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CARTAN-EILENBERG PROJECTIVE, INJECTIVE AND FLAT COMPLEXES

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Abstract. Let R be an associative ring with identity and $\mathcal{F}$ a class of R-modules. In this article: we first give a detailed treatment of Cartan-Eilenberg $\mathcal{F}$ complexes and extend the basic properties of the class $\mathcal{F}$ to the class $\mathrm{CE}(\mathcal{F})$. Secondly, we study and give some equivalent characterizations of Cartan-Eilenberg projective, injective and flat complexes which are similar to projective, injective and flat modules, respectively. As applications, we characterize some classical rings in terms of these complexes, including coherent, Noetherian, von Neumann regular rings, QF rings, semisimple rings, hereditary rings and perfect rings.

Keywords: Cartan-Eilenberg projective complex; Cartan-Eilenberg injective complex; Cartan-Eilenberg flat complex

MSC 2010: 18G10, 18G25, 18G35

1. Introduction and preliminaries

In his thesis Verdier [16] introduced the notion of Cartan-Eilenberg injective complexes. Of course, there is an obvious dual notion, that of Cartan-Eilenberg projective complexes. Cartan-Eilenberg projective and injective complexes have origin in [4] to give the definitions of projective and injective resolutions of a complex of modules. Furthermore, Enochs [7] considered Cartan-Eilenberg flat complexes which are extensions of Cartan-Eilenberg projective complexes and showed that they are precisely the direct limits of finitely generated Cartan-Eilenberg projective complexes. Recently, Yang and Liang [19] gave some characterizations of Cartan-Eilenberg flat complexes and proved that a ring R is right coherent if and only if every complex of R-modules has a Cartan-Eilenberg flat preenvelope (for the rest of the article, we will use the abbreviation CE for Cartan-Eilenberg).

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Motivated by the above work, in this article we continue to study some basic properties of CE projective, injective and flat complexes. Also, we will give some equivalent characterizations of these CE complexes which are similar to projective, injective and flat modules, respectively. As applications, we will characterize some classical rings by CE projective, injective and flat complexes, e.g. coherent rings, Noetherian rings, von Neumann regular rings, semisimple rings, hereditary rings and perfect rings.

Throughout this article, R denotes an associative ring with identity. Unless stated otherwise, an R-module will be understood to be a left R-module, an R-complex (complex of R-modules) will be understood to be a left R-complex. We use R-Mod to denote the category of R-modules, and $C(R\text{-Mod})$ to denote the category of R-complexes.

An R-complex

$$C = \dots C_{n+2} \xrightarrow{\delta^C_{n+2}} C_{n+1} \xrightarrow{\delta^C_{n+1}} C_n \xrightarrow{\delta^C_{n}} C_{n-1} \xrightarrow{\delta^C_{n-1}} C_{n-2} \xrightarrow{\delta^C_{n-2}} \dots$$

will be denoted by (C, δ) or C. The nth cycle module is defined as $\operatorname{Ker}\delta^C_n$ and is denoted by $Z_n(C)$, nth boundary module is defined as $\operatorname{Im}\delta^C_{n+1}$ and is denoted by $B_n(C)$, and nth homology module is $H_n(C) = Z_n(C)/B_n(C)$. The complexes of cycles and boundaries, and the homology complex of C are denoted by $Z(C), B(C)$ and $H(C)$, respectively. For any $i \in \mathbb{Z}$, $\Sigma^i C$ denotes the complex with the degree-n term $(\Sigma^i C)_n = C_{n-i}$ whose boundary operators are $(-1)^i \delta^C_{n-i}$. We set $\Sigma C = \Sigma^1 C$.

Given an R-module M, we will denote by $\overline{M}$ the complex

$$\dots \longrightarrow 0 \longrightarrow M \xrightarrow{\text{id}} M \longrightarrow 0 \longrightarrow \dots$$

with M in the 0th and 1st position. Also, by $\underline{M}$ we mean the complex with M in the 0th place and 0 elsewhere.

