Ammar Derbazi; Mohamed Dalah; Amar Megrous

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THE WEAK SOLUTION OF AN ANTIPLANE CONTACT PROBLEM FOR ELECTRO-VISCOELASTIC MATERIALS WITH LONG-TERM MEMORY

Ammar Derbazi, Bordj Bou Arréridj, Mohamed Dalah, Constantine,
Amar Megrous, Constantine

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Abstract. We study a mathematical model which describes the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. The material is assumed to be electro-viscoelastic with long-term memory, and the friction is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First we derive the classical variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field with a time-dependent variational equation for the potential field. Then we prove the existence of a unique weak solution to the model. Moreover, the proof is based on arguments of evolution equations and on the Banach fixed-point theorem.

Keywords: weak solution; variational formulation; antiplane shear deformation; electro-viscoelastic material; Tresca's friction; fixed point; variational inequality

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1. Introduction

Antiplane problems play a useful role as pilot problems, allowing for various aspects of solutions in solid mechanics to be examined in a particularly simple setting. Considerable attention has been paid to the modelling of such kind of problems, see for instance [9], [10], [11], and the references therein. In particular, the review article [9] deals with modern developments for the antiplane shear model involving linear and nonlinear solid materials, various constitutive settings and applications. Antiplane frictional contact problems are used in geophysics in order to describe pre-earthquake evolution of the regions of high tectonic activity, see for instance [5], [6] and the references therein. The mathematical analysis of models for antiplane frictional contact problems can be found in [1], [8], [13], [14], [17], [24].

Currently there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, i.e., materials characterized by the coupling of mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials and piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. General models for piezoelectric materials can be found in [2], [12], [23]. Static frictional contact problems for elastic and viscoelastic materials were studied in [3], [16], [18], [22], [21], [19], [20], [7], under the assumption that the foundation is insulated. Contact problems with normal compliance for electro-viscoelastic materials were investigated in [15], [25]. There, variational formulations of the problems were considered and their unique solvability was proved. Antiplane problems for piezoelectric materials were considered in [4], [24], [27].

In paper [26], the authors have studied an antiplane contact problem for viscoelastic materials with long-term memory. This mechanical problem leads to an integro-differential variational inequality. Unlike [26], in the present paper we deal with an antiplane contact problem for an electro-viscoelastic cylinder, which leads to a new mathematical model, different to the one presented in [26]. The novelty in this paper consists in the fact that we model the friction with Tresca's law and the material's behavior with a viscoelastic constitutive law with long-term memory. We neglect the inertial term in the equation of motion to obtain a quasistatic approximation of the process. The main result we provide concerns the existence of a unique weak solution to the model. Its proof is carried out in several steps, and is based on arguments of evolutionary variational inequalities and Banach's fixed-point theorem.

The rest of the paper is structured as follows. In Section 2 we describe the model of the frictional contact process between an electro-viscoelastic body and a conductive deformable foundation. In Section 3 we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.1. The proof of the theorem is provided in Section 4, where it is based on arguments of evolutionary inequalities, and a fixed-point theorem. The paper concludes in Section 5.

2. The model of the antiplane contact problem

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region Ω in the x_1, x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, \infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2, and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measures of Γ_1 and Γ_a, denoted $\operatorname{meas}\Gamma_1$ and $\operatorname{meas}\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, \infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (-\infty, \infty)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (-\infty, \infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b \times (-\infty, \infty)$. The cylinder is in contact over $\Gamma_3 \times (-\infty, \infty)$ with a conductive obstacle, the so-called foundation. The contact is frictional and is modeled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$. We assume that

$$\mathbf{f}_0 = (0, 0, f_0) \quad \text{with } f_0 = f_0(x_1, x_2, t) \colon\ \Omega \times [0, T] \to \mathbb{R}, \tag{2.1}$$

$$\mathbf{f}_2 = (0, 0, f_2) \quad \text{with } f_2 = f_2(x_1, x_2, t) \colon\ \Gamma_2 \times [0, T] \to \mathbb{R}, \tag{2.2}$$

$$q_0 = q_0(x_1, x_2, t) \colon\ \Omega \times [0, T] \to \mathbb{R}, \tag{2.3}$$

$$q_2 = q_2(x_1, x_2, t) \colon\ \Gamma_b \times [0, T] \to \mathbb{R}. \tag{2.4}$$

The forces (2.1), (2.2) and the electric charges (2.3), (2.4) are expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent of x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with } u = u(x_1, x_2, t) \colon\ \Omega \times [0, T] \to \mathbb{R}, \tag{2.5}$$

$$\varphi = \varphi(x_1, x_2, t) \colon\ \Omega \times [0, T] \to \mathbb{R}. \tag{2.6}$$

Such kind of deformation, associated to a displacement field of the form (2.5), is called an *antiplane shear*, see for instance [11] and [13] for details.

