Leonid Berezansky; Sandra Pinelas

Oscillation properties for a scalar linear difference equation of mixed type

Mathematica Bohemica, Vol. 141 (2016), No. 2, 169–182

Persistent URL: http://dml.cz/dmlcz/145711

OSCILLATION PROPERTIES FOR A SCALAR LINEAR DIFFERENCE EQUATION OF MIXED TYPE

Leonid Berezansky, Beer Sheva, Sandra Pinelas, Amadora

Received August 6, 2015
Communicated by Josef Diblík

Dedicated to Professor Jaroslav Kurzweil on the occasion of his 90th birthday

Abstract. The aim of this work is to study oscillation properties for a scalar linear difference equation of mixed type

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n)x(n+k) = 0, \quad n > n_0,$$

where $\Delta x(n) = x(n+1) - x(n)$ is the difference operator and $\{a_k(n)\}$ are sequences of real numbers for $k = -p, \ldots, q$, and $p > 0$, $q \geqslant 0$. We obtain sufficient conditions for the existence of oscillatory and nonoscillatory solutions. Some asymptotic properties are introduced.

Keywords: oscillation; difference equation; mixed type; asymptotic behavior

MSC 2010: 39A21, 39A99

1. Introduction

The aim of this work is to study oscillation properties for a scalar linear difference equation of mixed type

$$\Delta x(n) + \sum_{k=-p}^{q} a_k(n)x(n+k) = 0, \quad n > n_0, \tag{1}$$

where $\Delta x(n) = x(n+1) - x(n)$ is the difference operator and $\{a_k(n)\}$ are sequences of real numbers for $k = -p, \ldots, q$, and $p > 0$, $q \geqslant 0$.

Differential equations with delayed and advanced arguments (also called mixed differential equations or equations with mixed arguments) occur in many problems

DOI: 10.21136/MB.2016.14

of economy, biology and physics (see for example [2], [6], [8], [9], [11]), because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to the memory of a system, past events are important for the current behavior, and the concept of advance is related to potential future events which can be known at the current time and could be useful for decision-making. The study of various problems for differential equations with mixed arguments can be found in [3], [5], [4], [7], [10], [12], [13], [14]. It is well known that the solutions of these types of equations cannot be obtained in closed-form. In the absence of closed-form solutions a rewarding alternative is to resort to the qualitative study of the solutions of these types of differential equations. But it is not quite clear how to formulate an initial value problem for such equations and the existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equations on the half line.

2. Oscillatory behavior

As is customary, a solution is called nonoscillatory if it is eventually positive or eventually negative. Otherwise it is oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

First, we will consider the coefficients $a_k(n)$ nonnegative for all $k \in \{-p, \ldots, q\}$ and $n > n_0$.

Theorem 1. *Let $a_k(n)$ be nonnegative for all $k \in \{-p, \ldots, q\}$ and $n > n_0$. If*

$$\sum_{k=-p}^{-1} a_k(n+1) + \sum_{k=-p}^{0} a_k(n) \geqslant 1 \tag{2}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

P r o o f. Assume, for the sake of contradiction, that equation (1) has a nonoscillatory solution. Without loss of generality we may assume that $\{x(n)\}$ is eventually positive, i.e., there exists $n_1 \geqslant n_0$ such that $x(n) > 0$ for $n \geqslant n_1$. By (1) it is easy to see that $\{x(n)\}$ is decreasing, since

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n)x(n+k) \leqslant 0.$$

So,

$$0 = x(n+2) - x(n+1) + \sum_{k=-p}^{q} a_k(n+1)x(n+k+1)$$

$$
\begin{aligned}
&\geqslant -x(n+1)+\sum_{k=-p}^{q} a_k(n+1)x(n+k+1)\\
&> -x(n+1)+\sum_{k=-p}^{-1} a_k(n+1)x(n+k+1)\\
&\geqslant -x(n+1)+\sum_{k=-p}^{-1} a_k(n+1)x(n).
\end{aligned}
$$

Consequently,

$$
x(n+1) > \sum_{k=-p}^{-1} a_k(n+1)x(n). \tag{3}
$$

On the other hand, we have

$$
\begin{aligned}
0 &= x(n+1)-x(n)+\sum_{k=-p}^{q} a_k(n)x(n+k)\\
&\geqslant x(n+1)-x(n)+\sum_{k=-p}^{0} a_k(n)x(n+k)\\
&\geqslant x(n+1)-x(n)+\sum_{k=-p}^{0} a_k(n)x(n).
\end{aligned}
$$