For objects X and Y of $C(R\text{-Mod})$, we will denote by $\mathcal{H}om(X, Y)$ the complex of abelian groups with $\mathcal{H}om(X, Y)_n = \prod_{i\in\mathbb{Z}} \operatorname{Hom}(X_i, Y_{i+n})$ such that if $f = (f_i)_{i\in\mathbb{Z}} \in \mathcal{H}om(X, Y)_n$ then $\delta_n(f) = (\delta^Y_{i+n} f_i - (-1)^n f_{i-1}\delta^X_i)_{i\in\mathbb{Z}}$. A map f is called a chain map of degree n if $\delta_n(f) = 0$. A chain map of degree 0 is called a morphism. We will use $\operatorname{Hom}(X, Y)$ to denote the abelian group of morphisms from X to Y and Ext^i for $i \geqslant 1$ will denote the groups we get from the right derived functor of Hom. Let $\underline{\operatorname{Hom}}(X, Y) = Z(\mathcal{H}om(X, Y))$. It is not hard to see that $\underline{\operatorname{Hom}}(X, Y)$ is the complex of $\mathbb{Z}$-modules with the nth component $\underline{\operatorname{Hom}}(X, Y)_n = \operatorname{Hom}(X, \Sigma^{-n} Y) = \operatorname{Hom}(\Sigma^n X, Y)$.

If C is a complex of right R-modules and D is a complex of left R-modules, the tensor product of C and D is the complex of abelian groups $C \dot{\otimes} D$ with $(C \dot{\otimes} D)_n = \bigoplus_{i\in\mathbb{Z}} (C_i \otimes D_{n-i})$. Define $C \otimes D = C \dot{\otimes} D / B(C \dot{\otimes} D)$.

We recall some notion and results needed in the article.

Definition 1.1 ([20]). Let $\mathcal{F}$ be a class of R-modules. A complex A is called a CE $\mathcal{F}$ complex if A, $Z(A)$, $B(A)$ and $H(A)$ are all in $\mathcal{C}(\mathcal{F})$, where $\mathcal{C}(\mathcal{F})$ is the class of complexes with all components in $\mathcal{F}$. We let $\mathrm{CE}(\mathcal{F})$ denote the class of CE $\mathcal{F}$ complexes.

Definition 1.2 ([7]). A sequence of complexes $\ldots \to C_1 \to C_0 \to C_{-1} \to \ldots$ is said to be CE exact if
(1) $\ldots \to C_1 \to C_0 \to C_{-1} \to \ldots$,
(2) $\ldots \to Z(C_1) \to Z(C_0) \to Z(C_{-1}) \to \ldots$,
(3) $\ldots \to B(C_1) \to B(C_0) \to B(C_{-1}) \to \ldots$,
(4) $\ldots \to C_1/Z(C_1) \to C_0/Z(C_0) \to C_{-1}/Z(C_{-1}) \to \ldots$,
(5) $\ldots \to C_1/B(C_1) \to C_0/B(C_0) \to C_{-1}/B(C_{-1}) \to \ldots$,
(6) $\ldots \to H(C_1) \to H(C_0) \to H(C_{-1}) \to \ldots$ are all exact.

Remark 1.3. From [7], Lemma 5.2, we know that if (1) and (2), or (1) and (5) in the above definition are exact then all of (1)–(6) are exact.

Let X and Y be two R-complexes. In [7], Theorems 5.5 and 5.7, Enochs defined CE resolutions in terms of preenvelopes and precovers by CE injective and CE projective complexes. By [7], Proposition 6.3, we can compute derived functors of $\mathrm{Hom}(-,-)$ using either of the two CE resolutions. We denote these derived functors as $\overline{\mathrm{Ext}}^i(X,Y)$. If C is a complex of right R-modules and D is a complex of left R-modules, by [19], Lemma 2.4 and Theorem 2.6, we can compute left derived functors of -$\otimes$- using the CE flat resolution of either C or D. We denote these derived functors as $\overline{\mathrm{Tor}}_i(C,D)$.

2. CE projective, injective and flat complexes

In this section we give a detailed treatment of the CE $\mathcal{F}$ complex and extend the basic properties of the class $\mathcal{F}$ to the class $\mathrm{CE}(\mathcal{F})$. The main purpose is to give some equivalent characterizations of CE projective, injective and flat complexes which are similar to projective, injective and flat modules, respectively.