Below, the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable; also, a dot above represents the time derivative. We use $\mathcal{S}^3$ for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and "$\cdot$", $\|\cdot\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and $\mathcal{S}^3$; we have:

$$\mathbf{u}\cdot\mathbf{v} = u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}\cdot\mathbf{v})^{1/2} \quad \forall\,\mathbf{u} = (u_i),\ \mathbf{v} = (v_i) \in \mathbb{R}^3,$$
$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \quad \forall\,\boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathcal{S}^3.$$

The infinitesimal strain tensor is denoted by $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_{ij})$. We also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_i(\varphi) = -\varphi_{,i}\,.$$

The material's behavior is modeled by the following electro-viscoelastic constitutive law with long-term memory:

$$\boldsymbol{\sigma} = \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + 2\int_0^t \theta(t-s)\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{s}))\,\mathrm{d}s - \mathcal{E}^*\mathbf{E}(\varphi), \tag{2.7}$$

$$\mathbf{D} = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \beta\mathbf{E}(\varphi), \tag{2.8}$$

where λ and μ are the Lamé coefficients, $\theta\colon [0,T] \to \mathbb{R}$ is the viscosity coefficient, $\operatorname{tr}\boldsymbol{\varepsilon}(\mathbf{u}) = \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, $\mathcal{E}$ represents the third-order piezoelectric tensor, and $\mathcal{E}^*$ is its transpose. In the antiplane context (2.5), (2.6), using the constitutive equations (2.7), (2.8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & 0 & \boldsymbol{\sigma}_{13} \\ 0 & 0 & \boldsymbol{\sigma}_{23} \\ \boldsymbol{\sigma}_{31} & \boldsymbol{\sigma}_{32} & 0 \end{pmatrix}, \tag{2.9}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix}, \tag{2.10}$$

where, β is the electric permittivity constant, e is a piezoelectric coefficient,

$$\boldsymbol{\sigma}_{13} = \boldsymbol{\sigma}_{31} = \mu\partial_{x_1}u + \int_0^t \theta(t-s)\partial_{x_1}u(s)\,\mathrm{d}s,$$

and

$$\boldsymbol{\sigma}_{23} = \boldsymbol{\sigma}_{32} = \mu\partial_{x_2}u + \int_0^t \theta(t-s)\partial_{x_2}u(s)\,\mathrm{d}s.$$

We assume that

$$\mathcal{E}\boldsymbol{\varepsilon} = \begin{pmatrix} e(\varepsilon_{13}+\varepsilon_{31}) \\ e(\varepsilon_{23}+\varepsilon_{32}) \\ e\varepsilon_{33} \end{pmatrix} \quad \forall\, \boldsymbol{\varepsilon} = (\varepsilon_{ij}) \in \mathcal{S}^3. \tag{2.11}$$

We also assume that the coefficients θ, μ, β, and e depend on the spatial variables x_1, x_2, but are independent of the spatial variable x_3. Since $\mathcal{E}\boldsymbol{\varepsilon}\cdot\mathbf{v} = \boldsymbol{\varepsilon}\cdot\mathcal{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon}\in\mathcal{S}^3$, $\mathbf{v}\in\mathbb{R}^3$, it follows from (2.11) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall\, \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{2.12}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\operatorname{Div}\boldsymbol{\sigma} + \mathbf{f}_0 = \mathbf{0}, \quad D_{i,i} - q_0 = 0 \quad \text{in } \mathcal{B}\times(0,T),$$

where $\operatorname{Div}\boldsymbol{\sigma} = (\sigma_{ij,j})$ represents the divergence of the tensor field $\boldsymbol{\sigma}$. Taking into account (2.1), (2.3), (2.5), (2.6), (2.9), and (2.10), the equilibrium equations above reduce to the following scalar equations:

$$\operatorname{div}(\mu\nabla u) + \int_0^t \theta(t-s)\operatorname{div}(\nabla u(s))\,\mathrm{d}s + \operatorname{div}(e\nabla\varphi) + f_0 = 0 \quad \text{in } \Omega\times(0,T), \tag{2.13}$$

$$\operatorname{div}(e\nabla u - \beta\nabla\varphi) = q_0. \tag{2.14}$$

Here and below we use the notation

$$\operatorname{div}\boldsymbol{\tau} = \tau_{1,1} + \tau_{1,2} \quad \text{for } \boldsymbol{\tau} = (\tau_1(x_1,x_2,t), \tau_2(x_1,x_2,t)),$$
$$\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu v = v_{,1}\,\nu_1 + v_{,2}\,\nu_2 \quad \text{for } v = v(x_1,x_2,t).$$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1\times(-\infty,\infty)$ and the electric potential vanishes on $\Gamma_1\times(-\infty,\infty)$; thus, (2.5) and (2.6) imply that

$$u = 0 \quad \text{on } \Gamma_1\times(0,T), \tag{2.15}$$

$$\varphi = 0 \quad \text{on } \Gamma_a\times(0,T). \tag{2.16}$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma\times(-\infty,\infty)$. We have

$$\boldsymbol{\nu} = (\nu_1,\nu_2,0) \quad \text{with } \nu_i = \nu_i(x_1,x_2)\colon\ \Gamma\to\mathbb{R},\ i=1,2. \tag{2.17}$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}\cdot\boldsymbol{\nu}, \quad \mathbf{v}_\tau = \mathbf{v} - v_\nu\boldsymbol{\nu}. \tag{2.18}$$

For a given stress field $\boldsymbol{\sigma}$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components on the boundary, that is,

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}. \tag{2.19}$$

From (2.9), (2.10), and (2.17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \left(0, 0, \mu\partial_\nu u + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right), \quad \mathbf{D}\cdot\boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu\varphi. \tag{2.20}$$

