d}X \\ &\quad - \int_Q \vartheta^{2m}(x) \varrho_p'(t) \big(\tau_{r,h} D_s u, \varrho_p(\tau_{r,h} D_s u * g_s)\big) \, \mathrm{d}X = 0, \end{aligned}$$

and then, adding and subtracting the term

$$(A_{\alpha\beta}(D^{(m)}u(x,t)) D^\beta D_s u(x+he^r,t), D^\alpha \psi(x,t))$$

we get

$$\begin{aligned} &\int_Q \Big[\sum_{|\alpha|=m} \sum_{|\beta|=m} (\tau_{r,h} A_{\alpha\beta}(D^{(m)}u) D^\beta D_s u(x+he^r,t), D^\alpha \psi(x,t)) \\ &\quad + \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(D^{(m)}u(x,t)) \tau_{r,h} D^\beta D_s u, D^\alpha \psi(x,t)) \Big] \, \mathrm{d}X \\ &\quad - \int_Q \vartheta^{2m}(x) p_p'(t)(\tau_{r,h} D_s u, \varrho_p(\tau_{r,h} D_s u * g_s)) \, \mathrm{d}X = 0. \end{aligned}$$

Taking into account (1.5) and (1.6), we obtain, with obvious meaning of the mathematical symbol,

$$\begin{aligned}\tau_{r,h}A_{\alpha\beta}(D^{(m)}u(x,t)) &= \int_0^1 \frac{\partial A_{\alpha\beta}(D^{(m)}u+\eta\tau_{r,h}D^{(m)}u)}{\partial\eta}\,\mathrm{d}\eta\\ &= \int_0^1 \sum_{l=1}^{N}\sum_{|\gamma|=m}\frac{\partial A_{\alpha\beta}(D^{(m)}u+\eta\tau_{r,h}D^{(m)}u)}{\partial p_l^{\gamma}}\tau_{r,h}D^{\gamma}u_l\,\mathrm{d}\eta\\ &= \sum_{|\gamma|=m}\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^{\gamma}u,\end{aligned}$$

and hence

$$\begin{aligned}\int_Q\Bigg\{&\sum_{|\alpha|=m}\sum_{|\beta|=m}\sum_{|\gamma|=m}(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^{\gamma}uD^{\beta}D_su(x+he^r,t),D^{\alpha}\psi(x,t))\\ &+\sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}(D^{(m)}u)\tau_{r,h}D^{\beta}D_su,D^{\alpha}\psi(x,t))\\ &-\vartheta^{2m}(x)p_p'(t)(\tau_{r,h}D_su,\varrho_p(\tau_{r,h}D_su*g_s))\Bigg\}\,\mathrm{d}X=0.\end{aligned}$$

Adding and subtracting the term

$$(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^{\gamma}uD^{\beta}D_su,D^{\alpha}\psi(x,t))$$

we get

$$\begin{aligned}\int_Q\Bigg\{&\sum_{|\alpha|=m}\sum_{|\beta|=m}\sum_{|\gamma|=m}(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^{\gamma}u\tau_{r,h}D^{\beta}D_su,D^{\alpha}\psi(x,t))\\ &+\sum_{|\alpha|=m}\sum_{|\beta|=m}\sum_{|\gamma|=m}(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^{\gamma}uD^{\beta}D_su,D^{\alpha}\psi(x,t))\\ &+\sum_{|\alpha|=m}\sum_{|\beta|=m}(A_{\alpha\beta}(D^{(m)}u)\tau_{r,h}D^{\beta}D_su,D^{\alpha}\psi(x,t))\\ &-\vartheta^{2m}(x)\varrho_p'(t)\varrho_p(t)(\tau_{r,h}D_su,(\tau_{r,h}D_su*g_s))\Bigg\}\,\mathrm{d}X=0.\end{aligned}$$

Since

$$\begin{aligned}D^{\alpha}\psi(x,t) &= D^{\alpha}(\vartheta^{2m}\varrho_p[(\varrho_p\tau_{r,h}D_su)*g_s])\\ &= \vartheta^{2m}\varrho_p[(\varrho_p\tau_{r,h}D_su)*g_s]\\ &\quad+\vartheta^{2m}\varrho_p\sum_{\delta<\alpha}C_{\alpha\delta}(\vartheta)[(\varrho_p\tau_{r,h}D^{\delta}D_su)*g_s]\end{aligned}$$