Consequently,

$$
x(n+1) \leqslant x(n)-\sum_{k=-p}^{0} a_k(n)x(n). \tag{4}
$$

Using inequalities (3) and (4), we get

$$
\sum_{k=-p}^{-1} a_k(n+1)x(n) < x(n)-\sum_{k=-p}^{0} a_k(n)x(n)
$$

or

$$
\sum_{k=-p}^{-1} a_k(n+1)+\sum_{k=-p}^{0} a_k(n) < 1.
$$

This way we obtain a contradiction. □

Corollary 2. *Let $a_k(n)$ be nonnegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If there exists $m \in \{-p, \dots, 0\}$ such that*

$$\sum_{k=-p}^{m} a_k(n) > 1 \tag{5}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

Agarwal, in [1], Section 6.4, studies the oscillatory behavior of the difference equation

$$p(k)u(k+1) + p(k-1)u(k-1) = q(k)u(k), \quad k \in \mathbb{N}(1), \tag{6}$$

where the functions p and q are defined on $\mathbb{N}$ and $\mathbb{N}(1)$, respectively, and $p(k)$, $q(k) > 0$ for all $k \in \mathbb{N}$.

Equation (1) generalizes equation (6) and Theorem 1 extends Theorem 6.4.1 of [13] as we can see in next corollary.

Corollary 3. *Let $a_k(n)$ be nonegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If*

$$a_0(n) \geqslant 1 - p\min\{a_k(n)\colon\ k = -p, \dots, -1\} > 0 \tag{7}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

P r o o f. We have

$$\sum_{k=-p}^{0} a_k(n) = a_0(n) + \sum_{k=-p}^{-1} a_k(n) \geqslant a_0(n) + p\min\{a_k(n)\colon\ k = -p, \dots, -1\} \geqslant 1.$$

□

Corollary 4. *Let $a_k(n)$ be nonegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If*

$$a_{-p}(n) < \dots < a_{-1}(n) \quad \text{and} \quad a_0(n) > 1 - pa_{-p}(n) \tag{8}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

Corollary 5. *Let $a_k(n)$ be nonegative for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If*

$$a_{-p}(n) > \dots > a_{-1}(n) \quad \text{and} \quad a_0(n) > 1 - pa_{-1}(n) \tag{9}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

E x a m p l e 6. Consider the difference equation

$$\Delta x(n) + \sum_{k=-2}^{0} \frac{n-1}{|k-2|(n+1)} x(n+k) + \sum_{k=1}^{q} b_k(n) x(n+k) = 0, \quad n \geqslant 4, \tag{10}$$

where $b_k(n)$ is a nonnegative sequence for all $k \in \{1, \ldots, q\}$ and $q \geqslant 1$. Since

$$\frac{n}{4(n+2)} + \frac{n}{3(n+2)} + \frac{n-1}{4(n+1)} + \frac{n-1}{3(n+1)} + \frac{n-1}{2(n+1)} > 1$$

for $n \geqslant 2$, condition (2) holds, by Theorem 1 we can conclude that equation (10) is oscillatory.

We consider now the case when the coefficients $a_k(n)$ are nonpositive for all $k \in \{-p, \ldots, q\}$.

Theorem 7. *Let $a_k(n)$ be nonpositive for all $k \in \{-p, \ldots, q\}$, $q \geqslant 2$ and $n > n_0$. If*

$$\sum_{k=1}^{q} a_k(n) + \sum_{k=2}^{q} a_k(n-1) \leqslant -1 \tag{11}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

P r o o f. Assume, for the sake of contradiction, that equation (1) has a nonoscillatory solution. Without loss of generality we may assume that $x(n)$ is eventually positive, i.e., there exists $n_1 \geqslant n_0$ such that $x(n) > 0$ for $n \geqslant n_1$. By (1) it is easy to see that $\{x(n)\}$ is increasing since

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n) x(n+k) \geqslant 0.$$

So,

$$\begin{aligned}
0 &= x(n) - x(n-1) + \sum_{k=-p}^{q} a_k(n-1) x(n+k-1) \\
&< x(n) + \sum_{k=-p}^{q} a_k(n-1) x(n+k-1) \\
&\leqslant x(n) + \sum_{k=2}^{q} a_k(n-1) x(n+k-1) \\
&\leqslant x(n) + \sum_{k=2}^{q} a_k(n-1) x(n+1).
\end{aligned}$$

