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AVERAGING FOR ORDINARY DIFFERENTIAL EQUATIONS PERTURBED BY A SMALL PARAMETER

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his 90th birthday

Abstract. In this paper, we prove and discuss averaging results for ordinary differential equations perturbed by a small parameter. The conditions we assume on the right-hand sides of the equations under which our averaging results are stated are more general than those considered in the literature. Indeed, often it is assumed that the right-hand sides of the equations are uniformly bounded and a Lipschitz condition is imposed on them. Sometimes this last condition is relaxed to the uniform continuity in the second variable uniformly with respect to the first one. In our results, we assume only that the right-hand sides of the equations are bounded by some locally Lebesgue integrable functions with the property that their indefinite integrals satisfy a Lipschitz-type condition. Also, we consider that they are only continuous in the second variable uniformly with respect to the first one.

Keywords: ordinary differential equation; method of averaging

MSC 2010: 34C15, 34C29, 34K25

1. Introduction

The method of averaging is an important tool for analysis of nonlinear differential equations perturbed by a small parameter. It allows to replace a time-varying small perturbation, acting on a long time interval, by a time-invariant perturbation, while introducing only a small error. For significant results on the method of averaging with developments and examples we refer the interested reader to the books [1], [5], [10]–[12] and the references given therein.

In the present paper, we establish averaging results for ordinary differential equations perturbed by a small parameter. The conditions we assume on the right-hand

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sides of the equations under which our averaging results are stated are more general than those often considered in the existing literature as in [2], [4], [6]–[9], for instance (see also the survey paper [3]). Indeed, in all the references cited above, it is assumed that the right-hand sides of the equations are uniformly bounded. In addition, in [2], [6]–[8] a Lipschitz condition is imposed on them, whereas in [4], [9], this condition is relaxed to the uniform continuity in the second variable uniformly with respect to the first one. In our results, we assume only that the right-hand sides of the equations are bounded by some locally Lebesgue integrable functions with the property that their indefinite integrals satisfy a Lipschitz-type condition. Also, we consider that they are only continuous in the second variable uniformly with respect to the first one.

2. Averaging results

Consider the following initial value problem associated to an ordinary differential equation with a small parameter

$$\dot{x} = f\Big(\frac{t}{\varepsilon}, x\Big), \quad x(0) = x_0, \tag{2.1}$$

where $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ is a small parameter.

To the problem (2.1) we associate the averaged initial value problem

$$\dot{y} = f^0(y), \quad y(0) = x_0, \tag{2.2}$$

where the function $f^0\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that for any $x \in \mathbb{R}^n$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\tau, x)\,\mathrm{d}\tau = f^0(x). \tag{2.3}$$

The main result of this paper establishes the approximation of solutions of problems (2.1) by those of the averaged problem (2.2) on finite time intervals, and reads as follows.

Theorem 2.1. *Let $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a function and $x_0 \in \mathbb{R}^n$. Suppose that the following conditions are satisfied:*

(H1) *the function f is continuous on $\mathbb{R}_+ \times \mathbb{R}^n$;*
(H2) *the continuity of f in $x \in \mathbb{R}^n$ is uniform with respect to $t \in \mathbb{R}_+$;*
(H3) *there exist a locally Lebesgue integrable function $m\colon \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $M > 0$ such that*

$$|f(t,x)| \leqslant m(t), \quad \forall\, t \in \mathbb{R}_+,\ \forall\, x \in \mathbb{R}^n$$

with

$$\int_{t_1}^{t_2} m(t)\,\mathrm{d}t \leqslant M(t_2 - t_1), \quad \forall\, t_1, t_2 \in \mathbb{R}_+;$$

(H4) *for all* $x \in \mathbb{R}^n$, *the limit* (2.3) *exists.*

Then, for any $L > 0$ *and* $\delta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(x_0, L, \delta) > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0]$ *and any solution* x_ε *of* (2.1), *there exists a solution* y *of* (2.2) *which satisfies* $|x_\varepsilon(t) - y(t)| < \delta$ *for all* $t \in [0, L]$.

Notice that by the conditions (H1) and (H3) the initial value problem (2.1) is well defined and all its solutions exist for all $t \geqslant 0$. On the other hand, from conditions (H1)–(H4) we deduce that the average of the function f, that is, the function $f^0\colon \mathbb{R}^n \to \mathbb{R}^n$ in (H4), is continuous and bounded (see Lemma 2.1 below). So, the averaged initial value problem (2.2) is also well defined and all its solutions exist for all $t \geqslant 0$.