with

$$|C_{\alpha\delta}(\vartheta)|\leqslant c\sigma^{|\delta|-m}$$

we obtain

$$\int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \sum_{|\gamma|=m} (\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p \tau_{r,h} D^\beta D_s u, (\varrho_p \tau_{r,h} D^\alpha D_s u) * g_s)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^m \sum_{|\alpha|=m} \sum_{|\beta|=m}$$
$$\sum_{|\gamma|=m} \Big(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p \tau_{r,h} D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta)[(\varrho_p \tau_{r,h} D^\delta D_s u) * g_s]\Big)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \sum_{|\gamma|=m} (\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p D^\beta D_s u, (\varrho_p \tau_{r,h} D^\alpha D_s u) * g_s)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^m \sum_{|\alpha|=m} \sum_{|\beta|=m}$$
$$\sum_{|\gamma|=m} \Big(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p \tau_{r,h} D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta)[(\varrho_p \tau_{r,h} D^\delta D_s u) * g_s]\Big)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}\varrho_p \tau_{r,h} D^\beta D_s u, (\varrho_p \tau_{r,h} D^\alpha D_s u) * g_s)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \Big(A_{\alpha\beta}\varrho_p \tau_{r,h} D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta)[(\varrho_p \tau_{r,h} D^\delta D_s u) * g_s]\Big)\,\mathrm{d}X$$
$$= \int_Q \vartheta^{2m} \varrho_p'(\tau_{r,h} D_s u, (\varrho_p \tau_{r,h} D_s u) * g_s)\,\mathrm{d}X.$$

Taking the limit for $s \to \infty$, we obtain:

(2.2)
$$\int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \sum_{|\gamma|=m} (\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p \tau_{r,h} D^\beta D_s u, \varrho_p \tau_{r,h} D^\alpha D_s u)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^m \sum_{|\alpha|=m} \sum_{|\beta|=m}$$
$$\sum_{|\gamma|=m} \Big(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p \tau_{r,h} D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta) \varrho_p \tau_{r,h} D^\delta D_s u\Big)$$
$$+ \int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \sum_{|\gamma|=m} (\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p D^\beta D_s u, \varrho_p \tau_{r,h} D^\alpha D_s u)\,\mathrm{d}X$$
$$+ \int_Q \vartheta^m \sum_{|\alpha|=m} \sum_{|\beta|=m}$$
$$\sum_{|\gamma|=m} \Big(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u \varrho_p D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta) \varrho_p \tau_{r,h} D^\delta D_s u\Big)\,\mathrm{d}X$$

$$
\begin{aligned}
&+\int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta} \varrho_p \tau_{r,h} D^\beta D_s u, \varrho_p \tau_{r,h} D^\alpha D_s u) \, \mathrm{d}X \\
&+\int_Q \vartheta^{m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \Big(A_{\alpha\beta} \varrho_p \tau_{r,h} D^\beta D_s u, \sum_{\delta<\alpha} C_{\alpha\delta}(\vartheta) \varrho_p \tau_{r,h} D^\delta D_s u\Big) \, \mathrm{d}X \\
=\ &\int_Q \vartheta^{2m} \varrho_p' \varrho_p (\tau_{r,h} D_s u, \tau_{r,h} D_s u) \, \mathrm{d}X.
\end{aligned}
$$

Let us observe now that, since $D^\alpha u \in L^3_{\mathrm{loc}}(Q, \mathbb{R}^N)$, $|\alpha| \leqslant m+1$, it results for the first integral of (2.2)

$$
\begin{aligned}
&\Big| \int_Q \vartheta^{2m} \sum_{|\alpha|=m} \sum_{|\beta|=m} \sum_{|\gamma|=m} (\mathscr{A}_{\alpha\beta\gamma} \tau_{r,h} D^\gamma u \tau_{r,h} D^\beta D_s u, \tau_{r,h} D^\alpha D_s u) \, \mathrm{d}X \Big| \\
&\quad \leqslant M' \int_Q \vartheta^{2m} \varrho_p^2 \sum_{|\alpha|=m} \|\tau_{r,h} D^\alpha D_s u\|^2 \sum_{|\gamma|=m} \|\tau_{r,h} D^\gamma u\| \, \mathrm{d}X \\
&\quad \leqslant M' \Big(\int_Q (\vartheta^m \varrho_p)^{3/2} \sum_{|\alpha|=m} (\|\tau_{r,h} D^\alpha D_s u\|^2)^{3/2} \, \mathrm{d}X \Big)^{2/3} \\
&\qquad \times \Big(\int_Q (\vartheta^m \varrho_p)^3 \sum_{|\gamma|=m} \|\tau_{r,h} D^\gamma u\|^3 \, \mathrm{d}X \Big)^{1/3} \\
&\quad \leqslant c(M') \Big[\Big(\int_Q (\vartheta^m \varrho_p)^{3/2} \sum_{|\alpha|=m} \|D^\alpha D_s u(x + he^r, t)\|^3 \, \mathrm{d}x \Big)^{2/3} \\
&\qquad + \Big(\int_Q (\vartheta^m \varrho_p)^{3/2} \sum_{|\alpha|=m} \|D^\alpha D_s u(x, t)\|^3 \, \mathrm{d}X \Big)^{2/3} \Big] \\
&\qquad \times \Big(\int_Q (\vartheta^m \varrho_p)^3 \sum_{|\gamma|=m} \|\tau_{r,h} D^\gamma u\|^3 \, \mathrm{d}X \Big)^{1/3}.
\end{aligned}
$$