Taking into account (2.2), (2.4), and (2.20), the traction condition on $\Gamma_2\times(-\infty,\infty)$ and the electric conditions on $\Gamma_b\times(-\infty,\infty)$ are given by

$$\mu\partial_\nu u + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2\times(0,T), \tag{2.21}$$

$$e\partial_\nu u - \beta\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b\times(0,T). \tag{2.22}$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3\times(-\infty,\infty)$. First, from (2.5) and (2.17) we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during the whole process. Using now (2.5) and (2.17)–(2.19), we conclude that

$$\mathbf{u}_\tau = (0,0,u), \quad \boldsymbol{\sigma}_\tau = (0,0,\sigma_\tau), \tag{2.23}$$

where

$$\sigma_\tau = \left(0, 0, \mu\partial_\nu u + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right).$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is,

$$\begin{cases} |\boldsymbol{\sigma}_\tau(t)| \leqslant g, \\ |\boldsymbol{\sigma}_\tau(t)| < g \Rightarrow \dot{\boldsymbol{u}}_\tau(t) = \mathbf{0}, \\ |\boldsymbol{\sigma}_\tau(t)| = g \Rightarrow \exists\,\beta \geqslant 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\beta\dot{\boldsymbol{u}}_\tau \end{cases} \quad \text{on } \Gamma_3\times(0,T). \tag{2.24}$$

Here $g\colon \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary. Using now (2.23), it is straightforward to see that the friction law (2.24) implies

$$
\begin{cases}
\left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| \leqslant g, \\
\left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| < g \Rightarrow \dot{u}(t) = 0, \quad \text{on } \Gamma_3 \times (0,T). \\
\left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| = g \Rightarrow \\
\qquad \exists\,\beta \geqslant 0 \text{ such that } \mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi = -\beta\dot{u}.
\end{cases} \tag{2.25}
$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}\cdot\boldsymbol{\nu} = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (2.20) and the previous equality to obtain

$$e\partial_\nu u - \beta\partial_\nu\varphi = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T). \tag{2.26}$$

Finally, we prescribe the initial displacement

$$u(0) = u_0 \quad \text{in } \Omega, \tag{2.27}$$

where u_0 is a given function on Ω.

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem $\mathcal{P}$. Find the displacement field $u\colon \Omega \times [0,T] \to \mathbb{R}$ and the electric potential $\varphi\colon \Omega \times [0,T] \to \mathbb{R}$ such that

$$\operatorname{div}(\mu\nabla u) + \int_0^t \theta(t-s)\operatorname{div}(\nabla u(s))\,\mathrm{d}s + \operatorname{div}(e\nabla\varphi) + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{2.28}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \text{in } \Omega \times (0,T), \tag{2.29}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.30}$$

$$\mu\partial_\nu u + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.31}$$

$$\left\{\begin{array}{l} \left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| \leqslant g, \\ \left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| < g \Rightarrow \dot{u}(t) = 0, \qquad \text{on } \Gamma_3 \times (0,T), \\ \left|\mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi\right| = g \Rightarrow \\ \qquad \exists\,\beta \geqslant 0 \text{ such that } \mu\partial_\nu u + \displaystyle\int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi = -\beta\dot{u}, \end{array}\right. \tag{2.32}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \quad \text{on } \Gamma_b \times (0,T), \tag{2.33}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0,T), \tag{2.34}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.35}$$

Note that once the displacement field u and the electric potential φ which solve Problem $\mathcal{P}$ are known, then the stress tensor $\boldsymbol{\sigma}$ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (2.9) and (2.10), respectively.

3. Variational formulation and main result

We derive now the variational formulation of Problem $\mathcal{P}$. To this end we introduce the function spaces

$$V = \{v \in H^1(\Omega)\colon\ v = 0 \text{ on } \Gamma_1\}, \quad W = \{\psi \in H^1(\Omega)\colon\ \psi = 0 \text{ on } \Gamma_a\},$$

where, here and below, we write w for the trace γw of a function $w \in H^1(\Omega)$ on Γ. Since $\operatorname{meas}\Gamma_1 > 0$ and $\operatorname{meas}\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x \quad \forall\, u,v \in V, \quad (\varphi,\psi)_W = \int_\Omega \nabla\varphi \cdot \nabla\psi\,\mathrm{d}x \quad \forall\,\varphi,\ \psi \in W.$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \quad \forall\, v \in V, \quad \|\psi\|_W = \|\nabla\psi\|_{L^2(\Omega)^2} \quad \forall\,\psi \in W \tag{3.1}$$

are equivalent on V and W, with the usual norm $\|\cdot\|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist two positive constants $c_V > 0$ and $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \leqslant c_V \|v\|_V \quad \forall\, v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leqslant c_W \|\psi\|_W \quad \forall\, \psi \in W. \tag{3.2}$$

For a real Banach space $(X, \|\cdot\|_X)$ where $X = V \times W$, we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leqslant p \leqslant \infty$, $k = 1, 2, \ldots$ We also denote by $C([0,T];X)$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t \in [0,T]} \|x(t)\|_X$$

and we use the standard notations for the Lebesgue space $L^2(0,T;X)$ as well as the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|^2_X \,\mathrm{d}t$$

and the norm on the space $W^2(0,T;X)$ is given by the formula

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|^2_X \,\mathrm{d}t + \int_0^T \|\dot{u}(t)\|^2_X \,\mathrm{d}t.$$

Finally, we suppress the argument X when $X = \mathbb{R}$; thus, for example, we use the notation $W^2(0,T)$ for the space $W^2(0,T;\mathbb{R})$ and the notation $\|\cdot\|_{W^2(0,T)}$ for the norm $\|\cdot\|_{W^2(0,T;\mathbb{R})}$.