We point out also that under the conditions (H1) and (H2) in Theorem 2.1, it is only possible to obtain unilateral approximations, that is, the approximation of solutions of problems (2.1) by those of the averaged problem (2.2). The converse approximation is, in general, false as showed in [3], page 356, Example 1. However, when the problem (2.2) has a unique solution, this solution is approximated by any one of the problem (2.1) as it is stated by the following interesting result which is a particular case of Theorem 2.1.

Corollary 2.1. *Let* $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *be a function and* $x_0 \in \mathbb{R}^n$. *Suppose that the conditions* (H1)–(H4) *in Theorem* 2.1 *are satisfied. Suppose also that*

(H5) *the initial value problem* (2.2) *has a unique solution.*

Let y *be the* (*unique*) *solution of* (2.2). *Then, for any* $L > 0$ *and* $\delta > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(x_0, L, \delta) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$, *any solution* x_ε *of* (2.1) *satisfies* $|x_\varepsilon(t) - y(t)| < \delta$ *for all* $t \in [0, L]$.

When the function f is periodic or more generally almost periodic in the first variable, some of the conditions in Theorem 2.1 and Corollary 2.1 can be removed. Indeed, if f is periodic in t, from continuity and periodicity properties it is easy to deduce condition (H2). Periodicity also implies the condition (H4) in an obvious way. The average of f is then given, for any $x \in \mathbb{R}^n$, by

$$\frac{1}{T}\int_0^T f(\tau, x)\,\mathrm{d}\tau = f^0(x), \tag{2.4}$$

where T is the period. If f is almost periodic in t it is well-known that for all $x \in \mathbb{R}^n$, the limit

$$\lim_{T\to\infty} \frac{1}{T}\int_s^{s+T} f(\tau, x)\,\mathrm{d}\tau = f^0(x) \tag{2.5}$$

exists uniformly with respect to $s \in \mathbb{R}$. So, condition (H4) is satisfied when $s = 0$. We point out also that in a number of cases encountered in applications the function f is a finite sum of periodic functions in t. As in the periodic case above, condition (H2) is then satisfied. Hence we have the following result.

Corollary 2.2 (Periodic and almost periodic cases). *The conclusion of Theorem* 2.1 (*respectively, Corollary* 2.1) *holds in both of the following cases:*

(1) *The function f satisfies conditions* (H1), (H3) (*respectively,* (H1), (H3), (H5)) *and is periodic or a sum of periodic functions in the first variable.*
(2) *The function f satisfies conditions* (H1), (H2), (H3) (*respectively,* (H1), (H2), (H3), (H5)) *and is almost periodic in the first variable.*

2.1. Technical lemmas. In what follows we will prove some results we need for the proof of Theorem 2.1.

Lemma 2.1. *Let $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. Suppose that f satisfies the conditions* (H1)–(H4) *in Theorem* 2.1. *Then the function $f^0\colon \mathbb{R}^n \to \mathbb{R}^n$ in* (2.3) *is continuous and is uniformly bounded by the constant M in condition* (H4).

Proof. *Continuity of f^0.* Let $x_0 \in \mathbb{R}^n$. By the condition (H2), for any $\xi > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}^n$, $|x - x_0| \leqslant \delta$ implies that

$$|f(\tau, x) - f(\tau, x_0)| \leqslant \xi, \quad \forall\, \tau \in \mathbb{R}_+. \tag{2.6}$$

Now, by the condition (H4), we can easily deduce that for any $\eta > 0$ there exists $T_0 = T_0(x_0, x, \eta) > 0$ such that for all $T \geqslant T_0$ we have

$$\begin{aligned}
|f^0(x) - f^0(x_0)| &\leqslant \left| f^0(x) - \frac{1}{T}\int_0^T f(\tau, x)\,\mathrm{d}\tau \right| \\
&\quad + \left| \frac{1}{T}\int_0^T f(\tau, x)\,\mathrm{d}\tau - \frac{1}{T}\int_0^T f(\tau, x_0) \right| \mathrm{d}\tau \\
&\quad + \left| f^0(x_0) - \frac{1}{T}\int_0^T f(\tau, x_0)\,\mathrm{d}\tau \right| \\
&\leqslant 2\eta + \frac{1}{T}\int_0^T |f(\tau, x) - f(\tau, x_0)|\tau \leqslant 2\eta + \xi.
\end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain that $|f^0(x) - f^0(x_0)| \leqslant \xi$, which completes the proof of the continuity of f^0 at the point x_0.