Where, here and in the sequel $c(M')$ denotes a positive constant depending on M'.

Since

$$
\begin{aligned}
&\Big(\int_Q (\vartheta^m \varrho_p)^{3/2} \sum_{|\alpha|=m} \|D^\alpha D_s u(x + he^r, t\|^3 \, \mathrm{d}X \Big)^{2/3} \\
&\qquad \leqslant \int_{-2a}^0 \mathrm{d}t \int_{B(3\sigma/2)} (\vartheta^m \varrho_p)^{3/2} \sum_{|\alpha|=m} \|D^\alpha D_s u(x, t)\|^3 \, \mathrm{d}x \Big)^{2/3}
\end{aligned}
$$

we also have

$$\left|\int_Q \vartheta^{2m}\varrho_p^2 \sum_{|\alpha|=m}\sum_{|\beta|=m}\sum_{|\gamma|=m}(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u\tau_{r,h}D^\beta D_s u, \tau_{r,h}D^\alpha D_s u)\,\mathrm{d}X\right|$$

$$\leqslant c(M')\left(\int_{-2a}^0 \mathrm{d}t\int_{B(3\sigma/2)}(\vartheta^m\varrho_p)^{3/2}\sum_{|\alpha|=m}\|D^\alpha D_s u(x,t)\|^3\,\mathrm{d}x\right)^{2/3}$$

$$\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B(3\sigma/2)}(\vartheta^m\varrho_p)^3\sum_{|\gamma|=m}\|\tau_{r,h}D^\gamma u\|^3\,\mathrm{d}x\right)^{1/3}$$

$$\leqslant |h|c(M')\left(\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\sum_{|\alpha|=m}\|D^\alpha D_s u(x,t)\|^3\,\mathrm{d}x\right)^{2/3}$$

$$\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\sum_{|\gamma|=m}\|D_r D^\gamma u\|^3\,\mathrm{d}x\right)^{1/3}$$

$$\leqslant |h|c(M')\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\sum_{|\alpha|=m+1}\|D^\alpha u(x,t)\|^3\,\mathrm{d}x,$$

having first increased ϑ and ϱ_p with 1 and then applied a Nirenberg lemma (see [2], Chapter I, Lemma 3.IV).

In a similar way an analogous inequality can be obtained for the third integral of the first term of (2.2) while for what concerns the second integral of (2.2) we get

$$\left|\int_Q \vartheta^m\varrho_p^2\sum_{|\alpha|=m}\sum_{|\beta|=m}\right.$$

$$\left.\sum_{|\gamma|=m}\Big(\mathscr{A}_{\alpha\beta\gamma}\tau_{r,h}D^\gamma u\tau_{r,h}D^\beta D_s u, \sum_{\delta<\alpha}C_{\alpha\delta}(\vartheta)\tau_{r,h}D^\delta D_s u\Big)\,\mathrm{d}X\right|$$

$$\leqslant \sum_{|\alpha|=m}\sum_{\delta<\alpha}\frac{c(M')}{\sigma^{(m-|\delta|)}}\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\|D^\alpha D_s u\|^{3/2}\|D^\delta D_s u\|^{3/2}\,\mathrm{d}x\right)^{2/3}$$

$$\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B\left(\frac{3}{2}\sigma\right)}\sum_{|\gamma|=m}\|\tau_{r,h}D^\gamma u\|^3\,\mathrm{d}x\right)^{1/3}$$

$$\leqslant c(M')|h|\sum_{|\alpha|=m}\sum_{\delta<\alpha}\frac{1}{\sigma^{(m-|\delta|)}}$$

$$\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\|D^\alpha D_s u\|^{3/2}\|D^\delta D_s u\|^{3/2}\,\mathrm{d}x\right)^{2/3}$$