Consequently,

$$x(n) > -\sum_{k=2}^{q} a_k(n-1)x(n+1). \tag{12}$$

On the other hand, we have

$$\begin{aligned} 0 &= x(n+1) - x(n) + \sum_{k=-p}^{q} a_k(n)x(n+k) \\ &< x(n+1) - x(n) + \sum_{k=1}^{q} a_k(n)x(n+k) \\ &\leqslant x(n+1) - x(n) + \sum_{k=1}^{q} a_k(n)x(n+1). \end{aligned}$$

Consequently

$$x(n) < x(n+1) + \sum_{k=1}^{q} a_k(n)x(n+1). \tag{13}$$

Using the inequalities (12) and (13), we get

$$-\sum_{k=2}^{q} a_k(n-1)x(n+1) < x(n+1) + \sum_{k=1}^{q} a_k(n)x(n+1)$$

or

$$\sum_{k=1}^{q} a_k(n) + \sum_{k=2}^{q} a_k(n-1) > -1.$$

This way we obtain a contradiction. □

Corollary 8. *Let $a_k(n)$ be nonpositive for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If there exists $m \in \{1, \dots, q\}$ such that*

$$\sum_{k=1}^{m} a_k(n) \leqslant -1 \tag{14}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) are oscillatory.

Corollary 9. *Let $a_k(n)$ be nonpositive for all $k \in \{-p, \dots, q\}$ and $n > n_0$. If*

$$a_1(n) \leqslant -1 - q \max\{a_k(n)\colon\ k = 2, \dots, q\} \tag{15}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) are oscillatory.

Corollary 10. *Let $a_k(n)$ be nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$. If*

$$a_2(n) < \ldots < a_q(n) \quad \text{and} \quad a_1(n) < -1 - q a_q(n) \tag{16}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

Corollary 11. *Let $a_k(n)$ be nonpositive for all $k \in \{-p, \ldots, q\}$ and $n > n_0$. If*

$$a_2(n) > \ldots > a_q(n) \quad \text{and} \quad a_1(n) < -1 - q a_2(n) \tag{17}$$

for all $n > n_1 \geqslant n_0$, then all solutions of the difference equation (1) *are oscillatory.*

Example 12. Consider the difference equation

$$\Delta x(n) + \sum_{k=-p}^{0} c_k(n) x(n+k) + \sum_{k=1}^{3} \frac{k(\mathrm{e}^{-n} - 1)}{k+1} x(n+k) = 0, \quad n \geqslant 1, \tag{18}$$

where $c_k(n)$ is a nonpositive sequence for all $k \in \{-p, \ldots, 0\}$ and $p \geqslant 0$. Since

$$\frac{\mathrm{e}^{-n} - 1}{2} + \frac{2(\mathrm{e}^{-n} - 1)}{3} + \frac{3(\mathrm{e}^{-n} - 1)}{4} + \frac{2(\mathrm{e}^{-(n-1)} - 1)}{3} + \frac{3(\mathrm{e}^{-(n-1)} - 1)}{4} < -1$$

for $n \geqslant 1$, by Theorem 7 we can conclude that equation (18) is oscillatory.

In the next theorem we will establish a condition to get oscillatory solutions independently of the coefficients' sign.

Theorem 13. *Assume that for each $k \in \{-p, \ldots, q\}$ there exists the limit*

$$\lim_{n \to \infty} a_k(n) = a_k \neq 0. \tag{19}$$

If all roots $\lambda_1, \lambda_2, \ldots, \lambda_{q+p}$ of the equation

$$\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^k = 0 \tag{20}$$

satisfy $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_{q+p}|$ and n of them are negative (where $n \leqslant q+p$), then equation (1) *has n oscillatory solutions.*

Proof. Denote by λ_k the real negative root of (20). By Perron's theorem (see [13]), (1) has a solution u such that

$$\lim_{n \to \infty} \frac{u_k(n+1)}{u_k(n)} = \lambda_k < 0.$$

Thus such a solution is necessarily an oscillatory solution of (1). □

E x a m p l e 14. Consider the difference equation

$$\Delta x(n) - \frac{6n}{n+6}x(n-2) + \frac{5n+1}{n}x(n-1) + 6(\mathrm{e}^{-n}+1)x(n) + 6(\mathrm{e}^{-n}-1)x(n+1) + \frac{n}{n+1}x(n+2) = 0, \tag{21}$$