In the study of Problem $\mathcal{P}$ we assume that the viscosity coefficient satisfies

$$\theta \in W^{1,2}(0,T) \tag{3.3}$$

and the electric permittivity coefficient satisfies

$$\alpha \in L^\infty(\Omega) \text{ and there exists } \alpha^* > 0 \text{ such that } \ \alpha(\mathbf{x}) \geqslant \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{3.4}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\mathbf{x}) > 0 \quad \text{a.e. } \mathbf{x} \in \Omega, \tag{3.5}$$

$$e \in L^\infty(\Omega). \tag{3.6}$$

The forces, tractions, volume, and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)), \tag{3.7}$$

$$q_0 \in W^{1,2}(0,T;L^2(\Omega)), \quad q_2 \in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{3.8}$$

The electric conductivity coefficient and the friction bound function g satisfy the following properties:

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\mathbf{x}) \geqslant 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3, \tag{3.9}$$

$$g \in L^\infty(\Gamma_3) \quad \text{and} \quad g(\mathbf{x}) \geqslant 0 \quad \text{a.e. } \mathbf{x} \in \Gamma_3. \tag{3.10}$$

Finally, we assume that the electric potential of the foundation and the initial displacement are such that

$$\varphi_F \in W^{1,2}(0,T; L^2(\Gamma_3)). \tag{3.11}$$

The initial data are chosen such that

$$u_0 \in V \tag{3.12}$$

and, moreover,

$$a_\mu(u_0, v)_V + j(v) \geqslant (f(0), v)_V \quad \forall\, v \in V. \tag{3.13}$$

We define now the functional $j\colon\ [0,T] \to \mathbb{R}_+$ given by the formula

$$j(v) = \int_{\Gamma_3} g|v| \,\mathrm{d}a \quad \forall\, v \in V. \tag{3.14}$$

We also define the mappings $f\colon\ [0,T] \to V$ and $q\colon\ [0,T] \to W$ by

$$(f(t), v)_V = \int_\Omega f_0(t) v \,\mathrm{d}x + \int_{\Gamma_2} f_2(t) v \,\mathrm{d}a, \tag{3.15}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi \,\mathrm{d}x - \int_{\Gamma_b} q_2(t)\psi \,\mathrm{d}a + \int_{\Gamma_3} k\varphi_F(t)\psi \,\mathrm{d}a, \tag{3.16}$$
$$\forall\, v \in V, \psi \in W, t \in [0,T].$$

The definitions of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions (3.7)–(3.8) that the integrals above are well-defined and

$$f \in W^{1,2}(0,T; V), \tag{3.17}$$

$$q \in W^{1,2}(0,T; W). \tag{3.18}$$

Next, we define the bilinear forms $a_\mu\colon\ V \times V \to \mathbb{R}$, $a_e\colon\ V \times W \to \mathbb{R}$, $a_e^*\colon\ W \times V \to \mathbb{R}$, and $a_\alpha\colon\ W \times W \to \mathbb{R}$, by equalities

$$a_\mu(u, v) = \int_\Omega \mu \nabla u \cdot \nabla v \,\mathrm{d}x, \tag{3.19}$$

$$a_e(u, \varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi \,\mathrm{d}x = a_e^*(\varphi, u), \tag{3.20}$$

$$a_\alpha(\varphi, \psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi \,\mathrm{d}x + \int_{\Gamma_3} k\varphi\psi \,\mathrm{d}x, \tag{3.21}$$

for all $u, v \in V$, $\varphi, \psi \in W$. Assumptions (3.14)–(3.16) imply that the integrals above are well-defined and, using (3.1) and (3.2), it follows that the forms a_μ, a_e, and a_e^* are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W-elliptic, since

$$a_\alpha(\psi, \psi) \geqslant \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{3.22}$$

The variational formulation of Problem $\mathcal{P}$ is based on the following result.

Lemma 3.1. *If (u, φ) is a smooth solution to Problem $\mathcal{P}$, then $(u(t), \varphi(t)) \in X$ and*

$$\begin{aligned} &a_\mu(u(t), v - \dot{u}(t)) + \Big(\int_0^t \theta(t-s) u(s)\,\mathrm{d}s, v - \dot{u}(t)\Big)_V + a_e^*(\varphi(t), v - \dot{u}(t)) \\ &\quad + j(v) - j(\dot{u}(t)) \geqslant (f(t), v - \dot{u}(t))_V \quad \forall v \in V,\ t \in [0, T], \end{aligned} \tag{3.23}$$

$$a_\alpha(\varphi(t), \psi) - a_e(u(t), \psi) = (q(t), \psi)_W \quad \forall \psi \in W,\ t \in [0, T], \tag{3.24}$$

$$u(0) = u_0. \tag{3.25}$$

Proof. Let (u, φ) denote a smooth solution to Problem $\mathcal{P}$, we have $u(t) \in V$, $\dot{u}(t) \in V$ and $\varphi(t) \in W$ a.e. $t \in [0, T]$ and, from (2.28), (2.30), and (2.31), we obtain