Boundedness of f^0 by M. Let $x \in \mathbb{R}^n$. By the condition (H3), we deduce that for any $\eta > 0$ there exists $T_0 = T_0(x, \eta) > 0$ such that for all $T \geqslant T_0$ we have

$$\begin{aligned}|f^0(x)| &\leqslant \left| f^0(x) - \frac{1}{T}\int_0^T f(\tau, x)\,\mathrm{d}\tau \right| + \left| \frac{1}{T}\int_0^T f(\tau, x)\,\mathrm{d}\tau \right| \\ &\leqslant \eta + \frac{1}{T}\int_0^T |f(\tau, x)|\,\mathrm{d}\tau \leqslant \eta + M.\end{aligned}$$

Since the value of η is arbitrary, in the limit we obtain the desired result. $\square$

Lemma 2.2. *Let $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be a function. Suppose that f satisfies the condition* (H4) *in Theorem* 2.1. *Then, for all $x \in \mathbb{R}^n$, $t \geqslant 0$ and $\alpha > 0$, we have*

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon + \alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau = f^0(x).$$

P r o o f. Let $x \in \mathbb{R}^n$, $t \geqslant 0$ and $\alpha > 0$.

Case 1: $t = 0$. From the condition (H4), it follows immediately that

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\alpha} \int_0^{\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau = f^0(x).$$

Case 2: $t \in (0, L]$. By some calculations we obtain

$$\begin{aligned}\frac{\varepsilon}{\alpha}\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau &= \frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau - \frac{\varepsilon}{\alpha}\int_0^{t/\varepsilon} f(\tau, x)\,\mathrm{d}\tau \\ &= \frac{\varepsilon}{t+\alpha}\Big(\frac{t}{\alpha}+1\Big)\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau - \frac{t}{\alpha}\frac{\varepsilon}{t}\int_0^{t/\varepsilon} f(\tau, x)\,\mathrm{d}\tau \\ &= \frac{\varepsilon}{t+\alpha}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau \\ &\quad + \frac{t}{\alpha}\Big[\frac{\varepsilon}{t+\alpha}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau - \frac{\varepsilon}{t}\int_0^{t/\varepsilon} f(\tau, x)\,\mathrm{d}\tau\Big].\end{aligned} \tag{2.7}$$

From the condition (H4), we can easily deduce that

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t+\alpha}\int_0^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x)\,\mathrm{d}\tau = f^0(x)$$

and

$$\lim_{\varepsilon \to 0^+} \frac{\varepsilon}{t}\int_0^{t/\varepsilon} f(\tau, x)\,\mathrm{d}\tau = f^0(x).$$

Therefore the right-hand side of (2.7) tends to $f^0(x)$ as $\varepsilon \to 0^+$ and the result is proved. $\square$

Lemma 2.3. *Suppose that the function f in* (2.1) *satisfies the conditions* (H1) *and* (H3) *in Theorem* 2.1. *Let $x_0 \in \mathbb{R}^n$ and $L > 0$. Then the family $\{x_\varepsilon\}$ of solutions of the problem* (2.1) *converges uniformly on $[0, L]$ to a continuous function y when ε tends to 0^+.*

Proof. For $t, \tau \in [0, L]$ we have

$$|x_\varepsilon(t)| \leqslant |x_0| + ML \quad \text{and} \quad |x_\varepsilon(t) - x_\varepsilon(\tau)| \leqslant M|t - \tau|,$$

which proves that the family $\{x_\varepsilon\}$ is uniformly bounded and equicontinuous on $[0, L]$. Hence, by the Ascoli-Arzelà theorem, there exists a continuous function $y\colon [0, L] \to \mathbb{R}^n$ such that $\lim\limits_{\varepsilon\to 0^+} \sup\limits_{t\in[0,L]} |x_\varepsilon(t) - y(t)| = 0$. This finishes the proof of the lemma. □

Lemma 2.4. *Suppose that the function f in* (2.1) *satisfies the conditions* (H1), (H3) *and* (H4) *in Theorem* 2.1. *Let $x_0 \in \mathbb{R}^n$ and $L > 0$. Let $\{x_\varepsilon\}$ be the family of solutions of the problem* (2.1) *converging uniformly* (*by Lemma* 2.3) *to a continuous function y on $[0, L]$ when ε tends to 0^+. Then for all $t \in [0, L]$ and $\alpha > 0$ we have*

$$\lim_{\varepsilon\to 0^+} \frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t))\,\mathrm{d}\tau = f^0(y(t)). \tag{2.8}$$