$n \geqslant 1$. Since

$$\begin{aligned} a_{-2}(n) &= -\frac{6n}{n+6} \underset{n\to\infty}{\longrightarrow} -6, \\ a_{-1}(n) &= \frac{5n+1}{n} \underset{n\to\infty}{\longrightarrow} 5, \\ a_0(n) &= 6(\mathrm{e}^{-n}+1) \underset{n\to\infty}{\longrightarrow} 6, \\ a_1(n) &= 6(\mathrm{e}^{-n}-1) \underset{n\to\infty}{\longrightarrow} -6, \\ a_2(n) &= \frac{n}{n+1} \underset{n\to\infty}{\longrightarrow} 1, \end{aligned}$$

equation (20) gives

$$\lambda - 1 - 6\lambda^{-2} + 5\lambda^{-1} + 6 - 6\lambda + \lambda^2 = 0$$

or

$$\lambda^4 - 5\lambda^3 + 5\lambda^2 + 5\lambda - 6 = 0. \tag{22}$$

Equation (22) has four different roots: $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 1$, and $\lambda_4 = -1$, so by Theorem 13, equation (21) has an oscillatory solution.

3. Nonoscillatory behavior

In this section we will study the nonoscilatory behavior of the autonomous equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0, \quad n \geqslant 1. \tag{23}$$

According to Krisztin [9], the oscillatory behavior of equation (23) can be studied similarly as for delay equations.

The equation

$$\Delta x(n) + \sum_{k=-p}^{q} a_k x(n+k) = 0, \quad n > n_0 \tag{24}$$

is nonoscillatory if there exists $\lambda \in \mathbb{R}^+$ such that

$$\lambda - 1 + \sum_{k=-p}^{q} a_k \lambda^k = 0. \tag{25}$$

We define

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k. \tag{26}$$

Theorem 15. *If $a_{-p} a_q < 0$ for all $n > n_1 \geqslant n_0$, then equation (1) is nonoscillatory.*

P r o o f. Consider $a_{-p} < 0$ and $a_q > 0$. Notice that

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{-1} a_k \lambda^k - \sum_{k=0}^{q} a_k \lambda^k \underset{\lambda \to 0^+}{\longrightarrow} \infty \tag{27}$$

and

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{-1} a_k \lambda^k - \sum_{k=0}^{q} a_k \lambda^k \underset{\lambda \to \infty}{\longrightarrow} -\infty. \tag{28}$$

By the continuity, there exists $\lambda_0 > 0$ such that $N(\lambda_0) = 0$.

Analogously, we prove that there exists $\lambda_1 > 0$ such that $N(\lambda_1) = 0$, when $a_{-p} > 0$, $a_q < 0$. $\square$

E x a m p l e 16. Consider the equation

$$\Delta x(n) + a x(n-1) + (1-3a) x(n) - (1-3a) x(n+1) - a x(n+2) = 0, \quad n \geqslant 1. \tag{29}$$

By Theorem 15, the difference equation (21) has a nonoscillatory solution, since

$$a_{-1} = a = -a_2.$$

In fact $x(n) = n$ is a nonoscilatory solution of (29).

Theorem 17. *Let $a_k > 0$ for every $k \in \{-p, \ldots, q\}$ and*

$$\max\left\{ \left(p \sum_{k=-p}^{q} a_k \right)^{1/(p+1)}, \left(\sum_{k=-p}^{q} a_k \right)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}} \right\} < 1. \tag{30}$$

Then equation (1) is nonoscillatory.

P r o o f. Notice that

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k < 1 - \lambda$$

and consequently, $N(\lambda) < 0$ for $\lambda \geqslant 1$.