$$\begin{aligned} &\int_\Omega \mu \nabla u(t) \cdot \nabla(v - \dot{u}(t))\,\mathrm{d}x + \Big(\int_0^t \theta(t-s) u(s)\,\mathrm{d}s, v - \dot{u}(t)\Big)_V \\ &\qquad + \int_\Omega e \nabla \varphi(t) \cdot \nabla(v - \dot{u}(t))\,\mathrm{d}x \\ &\quad = \int_\Omega f_0(t)(v - \dot{u}(t))\,\mathrm{d}x + \int_{\Gamma_2} f_2(t)(v - \dot{u}(t))\,\mathrm{d}a \\ &\qquad + \int_{\Gamma_3} \Big(\Big|\mu \partial_\nu u(t) + \int_0^t \theta(t-s) \partial_\nu u(s)\,\mathrm{d}s + e \partial_\nu \varphi(t)\Big|\Big)(v - \dot{u}(t))\,\mathrm{d}a, \\ &\qquad\qquad \forall v \in V \ \ t \in (0, T), \end{aligned}$$

and from (2.29) and (2.33)–(2.34) we obtain

$$\begin{aligned} &\int_\Omega \alpha \nabla \varphi(t) \cdot \nabla \psi\,\mathrm{d}x - \int_\Omega e \nabla u(t) \cdot \nabla \psi\,\mathrm{d}x = \int_\Omega q_0(t) \psi\,\mathrm{d}x - \int_{\Gamma_b} q_2(t) \psi\,\mathrm{d}a \\ &\qquad + \int_{\Gamma_3} k \varphi_F(t) \psi\,\mathrm{d}a \quad \forall \psi \in W \ \ t \in (0, T). \end{aligned} \tag{3.26}$$

Using (3.14) and (2.32), we obtain

$$
\begin{aligned}
(3.27)\quad & a_\mu(u(t), v-\dot u(t)) + \Big(\int_0^t \theta(t-s)u(s)\,\mathrm{d}s, v-\dot u(t)\Big)_V + a_e^*(\varphi(t), v-\dot u(t)) \\
& - \int_{\Gamma_3} \Big(\Big|\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi(t)\Big|\Big)(v-\dot u(t))\,\mathrm{d}a \\
& = (f(t), v-\dot u(t))_V \quad \forall\, v\in V,\ t\in[0,T].
\end{aligned}
$$

Keeping in mind (3.16) and (3.20)–(3.21), we find the second equality in Lemma 3.1, i.e.,

$$
(3.28)\qquad a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall\,\psi\in W,\ t\in[0,T].
$$

Using the frictional contact condition (2.32) and (3.14) on $\Gamma_3\times(0,T)$, we deduce that for all $t\in[0,T]$

$$
(3.29)\qquad j(\dot u(t)) = -\int_{\Gamma_3} \Big(\Big|\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi(t)\Big|\Big)\dot u(t)\,\mathrm{d}a,
$$

it is clear that

$$
(3.30)\quad j(v) \geqslant -\int_{\Gamma_3} \Big(\Big|\mu\partial_\nu u(t) + \int_0^t \theta(t-s)\partial_\nu u(s)\,\mathrm{d}s + e\partial_\nu\varphi(t)\Big|\Big)v\,\mathrm{d}a \quad \forall\, v\in V.
$$

The first inequality in Lemma 3.1 follows now from (3.27) and (3.29)–(3.30).

Now, Lemma 3.1 and condition (3.25) lead to the following variational problem:

Problem $\mathcal{PV}$. Find a displacement field $u\colon [0,T]\to V$ and an electric potential field $\varphi\colon [0,T]\to W$ such that

$$
\begin{aligned}
(3.31)\ & a_\mu(u(t), v-\dot u(t)) + \Big(\int_0^t \theta(t-s)u(s)\,\mathrm{d}s, v-\dot u(t)\Big)_V + a_e^*(\varphi(t), v-\dot u(t)) \\
& \qquad + j(v) - j(\dot u(t)) \geqslant (f(t), v-\dot u(t))_V \quad \forall\, v\in V,\ t\in[0,T], \\
(3.32)\ & \qquad a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) = (q(t),\psi)_W \quad \forall\,\psi\in W,\ t\in[0,T], \\
(3.33)\ & \qquad\qquad u(0) = u_0.
\end{aligned}
$$

□

Our main existence and uniqueness result, which we state now and prove in the next section, is the following:

Theorem 3.1. *Assume that* (3.3)–(3.18) *hold. Then the variational problem* $\mathcal{PV}$ *possesses a unique solution* (u, φ) *satisfying*

$$u \in W^{1,2}(0,T;V), \quad \varphi \in W^{1,2}(0,T;W). \tag{3.34}$$

We note that an element (u, φ) which solves Problem $\mathcal{PV}$ is a weak solution of the antiplane contact Problem $\mathcal{P}$. Theorem 3.1 thus states that the antiplane contact Problem $\mathcal{P}$ has a unique weak solution, provided that (3.3)–(3.18) hold.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 will be carried out in several steps. In the rest of this section we assume that (3.3)–(3.18) hold.

Step 1: In the first step of the proof we introduce the set

$$\mathcal{W} = \{\eta \in W^{1,2}(0,T;X) \text{ such that } \eta(0) = 0_X\} \tag{4.1}$$

and we recall the following existence and unigueness result.