Proof. Let $t \in [0, L]$ and $\alpha > 0$. We have

$$\begin{aligned} &\left|\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t))\,\mathrm{d}\tau - f^0(y(t))\right| \\ &\qquad \leqslant \frac{\varepsilon}{\alpha}\left|\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t))\,\mathrm{d}\tau - \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, y(t))\,\mathrm{d}\tau\right| \\ &\qquad\quad + \left|\frac{\varepsilon}{\alpha} \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, y(t))\,\mathrm{d}\tau - f^0(y(t))\right|. \end{aligned} \tag{2.9}$$

By Lemma 2.2 the second term of the right-hand side of (2.9) tends to zero as $\varepsilon \to 0^+$. For the first term in the right-hand side of (2.9) we write

$$\begin{aligned} &\frac{\varepsilon}{\alpha}\left|\int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t))\,\mathrm{d}\tau - \int_{t/\varepsilon}^{t/\varepsilon+\alpha/\varepsilon} f(\tau, y(t))\,\mathrm{d}\tau\right| \\ &\qquad = \frac{1}{\alpha}\left|\int_t^{t+\alpha} f\Big(\frac{\tau}{\varepsilon}, x_\varepsilon(t)\Big)\,\mathrm{d}\tau - \int_t^{t+\alpha} f\Big(\frac{\tau}{\varepsilon}, y(t)\Big)\,\mathrm{d}\tau\right| := \frac{1}{\alpha}\xi. \end{aligned}$$

By Lemma 2.3 and the condition (H2), $\lim\limits_{\varepsilon\to 0^+} \xi = 0$. So, one can conclude that all terms of the right-hand side of the inequality (2.9) tend to zero as $\varepsilon \to 0^+$, which completes the proof of the lemma. □

Lemma 2.5. *Suppose that the function f in* (2.1) *satisfies the conditions* (H1)–(H4) *in Theorem* 2.1. *Let $x_0 \in \mathbb{R}^n$ and $L > 0$. Let $\{x_\varepsilon\}$ be the family of solutions of the problem* (2.1) *converging uniformly* (*by Lemma* 2.3) *to a continuous function y on $[0, L]$ when ε tends to 0^+. Then for all $L > 0$ we have*

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[0,L]} \left| \int_0^t f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_0^t f^0(y(\tau))\,\mathrm{d}\tau \right| = 0.$$

P r o o f. Let $L > 0$ and let $t_0 = 0 < t_1 < \ldots < t_m < \ldots < t_p = L$, $p \in \mathbb{N}$, be a partition of $[0, L]$ with $\alpha = \alpha(\varepsilon) := t_{m+1} - t_m$, $m = 0, \ldots, p-1$, and $\lim\limits_{\varepsilon\to 0^+} \alpha = 0$. Let $t \in [t_m, t_{m+1}]$ for any $m \in \{0, \ldots, p-1\}$. Then

$$\begin{aligned} \left| \int_0^t f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_0^t f^0(y(\tau))\,\mathrm{d}\tau \right| &\leqslant \sum_{i=0}^{m-1} \left| \int_{t_i}^{t_{i+1}} f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_{t_i}^{t_{i+1}} f^0(y(\tau))\,\mathrm{d}\tau \right| \\ &\quad + \left| \int_{t_m}^t f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_{t_m}^t f^0(y(\tau))\,\mathrm{d}\tau \right|. \end{aligned} \tag{2.10}$$

By the condition (H3) and Lemma 2.1 we have

$$\begin{aligned} \left| \int_{t_m}^t f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_{t_m}^t f^0(y(\tau))\,\mathrm{d}\tau \right| &\leqslant \int_{t_m}^t \left| f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr) \right| \mathrm{d}\tau + \int_{t_m}^t |f^0(y(\tau))|\,\mathrm{d}\tau \leqslant 2M\alpha. \end{aligned}$$

Now, for each $i = 0, \ldots, m-1$ and $\tau \in [t_i, t_{i+1}]$, by the condition (H3) we can easily deduce that $|x_\varepsilon(\tau) - x_\varepsilon(t_i)| \leqslant M\alpha$ so that by the condition (H2) and the continuity of f^0 (Lemma 2.2), it follows, respectively, that

$$\left| f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr) - f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(t_i)\Bigr) \right| \leqslant \gamma_i = \gamma_i(\varepsilon)$$

and $|f^0(x_\varepsilon(\tau)) - f^0(x_\varepsilon(t_i))| \leqslant \delta_i = \delta_i(\varepsilon)$, with $\lim\limits_{\varepsilon\to 0^+} \gamma_i = \lim\limits_{\varepsilon\to 0^+} \delta_i = 0$.