For any class $\mathcal{X}$ of R-modules, we say $\mathcal{X}$ is projectively resolving if $\mathcal{P}(\mathcal{R}) \subseteq \mathcal{X}$, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. We say $\mathcal{X}$ is injectively resolving if $\mathcal{I}(\mathcal{R}) \subseteq \mathcal{X}$ and if $X' \in \mathcal{X}$ then the conditions $X'' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent.

Lemma 2.1. *Let $\mathcal{F}$ be a class of R-modules. For an R-complex A, the following assertions hold:*

(1) *If $\mathcal{F}$ is projectively resolving, then the following conditions are equivalent:*
   (i) $A \in \mathrm{CE}(\mathcal{F})$.
   (ii) $B(A)$, $H(A) \in C(\mathcal{F})$.
   (iii) A, $A/B(A) \in C(\mathcal{F})$.

(2) *If $\mathcal{F}$ is injectively resolving, then the following conditions are equivalent:*
   (i) $A \in \mathrm{CE}(\mathcal{F})$.
   (ii) $B(A)$, $H(A) \in C(\mathcal{F})$.
   (iii) A, $Z(A) \in C(\mathcal{F})$.

Proof. (1): (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii). For each $i \in \mathbb{Z}$, consider the exact sequences of R-modules

$$0 \to H_i(A) \to A_i/B_i(A) \to B_{i-1}(A) \to 0, \tag{1.1}$$

$$0 \to B_i(A) \to A_i \to A_i/B_i(A) \to 0. \tag{1.2}$$

Since $\mathcal{F}$ is projectively resolving, $A/B(A) \in C(\mathcal{F})$ by (1.1), $A \in C(\mathcal{F})$ by (1.2).

(iii) $\Rightarrow$ (i). By (1.2) and (1.1) we have $B(A) \in C(\mathcal{F})$, $H(A) \in C(\mathcal{F})$. By the same argument we get that $Z(A) \in C(\mathcal{F})$ since $0 \to B_i(A) \to Z_i(A) \to H_i(A) \to 0$ is exact for all $i \in \mathbb{Z}$. Thus $A \in \mathrm{CE}(\mathcal{F})$ by Definition 1.1.

Dually, we can prove (2). □

Proposition 2.2. *Let $\mathcal{F}$ be a class of R-modules and let $0 \to A \to B \to C \to 0$ be a short* CE *exact sequence in $C(R\text{-Mod})$. Then the following results hold:*

(1) *If $\mathcal{F}$ is projectively resolving and $C \in \mathrm{CE}(\mathcal{F})$, then $A \in \mathrm{CE}(\mathcal{F})$ if and only if $B \in \mathrm{CE}(\mathcal{F})$.*

(2) *If $\mathcal{F}$ is injectively resolving and $A \in \mathrm{CE}(\mathcal{F})$, then $B \in \mathrm{CE}(\mathcal{F})$ if and only if $C \in \mathrm{CE}(\mathcal{F})$.*

Proof. We prove part (1); the proof of part (2) is dual.

By the hypothesis, for each $i \in \mathbb{Z}$ we have the exact sequences

$$0 \to A_i \to B_i \to C_i \to 0, \tag{1.3}$$

$$0 \to A_i/B_i(A) \to B_i/B_i(B) \to C_i/B_i(C) \to 0. \tag{1.4}$$

By Lemma 2.1, we have C_i, $C_i/B_i(C) \in \mathcal{F}$. Since $\mathcal{F}$ is projectively resolving, $A_i \in \mathcal{F}$ if and only if $B_i \in \mathcal{F}$ by (1.3), $A_i/B_i(A) \in \mathcal{F}$ if and only if $B_i/B_i(B) \in \mathcal{F}$ by (1.4). Therefore, $A \in \mathrm{CE}(\mathcal{F})$ if and only if $B \in \mathrm{CE}(\mathcal{F})$ by Lemma 2.1 again. □