Lemma 4.1. *For all* $\eta \in \mathcal{W}$ *there exists a unique element* $\eta \in W^{1,2}(0,T;X)$ *which satisfies the inequality and the data condition defined by the problem* $\mathcal{PV}_\eta^1$*:*

Problem $\mathcal{PV}_\eta^1$**.**

$$\begin{aligned} a(u_\eta(t), v - \dot{u_\eta}(t)) + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}_\eta(t)) \\ \geqslant (f(t), v - \dot{u_\eta}(t))_X \quad \forall\, v \in X,\ t \in [0,T], \end{aligned} \tag{4.2}$$

$$u_\eta(0) = u_0. \tag{4.3}$$

We use in the proof of Lemma 4.1 the following theorem:

Theorem 4.1 ([1], p. 117)**.** *Let* $(X, (\cdot,\cdot)_X)$ *be a real Hilbert space and let* $j\colon X \to (-\infty, \infty)$ *be a convex lower semicontinuous functional. Assume that* $j \neq \infty$*, that is,*

$$D(j) = \{v \in X \mid j(v) < \infty\} \neq \varphi.$$

Let $f \in W^{1,2}(0,T;X)$ *and* $u_0 \in X$ *be such that*

$$\sup_{v \in D(j)} = \{(f(0), v)_X - (u_0, v)_X - j(v)\} < \infty.$$

Then the variational problem $\mathcal{PV}$ possesses a unique solution (u, φ) satisfying $u(0) = u_0$ and

$$(u(t), v - \dot{u}(t))_X + j(v) - j(\dot{u}(t)) \leqslant (f(t), v - \dot{u}(t))_X \quad \forall\, v \in X \text{ a.e. } t \in (0, T).$$

P r o o f of Lemma 4.1. Let $a(\cdot,\cdot)$ be defined by

$$a(u, v) = (u, v)_a \quad \forall\, u, v \in X. \tag{4.4}$$

Notice that $(\cdot,\cdot)_a$ is an inner product on the space X and $\|\cdot\|_a$ is the associated norm which is equivalent to the norm $\|\cdot\|_X$ on the space X. Then $(X, (\cdot,\cdot)_a)$ is a real Hilbert space.

We define now the function $f_\eta\colon [0, T] \to X$ by the formula

$$(f_\eta(t), v)_a = (f(t), v)_X - (\eta(t), v)_X \quad \forall\, v \in V,\ t \in [0, T]. \tag{4.5}$$

It follows from (3.17) and (4.1) that

$$f_\eta(t) \in W^{1,2}(0, T; X). \tag{4.6}$$

Using now (4.5) at $t = 0$, we obtain

$$(f_\eta(0), v)_a = (f(0), v)_X - (\eta(0), v)_X \quad \forall\, v \in V,\ t \in [0, T]. \tag{4.7}$$

Moreover, rewriting (4.4) at $t = 0$, we have

$$a(u_0, v) = (u_0, v)_a \quad \forall\, v \in X. \tag{4.8}$$

On the other hand, taking into account (4.1), (4.7), and (4.8), we obtain the equality

$$(f_\eta(0), v)_a - (u_0, v)_a - j(v) = (f(0), v)_X - a(u_0, v) - j(v) \quad \forall\, v \in V. \tag{4.9}$$

From assumption (3.13), we find

$$\sup_{v \in D(j)} = \{(f_\eta(0), v)_a - (u_0, v)_a - j(v)\} < \infty. \tag{4.10}$$

Taking into account (3.13), (3.14), (4.6), and (4.10), we can use Theorem 4.1 on the space $(X, (\cdot,\cdot)_a)$, then there exists a unique element u_η satisfying

$$u_\eta \in W^{1,2}(0, T; X) \quad \text{such that } u_\eta = u_0 \tag{4.11}$$

and

$$\begin{aligned} &(u_\eta(t), v - \dot{u}_\eta(t))_a + j(v) - j(\dot{u}_\eta(t)) \\ &\qquad \geqslant (f_\eta(t), v - \dot{u}_\eta(t))_a \forall\, v \in X \quad \text{a.e. } t \in (0, T). \end{aligned} \tag{4.12}$$

Using (4.4) and (4.7), we obtain inequality (3.31) and initial data (3.33) defined in Problem $\mathcal{PV}$. This concludes the proof of Lemma 4.1. □

Step 2: In the second step, we use the displacement field u_η obtained in Lemma 4.1 to define the following variational problem for the electric potential field.

Problem $\mathcal{PV}^2_\eta$. Find an electrical potential $\varphi_\eta\colon [0,T]\to W$ such that

$$a_\alpha(\varphi_\eta(t),\psi)-a_e(u_\eta(t),\psi)=(q(t),\psi)_W\quad\forall\,\psi\in W,\ t\in[0,T]. \tag{4.13}$$

The proof of well-posedness of Problem $\mathcal{PV}^2_\eta$ follows.