Let $\lambda < 1$. So,

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > 1 - \lambda - \lambda^{-p} \sum_{k=-p}^{q} a_k.$$

The function

$$f(\lambda) = 1 - \lambda - \lambda^{-p} \sum_{k=-p}^{q} a_k$$

has a maximum for $\lambda_0 = \Bigl(p \sum\limits_{k=-p}^{q} a_k\Bigr)^{1/(p+1)}$ and

$$f(\lambda_0) = 1 - \Bigl(\sum_{k=-p}^{q} a_k\Bigr)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}} > 0.$$

Consequently, $N(\lambda_0) > 0$. □

Theorem 18. *Let $a_k < 0$ for every $k \in \{-p, \dots, q\}$ and*

$$\min\Biggl\{\Bigl(-q \sum_{k=-p}^{q} a_k\Bigr)^{-1/(q-1)}, \Bigl(-\sum_{k=-p}^{q} a_k\Bigr)^{-1/(q-1)} \frac{q+1}{q^{q/(q-1)}}\Biggr\} \geqslant 1. \tag{31}$$

Then equation (1) *is nonoscillatory.*

P r o o f. Notice that

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > 1 - \lambda$$

and consequently, $N(\lambda) > 0$ for $\lambda \leqslant 1$.

Let $\lambda > 1$. So,

$$N(\lambda) = 1 - \lambda - \sum_{k=-p}^{q} a_k \lambda^k > 1 - \lambda - \lambda^{q} \sum_{k=-p}^{q} a_k.$$

The function

$$g(\lambda) = 1 - \lambda - \lambda^q \sum_{k=-p}^{q} a_k$$

has a minimum for $\lambda_1 = \Big(-q \sum\limits_{k=-p}^{q} a_k\Big)^{-1/(q-1)}$ and

$$g(\lambda_1) = 1 - \left(-\sum_{k=-p}^{q} a_k\right)^{-1/(q-1)} \frac{q+1}{q^{q/(q-1)}} \leqslant 0.$$

Consequently, $N(\lambda_1) > 0$. $\square$

Example 19. Consider the equation

$$\Delta x(n) + \frac{253}{2048} x(n-1) + \frac{1}{512} x(n) + \frac{1}{256} x(n+1) + \frac{1}{64} x(n+3) = 0, \quad n \geqslant 1. \tag{32}$$

In this case we have $p = 1$,

$$\left(p \sum_{k=-p}^{q} a_k\right)^{1/(p+1)} = \sqrt{\frac{253}{2048} + \frac{1}{512} + \frac{1}{256} + \frac{1}{64}} = \sqrt{\frac{297}{2048}} < 1,$$

and

$$\left(\sum_{k=-p}^{q} a_k\right)^{1/(p+1)} \frac{p+1}{p^{p/(p+1)}} = 2\sqrt{\frac{297}{2048}} = \sqrt{\frac{297}{512}} < 1.$$

Consequently, by Theorem 17, the difference equation (32) has a nonoscillatory solution. In fact, $x(n) = 2^{-n}$ is a nonoscilatory solution of (32).

4. Asymptotic behavior

Now we will study the asymptotic behavior of the nonoscillatory solutions.

Theorem 20. *Let $a_k(n) \geqslant 0$ for every $k \in \{-p, \dots, q\}$, $n > n_0$ and*

$$\sum_{n=n_1}^{\infty} \sum_{k=-p}^{q} a_k(n) = \infty. \tag{33}$$

If x is an eventually positive solution of (1), then

$$\lim_{n\to\infty} x(n) = 0.$$

P r o o f. Suppose $x(n) > 0$ for $n > n_1$, so $\Delta x(n) < 0$ and consequently $\{x(n)\}$ is decreasing and has a finite limit. If

$$\lim_{n\to\infty} x(n) = d > 0,$$

then $x(n) > d$ for any $n > n_1$ and

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n)x(n+k) < -d\sum_{k=-p}^{q} a_k(n).$$

So,

$$x(n+1) < x(n_1) - d\sum_{i=n_1}^{n}\sum_{k=-p}^{q} a_k(i) \underset{n\to\infty}{\longrightarrow} -\infty.$$

This is a contradiction and this way we prove that $d = 0$. □

Theorem 21. *Let $a_k(n) \leqslant 0$ for every $k \in \{-p,\dots,q\}$, $n > n_0$ and*

$$\sum_{n=n_2}^{\infty}\sum_{k=-p}^{q} a_k(n) = -\infty. \tag{34}$$

If x is an eventually positive solution of (1), *then*

$$\lim_{n\to\infty} x(n) = 0.$$

P r o o f. Suppose $x(n) > 0$ for $n > n_1$, so $\Delta x(n) > 0$ and consequently $\{x(n)\}$ is increasing and

$$\Delta x(n) = -\sum_{k=-p}^{q} a_k(n)x(n+k) \geqslant -x(n-p)\sum_{k=-p}^{q} a_k(n).$$