Hence, from (2.10), it follows that

$$\begin{aligned} \left| \int_0^t f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Bigr)\,\mathrm{d}\tau - \int_0^t f^0(x_\varepsilon(\tau))\,\mathrm{d}\tau \right| &\leqslant \sum_{i=0}^{m-1} \left| \int_{t_i}^{t_{i+1}} f\Bigl(\frac{\tau}{\varepsilon}, x_\varepsilon(t_i)\Bigr)\,\mathrm{d}\tau - \int_{t_i}^{t_{i+1}} f^0(x_\varepsilon(t_i))\,\mathrm{d}\tau \right| \\ &\quad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (\gamma_i + \delta_i)\,\mathrm{d}\tau + 2M\alpha. \end{aligned} \tag{2.11}$$

For each $i = 0, \ldots, m-1$, we have

$$\begin{aligned}\beta_i &:= \left|\int_{t_i}^{t_{i+1}} f\Big(\frac{\tau}{\varepsilon}, x_\varepsilon(t_i)\Big)\,\mathrm{d}\tau - \int_{t_i}^{t_{i+1}} f^0(x_\varepsilon(t_i))\,\mathrm{d}\tau\right| \\ &= \alpha\left|\frac{\varepsilon}{\alpha}\int_{t_i/\varepsilon}^{t_i/\varepsilon+\alpha/\varepsilon} f(\tau, x_\varepsilon(t_i))\,\mathrm{d}\tau - f^0(x_\varepsilon(t_i))\right| := \alpha\varrho_i \leqslant \alpha\varrho_m,\end{aligned}$$

where $\varrho_m = \max\{\varrho_i = \varrho_i(\varepsilon)\colon\ i = 0, \ldots, m-1\}$ and, by Lemma 2.4, $\lim\limits_{\varepsilon\to 0^+} \varrho_i = 0$.

Then

$$\sum_{i=0}^{m-1} \beta_i \leqslant \varrho_m \sum_{i=0}^{m-1} \alpha = \varrho_m \sum_{i=0}^{m-1} (t_{i+1} - t_i) = \varrho_m t \leqslant \varrho_m L.$$

On the other hand, we have

$$\sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} (\gamma_i + \delta_i)\,\mathrm{d}\tau \leqslant \eta_m \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \mathrm{d}\tau = \eta_m t \leqslant \eta_m L,$$

where $\eta_m = \eta_m(\varepsilon) = \max\{\gamma_i + \delta_i\colon\ i = 0, \ldots, m-1\}$ and $\lim\limits_{\varepsilon\to 0^+} \eta_m = 0$.

Finally, from (2.11) we obtain

$$\sup_{t\in[0,L]}\left|\int_0^t f\Big(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Big)\,\mathrm{d}\tau - \int_0^t f^0(x_\varepsilon(\tau))\,\mathrm{d}\tau\right| \leqslant (\varrho + \eta)L + 2M\alpha, \tag{2.12}$$

where $\varrho = \max\{\varrho_m\colon\ m = 0, \ldots, p-1\}$ and $\eta = \max\{\eta_m\colon\ m = 0, \ldots, p-1\}$. As the right-hand side of (2.12) tends to zero as $\varepsilon \to 0^+$, the lemma is proved. $\square$

2.2. Proof of Theorem 2.1. We are now able to prove our main result (there is not much work left). We assume that the assumptions in Theorem 2.1 are fulfilled. Let $L > 0$. Let $\{x_\varepsilon\}$ be the family of solutions of the problem (2.1). By Lemma 2.3, there exists a continuous function $y\colon\ [0, L] \to \mathbb{R}^n$ such that

$$\lim_{\varepsilon\to 0^+} \sup_{t\in[0,L]} |x_\varepsilon(t) - y(t)| = 0. \tag{2.13}$$

For any $t \in [0, L]$ the function y is such that

$$\begin{aligned}&\left|y(t) - x_0 - \int_0^t f^0(y(\tau))\,\mathrm{d}\tau\right| \\ &\quad\leqslant |y(t) - x_\varepsilon(t)| + \left|x_\varepsilon(t) - x_0 - \int_0^t f^0(y(\tau))\,\mathrm{d}\tau\right| \\ &\quad\leqslant \sup_{t\in[0,L]} |y(t) - x_\varepsilon(t)| + \sup_{t\in[0,L]}\left|\int_0^t f\Big(\frac{\tau}{\varepsilon}, x_\varepsilon(\tau)\Big)\,\mathrm{d}\tau - \int_0^t f^0(y(\tau))\,\mathrm{d}\tau\right|.\end{aligned} \tag{2.14}$$

By (2.13) and Lemma 2.5, the right-hand side of (2.14) tends to zero as $\varepsilon \to 0^+$, so that one can conclude that the function y is a solution of the problem (2.2). The proof is complete. $\square$

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