Lemma 4.2. *There exists a unique solution $\varphi_\eta\in W^{1,2}(0,T;W)$ which satisfies* (4.13). *Moreover, if φ_{η_1} and φ_{η_2} are the solutions of* (4.13) *corresponding to $\eta_1,\eta_2\in C([0,T];V)$, then there exists $c>0$ such that*

$$\|\varphi_{\eta_1}(t)-\varphi_{\eta_2}(t)\|_W\leqslant c\|u_{\eta_1}(t)-u_{\eta_2}(t)\|_V\quad\forall\,t\in[0,T]. \tag{4.14}$$

Proof. Let $t\in[0,T]$. We use the properties of the bilinear form a_β and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t)\in W$ which solves (4.13) at any moment $t\in[0,T]$. Consider now $t_1,t_2\in[0,T]$; using (4.13), we get

$$a_\alpha(\varphi_\eta(t_1),\psi)-a_e(u_\eta(t_1),\psi)=(q(t_1),\psi)_W\quad\forall\,\psi\in W,\ t_1\in[0,T] \tag{4.15}$$

and

$$a_\alpha(\varphi_\eta(t_2),\psi)-a_e(u_\eta(t_2),\psi)=(q(t_2),\psi)_W\quad\forall\,\psi\in W,\ t_2\in[0,T]. \tag{4.16}$$

Using and (4.15), (4.16), and (3.22), we find that

$$\alpha^*\|\varphi(t_1)-\varphi(t_2)\|^2_W\leqslant(\|e\|_{L^\infty(\Omega)}\|u(t_1)-u(t_2)\|_V+\|q(t_1)-q(t_2)\|_W)\|\varphi(t_1)-\varphi(t_2)\|_W,$$

it follows from the previous inequality that

$$\|\varphi(t_1)-\varphi(t_2)\|_W\leqslant c(\|u(t_1)-u(t_2)\|_V+\|q(t_1)-q(t_2)\|_W). \tag{4.17}$$

Then, the regularity $u_\eta\in W^{1,2}(0,T;V)$ combined with (3.18) and (4.17) imply that $\varphi_\eta\in W^{1,2}(0,T;W)$ which concludes the proof. □

Now, for all $\eta\in\mathcal{W}$ we denote by u_η the solution of Problem $\mathcal{PV}^1_\eta$ obtained in Lemma 4.1 and by φ_η the solution of Problem $\mathcal{PV}^2_\eta$ obtained in Lemma 4.2.

Step 3: In the third step, we consider the operator $\Lambda\colon\mathcal{W}\to\mathcal{W}$.

We now use Riesz's representation theorem to define the element $\Lambda\eta(t)\in\mathcal{W}$ by the equality

$$(\Lambda\eta(t),w)_{\mathcal{W}}=\int_0^t\theta(t-s)u_\eta(s)\,\mathrm{d}s+a_e^*(\varphi_\eta(t),w)\qquad\forall\,\eta\in\mathcal{W},\ \forall\,w\in W,\ t\in[0,T]. \tag{4.18}$$

Clearly, for a given $\eta\in\mathcal{W}$ the function $t\mapsto\Lambda\eta(t)$ belongs to $\mathcal{W}$. In this step we show that the operator $\Lambda\colon\mathcal{W}\to\mathcal{W}$ has a unique fixed point.

Lemma 4.3. *There exists a unique $\eta^* \in \mathcal{W}$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let η_1, $\eta_2 \in \mathcal{W}$ and $t \in [0,T]$. In what follows we denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 4.1 and 4.2, for $i = 1,2$. Using (4.18) and (3.20), we obtain

$$\begin{aligned}(4.19)\qquad &\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2\\ &\leqslant C\Big(\int_0^t \|u_1(s) - u_2(s)\|_X^2 \,\mathrm{d}s + \|\varphi_1(t) - \varphi_2(t)\|_W^2\Big) \quad \forall\, t \in [0,T].\end{aligned}$$

The constant C represents a generic positive number which may depend on $\|\theta\|_{W^{1,2}(0,T)}$, T and e, and whose value may change from place to place.

Since $u_\eta \in W^{1,2}(0,T;V)$ and $\varphi_\eta \in W^{1,2}(0,T;W)$, we deduce from inequality (4.19) that $\Lambda\eta \in W^{1,2}(0,T;X)$. On the other hand, (4.14) and arguments similar to those used in the proof of (4.17) yield

$$(4.20)\qquad \|\varphi_1(t) - \varphi_2(t)\|_W \leqslant C\|u_1(t) - u_2(t)\|_V.$$

Using now (4.20) in (4.19), we get

$$\begin{aligned}(4.21)\qquad &\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^2\\ &\leqslant C\Big(\int_0^t \|u_1(s) - u_2(s)\|_X^2 \,\mathrm{d}s + \|u_1(t) - u_2(t)\|_V\Big), \text{ a.e. } t \in [0,T].\end{aligned}$$

Taking into account (3.31), we have the inequalities

$$\begin{aligned}&a(u_1(t), v - \dot{u}_1(t)) + (\eta_1(t), v - \dot{u}_1(t))_X + j(v) - j(\dot{u}_1(t))\\ &\qquad \geqslant (f(t), v - \dot{u}_1(t))_X \quad \forall\, v \in X,\ t \in [0,T]\end{aligned}$$

and

$$\begin{aligned}&a(u_2(t), v - \dot{u}_2(t)) + (\eta_2(t), v - \dot{u}_2(t))_X + j(v) - j(\dot{u}_2(t))\\ &\qquad \geqslant (f(t), v - \dot{u}_2(t))_X \quad \forall\, v \in X,\ t \in [0,T],\end{aligned}$$

for all $v \in X$, a.e. $s \in (0,T)$. We choose $v = \dot{u}_2(s)$ in the first inequality and $v = \dot{u}_1(s)$ in the second inequality, add the result to obtain

$$\frac{1}{2}\|u_1(s) - u_2(s)\|_X^2 \leqslant -(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s))_X \quad \text{a.e. } s \in (0,T).$$