Proposition 2.3. *Let $\mathcal{F}$ be a class of R-modules that is closed under direct limits. If $\mathcal{F}$ is closed under kernels of epimorphisms or cokernels of monomorphisms, then* $\mathrm{CE}(\mathcal{F})$ *is closed under direct limits.*

Proof. Suppose $\{C^i, (f_{ji})\colon i \in I\}$ is a direct system of CE $\mathcal{F}$ complexes. For each $n \in \mathbb{Z}$, we have

$$Z_n(\varinjlim C^i) \cong \varinjlim Z_n(C^i), \quad H_n(\varinjlim C^i) \cong \varinjlim H_n(C^i).$$

Since $\mathcal{F}$ is closed under direct limits, we have

$$\varinjlim C^i_n,\ Z_n(\varinjlim C^i), \quad H_n(\varinjlim C^i) \in \mathcal{F}.$$

If $\mathcal{F}$ is closed under kernels of epimorphisms, then we have

$$B_n(\varinjlim C^i) \in \mathcal{F}$$

due to the exact sequence

$$0 \to B_n(\varinjlim C^i) \to Z_n(\varinjlim C^i) \to H_n(\varinjlim C^i) \to 0.$$

If $\mathcal{F}$ is closed under cokernels of monomorphisms, then the same result holds by the exact sequence

$$0 \to Z_{n+1}(\varinjlim C^i) \to \varinjlim C^i_{n+1} \to B_n(\varinjlim C^i) \to 0.$$

Thus $\mathrm{CE}(\mathcal{F})$ is closed under direct limits by Definition 1.1. $\square$

Proposition 2.4. *Let $\mathcal{F}$ be a class of R-modules. Then the following statements hold:*

(1) *If $\mathcal{F}$ is closed under arbitrary direct summands, then* $\mathrm{CE}(\mathcal{F})$ *is closed under arbitrary direct summands.*

(2) *If $\mathcal{F}$ is closed under arbitrary direct sums (direct products), then* $\mathrm{CE}(\mathcal{F})$ *is closed under arbitrary direct sums (direct products).*

Proof. (1) Assume that Y is a direct summand of $X \in \mathrm{CE}(\mathcal{F})$. We wish to show that $Y \in \mathrm{CE}(\mathcal{F})$. Write $X = Y \oplus U$ for some complex U. For any $i \in \mathbb{Z}$, we have $X_i = (Y \oplus U)_i = Y_i \oplus U_i$, $Z_i(X) = Z_i(Y \oplus U) = Z_i(Y) \oplus Z_i(U)$, $B_i(X) = B_i(Y \oplus U) = B_i(Y) \oplus B_i(U)$. Hence $H_i(X) = H_i(Y \oplus U) \cong H_i(Y) \oplus H_i(U)$. Since $X \in \mathrm{CE}(\mathcal{F})$, we have X_i, $Z_i(X)$, $B_i(X)$, $H_i(X) \in \mathcal{F}$. If $\mathcal{F}$ is closed under arbitrary direct summands, then Y_i, $Z_i(Y)$, $B_i(Y)$, $H_i(Y) \in \mathcal{F}$, so $Y \in \mathrm{CE}(\mathcal{F})$ by Definition 1.1.

(2) Suppose $\{C^i\}_{i\in I}$ is a family of CE$\,\mathcal{F}$ complexes in $C(R\text{-Mod})$. We will show that $\bigoplus\limits_{i\in I} C^i$ is CE$\,\mathcal{F}$ complex. For any $n \in \mathbb{Z}$, we have $\Big(\bigoplus\limits_{i\in I} C^i\Big)_n = \bigoplus\limits_{i\in I}(C^i)_n$, $Z_n\Big(\bigoplus\limits_{i\in I} C^i\Big) = \bigoplus\limits_{i\in I} Z_n(C^i)$, $B_n\Big(\bigoplus\limits_{i\in I} C^i\Big) = \bigoplus\limits_{i\in I} B_n(C^i)$. Hence $H_n\Big(\bigoplus\limits_{i\in I} C^i\Big) \cong \bigoplus\limits_{i\in I} H_n(C^i)$. By the hypothesis, $\mathcal{F}$ is closed under arbitrary direct sums, so $\Big(\bigoplus\limits_{i\in I} C^i\Big)_n$, $Z_n\Big(\bigoplus\limits_{i\in I} C^i\Big)$, $B_n\Big(\bigoplus\limits_{i\in I} C^i\Big)$, $H_n\Big(\bigoplus\limits_{i\in I} C^i\Big) \in \mathcal{F}$. Thus $\bigoplus\limits_{i\in I} C^i$ is CE$\,\mathcal{F}$ complex by Definition 1.1. For the case of closing under direct products, one can proceed similarly. $\square$