So,

$$x(n+1) < x(n-p)\biggl(1 - \sum_{i=n-p}^{n}\sum_{k=-p}^{q} a_k(i)\biggr) \underset{n\to\infty}{\longrightarrow} \infty.$$

This completes the proof. □

E x a m p l e 22. Consider the equation

$$\Delta x(n) + 3^{-1-(2nk-k^2)/(2n+1)}x(n-k) + 3^{-1+(2nl+l^2)/(2n+1)}x(n+l) = 0, \quad n \geqslant 1, \tag{35}$$

where $k, l \geqslant 1$. Notice that

$$\sum_{n=1}^{\infty}(3^{-1-(2nk-k^2)/(2n+1)} + 3^{-1+(2nl+l^2)/(2n+1)}) = \frac{1}{3}\sum_{n=1}^{\infty}(3^{-(2nk-k^2)/(2n+1)} + 3^{(2nl+l^2)/(2n+1)}) = \infty,$$

since $\sum_{n=1}^{\infty} 3^{(2nl+l^2)/(2n+1)}$ is divergent. So, by Theorem 20, if x is an eventually positive solution of (35), then $\lim_{n\to\infty} x(n) = 0$. In fact, $x(n) = 3^{-n^2/(2n+1)}$ is a positive solution of (35) and $\lim_{n\to\infty} 3^{-n^2/(2n+1)} = 0$.

References

[1] *R. P. Agarwal*: Difference Equations and Inequalities: Theory, Methods, and Applications. Pure and Applied Mathematics 228, Marcel Dekker, New York, 2000.

[2] *T. Asada, H. Yoshida*: Stability, instability and complex behavior in macrodynamic models with policy lag. Discrete Dyn. Nat. Soc. *5* (2001), 281–295.

[3] *L. Berezansky, E. Braverman*: Some oscillation problems for a second order linear delay differential equation. J. Math. Anal. Appl. *220* (1998), 719–740.

[4] *J. Diblík, K. Janglajew, M. Kúdelčíková*: An explicit coefficient criterion for the existence of positive solutions to the linear advanced equation. Discrete Contin. Dyn. Syst. Ser. B *19* (2014), 2461–2467.

[5] *J. Diblík, M. Kúdelčíková*: New explicit integral criteria for the existence of positive solutions to the linear advanced equation $y' = c(t)y(t+\tau)$. Appl. Math. Lett. *38* (2014), 144–148.

[6] *D. Dubois, K. E. Stecke*: Dynamic analysis of repetitive decision-free discrete-event processes: applications to production systems. Ann. Oper. Res. *26* (1990), 323–347.

[7] *J. M. Ferreira, S. Pinelas*: Oscillatory mixed difference systems. Adv. Difference Equ. (2006), 1–18.

[8] *R. Frisch, H. Holme*: The characteristic solutions of a mixed difference and differential equation occurring in economic dynamics. Econometrica *3* (1935), 225–239.

[9] *G. Gandolfo*: Economic Dynamics. Springer, Berlin, 2010.

[10] *V. Iakovleva, C. J. Vanegas*: On the solution of differential equations with delayed and advanced arguments. Electron. J. Differ. Equ. *13* (2005), 57–63.

[11] *R. W. James, M. H. Belz*: The significance of the characteristic solutions of mixed difference and differential equations. Econometrica *6* (1938), 326–343.

[12] *T. Krisztin*: Nonoscillation for functional differential equations of mixed type. J. Math. Anal. Appl. *245* (2000), 326–345.

[13] *G. S. Ladde, V. Lakshmikantham, B. G. Zhang*: Oscillation Theory of Differential Equations with Deviating Arguments. Pure and Applied Mathematics 110, Marcel Dekker, New York, 1987.

[14] *Y. V. Rogovchenko*: Oscillation criteria for certain nonlinear differential equations. J. Math. Anal. Appl. *229* (1999), 399–416.

Authors' addresses: *Leonid Berezansky*, Department of Mathematics, Ben Gurion University of the Negev, P.O.Box 653, Beer Sheva 8410501, Israel, e-mail: brznsky@cs.bgu.ac.il; *Sandra Pinelas*, Academia Militar, Departamento de Ciencias Exactas e Naturais, Avenide Conde Castro Guimaraes, 2720-113 Amadora, Portugal, e-mail: sandra.pinelas@gmail.com.