Let $t \in [0,T]$. Integrating the previous inequality from 0 to t and using (3.33), we obtain

$$\begin{aligned}\frac{1}{2}\|u_1(t) - u_2(t)\|_X^2 \leqslant &-(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t))_X\\ &+ \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s))_X \,\mathrm{d}s.\end{aligned}$$

We deduce that

$$C\|u_1(t)-u_2(t)\|_X^2 \leqslant \|\eta_1(t)-\eta_2(t)\|_X\|u_1(t)-u_2(t)\|_X + \int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X\|u_1(s)-u_2(s)\|_X \,\mathrm{d}s.$$

Using Young's inequality, we get

$$\|u_1(t)-u_2(t)\|_X^2 \leqslant C\Big(\|\eta_1(t)-\eta_2(t)\|_X^2 + \int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s + \int_0^t \|u_1(s)-u_2(s)\|_X^2 \,\mathrm{d}s\Big). \tag{4.22}$$

On the other hand, as

$$\eta_1(t)-\eta_2(t) = \int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s,$$

we can obtain

$$\|\eta_1(t)-\eta_2(t)\|_X^2 \leqslant C\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s. \tag{4.23}$$

Using now (4.23) in (4.22), we have

$$\|u_1(t)-u_2(t)\|_X^2 \leqslant C\Big(\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s + \int_0^t \|u_1(s)-u_2(s)\|_X^2 \,\mathrm{d}s\Big).$$

Taking into account Gronwall's inequality, we deduce

$$\|u_1(t)-u_2(t)\|_X^2 \leqslant C\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s, \tag{4.24}$$

which yields

$$\int_0^t \|u_1(s)-u_2(s)\|_X^2 \,\mathrm{d}s \leqslant C\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s. \tag{4.25}$$

From (4.21), (4.24) and (4.25) we obtain

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_X^2 \leqslant C\int_0^t \|\dot\eta_1(s)-\dot\eta_2(s)\|_X^2 \,\mathrm{d}s.$$

Iterating the last inequality m times, we infer

$$\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\|_X^2 \leqslant C^m \int_0^t \int_0^{s_1} \dots \int_0^{s_{m-1}} \|\dot{\eta}_1(s_m) - \dot{\eta}_2(s_m)\|_X^2 \, \mathrm{d}s_m \dots \mathrm{d}s_1,$$

where Λ^m denotes the power of the operator Λ. The last inequality gives

$$\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\|_{W^{1,2}(0,T;X)}^2 \leqslant \frac{C^mT^m}{m!}\|\eta_1 - \eta_2\|_{W^{1,2}(0,T;X)}^2,$$

which implies that for m sufficietly large the power Λ^m of Λ is a contraction in the Banach space, since

$$\lim_{m\to\infty} \frac{C^mT^m}{m!} = 0,$$

it follows now from Banach's fixed-point theorem that there exists a unique element $\eta^* \in \mathcal{W}$ such that $\Lambda^m\eta^* = \eta^*$. Moreover, since

$$\Lambda^m(\Lambda\eta^*) = \Lambda(\Lambda^m\eta^*) = \Lambda\eta^*,$$

we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^m. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^* = \eta^*$, which shows that η^* is a fixed point, we conclude that $\Lambda\eta^* = \eta^*$. □

Step 4: In the fourth and last step of our demonstration, we have now all the ingredients to provide the proof of Theorem 3.1:

Existence. Let $\eta^* \in W^{1,2}(0,T;V)$ be the fixed point of the operator Λ, and let u_{η^*}, φ_{η^*} be the solutions of problems $\mathcal{PV}_\eta^1$ and $\mathcal{PV}_\eta^2$, respectively, for $\eta = \eta^*$. It follows from (4.18) that

$$(\eta^*(t), v)_V = \int_0^t \theta(t-s)u_\eta*(s)\,\mathrm{d}s + a_e^*(\varphi_\eta*(t), w) \quad \forall\, v \in V,\ t \in [0,T]$$

and, therefore, (3.31), (3.33), and (4.14) imply that $(u_{\eta^*}, \varphi_{\eta^*})$ is a solution of problem $\mathcal{PV}$. Regularity (3.34) of the solution follows from Lemmas 4.1 and 4.2.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed-point of the operator Λ. It can also be obtained by using arguments similar to those used in [26] and [15].

5. Conclusion

We presented a model for an antiplane contact problem for electro-viscoelastic materials with long-term memory. The problem was set as a variational inequality for the displacements and a variational equality for the electric potential. The existence of a unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities and a fixed-point theorem. This work opens the way to study further problems with other conditions for electrically conductive materials.

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Authors' addresses: *Ammar Derbazi*, Département de Matématiques, Faculté MI, Université Bachir el-Ibrahimi, 34 000 Bordj Bou Arréridj, Algérie, e-mail: `aderbazi@yahoo.fr`; *Mohamed Dalah* (corresponding author), Laboratoire Modélisation Mathématiques et Simulation (LMMS), Département de Matématiques, Faculté des Sciences, Université Mentouri Constantine, Route Ain El-Bey, Campus Hamana, 25 000 Constantine, Algérie, e-mail: `dalah-mohamed@hotmail.fr`; *Amar Megrous*, École Préparatoire de Constantine, EPSE-CSG, Département de Matématiques, Rue de Charco, 25 000 Constantine, Algérie, e-mail: `megrous.amar@yahoo.com`.