According to [13], a short exact sequence $0 \to A \to B \to C \to 0$ in $C(R\text{-Mod})$ is called pure if the sequence $0 \to F \otimes A \to F \otimes B$ is exact for any complex F of right R-modules. A subcomplex $S \subset C$ is pure if $0 \to S \to C \to S/C \to 0$ is a pure exact sequence.

Proposition 2.5. *Let $\mathcal{F}$ be a class of R-modules which is injectively resolving. If $\mathcal{F}$ is closed under pure submodules, then* $\mathrm{CE}(\mathcal{F})$ *is closed under pure subcomplexes and pure quotient complexes.*

Proof. Suppose $0 \to A \to B \to C \to 0$ is pure exact in $C(R\text{-Mod})$ with $B \in \mathrm{CE}(\mathcal{F})$. It was proved that $0 \to Z_i(A) \to Z_i(B) \to Z_i(C) \to 0$ and $0 \to A_i \to B_i \to C_i \to 0$ are pure exact for all $i \in \mathbb{Z}$ by [17], Lemma 2.6 and 3.7, which implies that $0 \to A \to B \to C \to 0$ is CE exact by Remark 1.3. Since $B \in \mathrm{CE}(\mathcal{F})$, one gets $Z_i(B), B_i \in \mathcal{F}$, and so $Z_i(A), A_i \in \mathcal{F}$, which implies that $A \in \mathrm{CE}(\mathcal{F})$ by Lemma 2.1 (2). Since $\mathcal{F}$ is injectively resolving, we get that $C \in \mathrm{CE}(\mathcal{F})$ by Proposition 2.2. $\square$

By [11], Remark 1, the class of injective R-modules is closed under pure submodules. Here we have the following result.

Corollary 2.6. $\mathrm{CE}(\mathcal{I}(\mathcal{R}))$ *is closed under pure subcomplexes and pure quotient complexes.*

Next, we give some new characterizations of CE projective, injective and flat complexes.

Proposition 2.7. *For an R-complex P, the following conditions are equivalent:*
(1) $P \in \mathrm{CE}(\mathcal{P}(\mathcal{R}))$.
(2) P, $P/B(P) \in C(\mathcal{P}(\mathcal{R}))$.
(3) $B(P)$, $H(P) \in C(\mathcal{P}(\mathcal{R}))$.
(4) $P = \Big(\bigoplus\limits_{i\in\mathbb{Z}} \Sigma^i \overline{B_i(P)}\Big) \oplus \Big(\bigoplus\limits_{i\in\mathbb{Z}} \Sigma^i \underline{H_i(P)}\Big)$ *with* $B_i(P)$, $H_i(P) \in \mathcal{P}(\mathcal{R})$.

(5) $\overline{\operatorname{Ext}}^1(P,-)=0$, *that is every short* CE *exact sequence* $0\to A\to B\to P\to 0$ *is split.*

(6) $\operatorname{Hom}(P,-)$ *is exact for any short* CE *exact sequence in* $C(R\text{-Mod})$.

(7) $\underline{\operatorname{Hom}}(P,-)$ *is exact for any short* CE *exact sequence in* $C(R\text{-Mod})$.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Lemma 2.1 (1).