$$\times\left(\int_{-2a}^0 \mathrm{d}t\int_{B(2\sigma)}\sum_{|\gamma|=m}\|D_r D^\gamma u\|^3\,\mathrm{d}x\right)^{1/3}$$

$$
\begin{aligned}
&\leqslant c(M')|h| \sum_{|\alpha|=m} \sum_{\delta<\alpha} \frac{1}{\sigma^{(m-|\delta|)}} \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha D_s u\|^3 \,\mathrm{d}x \Big)^{1/3} \\
&\quad \times \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha D_s u\|^3 \,\mathrm{d}x \Big)^{1/3} \\
&\quad \times \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \sum_{|\gamma|=m} \|D_r D^\gamma u\|^3 \,\mathrm{d}x \Big)^{1/3} \\
&\leqslant |h| c(M') \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \sum_{|\alpha|=m+1} \|D^\alpha u\|^3 \,\mathrm{d}x \Big)^{2/3} \\
&\quad \times \sum_{|\alpha|=m} \sum_{\delta<\alpha} \frac{1}{\sigma^{(m-|\delta|)}} \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha D_s u\|^3 \,\mathrm{d}x \Big)^{1/3}.
\end{aligned}
$$

Finally an analogous estimate holds also for the fourth integral of (2.2). By (2.2), using the estimates already obtained for the first four integrals of the left hand side and estimating the other therms by means of a well known technique (see [1], page 116, for an analogous calculation) we reach

$$
\begin{aligned}
(2.3)\qquad & \int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \sum_{|\alpha|=m} \|\tau_{r,h} D^\alpha D_s u\|^2 \,\mathrm{d}x \\
&\leqslant c(\nu, M, M', \sigma, m, n, a) \\
&\quad \times \Big\{ |h|^2 \int_{-a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D_r D_s u\|^2 \,\mathrm{d}x \\
&\quad + |h|^2 \sum_{1\leqslant|\alpha|=m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x \\
&\quad + |h| \sum_{|\alpha|=m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x \\
&\quad + |h| \sum_{|\alpha|=m} \sum_{\delta<\alpha} \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha D_s u\|^3 \,\mathrm{d}x \Big)^{1/3} \\
&\quad \times \Big(\sum_{|\alpha|=m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x \Big)^{2/3} \Big\}.
\end{aligned}
$$

From (2.3) for $|h| < h_0 = \min\{1, \sigma\}$ it follows that

$$
(2.4)\qquad \sum_{r=1}^{n} \int_{-a}^{0} \mathrm{d}t \int_{B(\sigma)} \sum_{|\alpha|=m} \|\tau_{r,h} D^\alpha D_s u\|^2 \,\mathrm{d}x \leqslant c(\nu, M, M', \sigma, m, n, a)|h|\mathscr{A}
$$

where we have set:

(2.5)

$$\begin{aligned}\mathscr{A} = \Big\{ &\sum_{1\leqslant|\alpha|\leqslant m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(3\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x + \sum_{|\alpha|=m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(3\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x \\ &+ \sum_{|\alpha|=m} \sum_{\delta<\alpha} \Big(\int_{-2a}^{0} \mathrm{d}t \int_{B(3\sigma)} \|D^\delta D_s u\|^3 \,\mathrm{d}x \Big)^{1/3} \\ &\times \Big(\sum_{|\alpha|=m+1} \int_{-2a}^{0} \mathrm{d}t \int_{B(2\sigma)} \|D^\alpha u\|^3 \,\mathrm{d}x \Big)^{2/3} \Big\}.\end{aligned}$$

The inequality (2.4) is also verified for $h_0 \leqslant |h| < \sigma$ and, then, it results for $\vartheta \in \big(0, 1/2\big)$

$$\begin{aligned}\sum_{r=1}^{n} \int_{-a}^{0} \mathrm{d}t \int_{-2\sigma}^{2\sigma} \frac{\mathrm{d}h}{|h|^{1+2\vartheta}} \int_{B(\sigma)} \|\tau_{r,h} D^\alpha D_s u\|^2 \,\mathrm{d}x \\ \leqslant c(\nu, M, M', \sigma, m, n, a)\mathscr{A}\end{aligned} \tag{2.6}$$

and, then, by means of Lemma II.3 of [3], we get the assertion:

$$D^\alpha u \in L^2(-a, 0, H^\vartheta(B(\sigma), \mathbb{R}^N)), \quad |\alpha| = m+1,$$

and

$$\int_{-a}^{0} |D^\alpha u|^2_{\vartheta, B(\sigma)} \,\mathrm{d}t \leqslant c(\nu, M, M', \sigma, m, n, a)\mathscr{A}. \tag{2.7}$$

□

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