(3) $\Rightarrow$ (4). For each $i\in\mathbb{Z}$, we have the exact sequences of R-modules $0\to B_i(P)\to Z_i(P)\to H_i(P)\to 0$ and $0\to Z_i(P)\to P_i\to B_{i-1}(P)\to 0$. Since $B(P)$, $H(P)\in C(\mathcal{P}(\mathcal{R}))$, each sequence splits, which allows us to write $P_i=B_i(P)\oplus H_i(P)\oplus B_{i-1}(P)$. Therefore, $P=\ldots\to P_{i+1}\to P_i\to P_{i-1}\to\ldots=\ldots\to B_{i+1}(P)\oplus H_{i+1}(P)\oplus B_i(P)\to B_i(P)\oplus H_i(P)\oplus B_{i-1}(P)\to B_{i-1}(P)\oplus H_{i-1}(P)\oplus B_{i-2}(P)\to\ldots=\Big(\bigoplus\limits_{i\in\mathbb{Z}}\Sigma^i\overline{B_i(P)}\Big)\oplus\Big(\bigoplus\limits_{i\in\mathbb{Z}}\Sigma^i\underline{H_i(P)}\Big)$.

(4) $\Rightarrow$ (5). Let C be an R-complex. By [7], Lemmas 9.1 and 9.2 and (4), $\overline{\operatorname{Ext}}^1(\Sigma^i\overline{B_i(P)},C)\cong\operatorname{Ext}^1(B_i(P),C_{i+1})=0$ and $\overline{\operatorname{Ext}}^1(\Sigma^i\underline{H_i(P)},C)\cong\operatorname{Ext}^1(H_i(P),Z_i(C))=0$, thus $\overline{\operatorname{Ext}}^1(P,C)=0$. Consider a CE exact sequence $0\to A\to B\to P\to 0$; since $\overline{\operatorname{Ext}}^1(P,A)=0$, it is split.

(5) $\Rightarrow$ (6) is trivial.

(6) $\Rightarrow$ (1). Let C be an R-complex. By [7], Theorem 5.6, there exists a CE exact sequence $0\to C\to I\to N\to 0$ with $I\in\mathrm{CE}(\mathcal{I}(\mathcal{R}))$. Then $0\to\operatorname{Hom}(P,C)\to\operatorname{Hom}(P,I)\to\operatorname{Hom}(P,N)\to\overline{\operatorname{Ext}}^1(P,C)\to\overline{\operatorname{Ext}}^1(P,I)=0$ is exact by [7], Theorem 9.4. On the other hand, the exactness of $0\to\operatorname{Hom}(P,C)\to\operatorname{Hom}(P,I)\to\operatorname{Hom}(P,N)\to 0$ yields $\overline{\operatorname{Ext}}^1(P,C)=0$ by Five Lemma. Therefore $P\in\mathrm{CE}(\mathcal{P}(\mathcal{R}))$ by [7], Theorem 9.4.

(6) $\Rightarrow$ (7). Assume that (6) holds, then $P\in\mathrm{CE}(\mathcal{P}(\mathcal{R}))$ by the above. For each $n\in\mathbb{Z}$, $\Sigma^nP\in\mathrm{CE}(\mathcal{P}(\mathcal{R}))$. Hence $\underline{\operatorname{Hom}}(P,-)$ is exact for any short CE exact sequence of R-complexes.

(7) $\Rightarrow$ (6) is trivial. □

Dual argument to the above gives the following results concerning the CE injective complexes.

Proposition 2.8. *For an R-complex I, the following conditions are equivalent:*

(1) $I\in\mathrm{CE}(\mathcal{I}(\mathcal{R}))$.

(2) I, $Z(I)\in C(\mathcal{I}(\mathcal{R}))$.

(3) $B(I)$, $H(I)\in C(\mathcal{I}(\mathcal{R}))$.

(4) $I=\Big(\bigoplus\limits_{i\in\mathbb{Z}}\Sigma^i\overline{B_i(I)}\Big)\oplus\Big(\bigoplus\limits_{i\in\mathbb{Z}}\Sigma^i\underline{H_i(I)}\Big)$ *with* $B_i(I)$, $H_i(I)\in\mathcal{I}(\mathcal{R})$.

(5) $\overline{\operatorname{Ext}}^1(-,I)=0$, *that is every short* CE *exact sequence* $0\to I\to B\to C\to 0$ *is split.*

(6) $\mathrm{Hom}(-, I)$ *is exact for any short* CE *exact sequence of R-complexes.*
(7) $\underline{\mathrm{Hom}}(-, I)$ *is exact for any short* CE *exact sequence of R-complexes.*

Proposition 2.9. *For an R-complex F, the following conditions are equivalent:*
(1) $F \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$.
(2) F, $F/B(F) \in C(\mathcal{F}(\mathcal{R}))$.
(3) $B(F)$, $H(F) \in C(\mathcal{F}(\mathcal{R}))$.
(4) $\overline{\mathrm{Tor}}_i(-, F) = 0$ *for all* $i \geqslant 1$.
(5) $- \bigotimes F$ *is exact for any short* CE *exact sequence of right R-complexes.*
(6) *Every short* CE *exact sequence* $0 \to A \to B \to F \to 0$ *is pure.*
(7) *There exists a pure exact sequence* $0 \to A \to P \to F \to 0$ *such that P is* CE *projective* (CE *flat*).

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Lemma 2.1 (1).
(1) $\Rightarrow$ (4) holds by [19], Remark 2.7.
(4) $\Rightarrow$ (5) is obvious.
(5) $\Rightarrow$ (1) follows from [19], Lemma 2.5.
(1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are immediate from [19], Proposition 2.11. $\square$

Given a complex C we let C^+ stand for the character complex $\underline{\mathrm{Hom}}(C, \overline{\mathbb{Q}/\mathbb{Z}})$ of C. By [10], Proposition 2.1 (3), we have $C^+ = \ldots \to \mathrm{Hom}(C_{-1}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(C_0, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(C_1, \mathbb{Q}/\mathbb{Z}) \to \ldots$ with ith component $C_i^+ = \mathrm{Hom}(C_{-i}, \mathbb{Q}/\mathbb{Z})$.

Lemma 2.10 ([19], Lemma 2.2)**.** *Let C be an R-complex. The following conditions hold for any $i \in \mathbb{Z}$:*
(1) $Z_i(C^+) \cong \mathrm{Hom}_{\mathbb{Z}}(C_{-i}/B_{-i}(C), \mathbb{Q}/\mathbb{Z}) \cong (C_{-i}/B_{-i}(C))^+$.
(2) $B_i(C^+) \cong \mathrm{Hom}_{\mathbb{Z}}(B_{-i-1}(C), \mathbb{Q}/\mathbb{Z}) \cong (B_{-i-1}(C))^+$.
(3) $H_i(C^+) \cong (H_{-i}(C))^+$.

Lemma 2.11. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $C(R\text{-}\mathrm{Mod})$. Then this sequence is* CE *exact if and only if $0 \to C^+ \to B^+ \to A^+ \to 0$ is* CE *exact.*

Proof. ($\Rightarrow$). We only need to prove the "only if" part; one can prove the "if" part similarly. For each $i \in \mathbb{Z}$, consider the exact sequences $0 \to A_{-i} \to B_{-i} \to C_{-i} \to 0$ and $0 \to A_{-i}/B_{-i}(A) \to B_{-i}/B_{-i}(B) \to C_{-i}/B_{-i}(C) \to 0$. Since $\mathrm{Hom}(-, \mathbb{Q}/\mathbb{Z})$ is an exact functor, $0 \to (C_{-i})^+ \to (B_{-i})^+ \to (A_{-i})^+ \to 0$ and $0 \to (C_{-i}/B_{-i}(C))^+ \to (B_{-i}/B_{-i}(B))^+ \to (A_{-i}/B_{-i}(A))^+ \to 0$ are exact. This implies $0 \to (C^+)_i \to (B^+)_i \to (A^+)_i \to 0$ and $0 \to Z_i(C^+) \to Z_i(B^+) \to Z_i(A^+) \to 0$ are exact by Lemma 2.10. Thus the desired result follows from Remark 1.3. $\square$

In this section we characterize some classical rings in terms of CE projective, injective and flat complexes.

Definition 3.1. A complex X is said to have CE injective dimension less than or equal to n, denoted by $\mathrm{CE\text{-}id}(X) \leqslant n$, if there is a CE exact sequence $0 \to X \to I^0 \to I^1 \to \ldots \to I^{n-1} \to I^n \to 0$ with each $I^i \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$. If n is the least, then we set $\mathrm{CE\text{-}id}(X) = n$ and if there is no such n, we set $\mathrm{CE\text{-}id}(X) = \infty$. CE projective dimension and CE flat dimension can be defined dually.

Using the definition and the proof of [20], Proposition 2.15, we have the following result.

Remark 3.2. For any R-complex X, the following conditions hold:
(1) $\mathrm{CE\text{-}id}(X) = \sup\{\mathrm{id}(B_i(X)), \mathrm{id}(H_i(X))\colon i \in \mathbb{Z}\}$.
(2) $\mathrm{CE\text{-}pd}(X) = \sup\{\mathrm{pd}(B_i(X)), \mathrm{pd}(H_i(X))\colon i \in \mathbb{Z}\}$.
(3) $\mathrm{CE\text{-}fd}(X) = \sup\{\mathrm{fd}(B_i(X)), \mathrm{fd}(H_i(X))\colon i \in \mathbb{Z}\}$.

Let $\mathcal{L}$ be a class of objects in an abelian category $\mathcal{C}$. Let M be an object of $\mathcal{C}$. Recall from [8] that a morphism $f\colon L \to M$ is an $\mathcal{L}$-precover of M if $L \in \mathcal{L}$ and $\mathrm{Hom}(L', L) \to \mathrm{Hom}(L', M)$ is exact for all $L' \in \mathcal{L}$. If, moreover, any $g\colon L \to L$ such that $fg = f$ is an automorphism of L then $f\colon L \to M$ is called an $\mathcal{L}$-cover of M. We say a class $\mathcal{L}$ of objective of $\mathcal{C}$ is a (pre)covering if every objective of $\mathcal{C}$ has an $\mathcal{L}$-(pre)covering. Dually, we have the concepts of an $\mathcal{L}$-(pre)envelope and an $\mathcal{L}$-(pre)enveloping class.

Coherent rings have been characterized in various ways. One of the deepest results is the one due to Chase [5] which claims that the ring R is right coherent if and only if products of flat R-modules are again flat if and only if products of copies of R are flat R-modules. Now we are in the position to give one of our main results.

Theorem 3.3. *Let X be a right R-complex. The following conditions are equivalent for a ring R:*
(1) *R is right coherent.*
(2) *$X \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$ if and only if $X^+ \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$.*
(3) *$\mathrm{CE\text{-}fd}(X^+) \leqslant \mathrm{CE\text{-}id}(X)$.*
(4) *$X \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$ if and only if $X^{++} \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$.*
(5) *The class $\mathrm{CE}(\mathcal{F}(\mathcal{R}))$ is preenveloping.*
(6) *Every direct product of CE flat complexes is CE flat.*

P r o o f. (1) $\Rightarrow$ (2) follows from [19], Corollary 2.3.

(2) $\Rightarrow$ (3). If $\mathrm{CE\text{-}id}(X)=\infty$, then (3) holds clearly. If $\mathrm{CE\text{-}id}(X)=n<\infty$, then there exists a CE exact sequence

$$0 \to X \to I^0 \to I^1 \to \ldots \to I^{n-1} \to I^n \to 0$$

with each $I^i \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$, which gives rise to the CE exactness of

$$0 \to (I^n)^+ \to (I^{n-1})^+ \to \ldots \to (I^1)^+ \to (I^0)^+ \to X^+ \to 0$$

by Lemma 2.11, where $(I^0)^+,\ldots,(I^n)^+$ are CE flat complexes, and therefore $\mathrm{CE\text{-}fd}(X^+) \leqslant n$.

(3) $\Rightarrow$ (4). Suppose that X is a CE flat complex. Then X^+ is CE injective by [19], Corollary 2.3, hence $\mathrm{CE\text{-}id}(X^+)=0$, so $\mathrm{CE\text{-}fd}(X^{++}) \leqslant \mathrm{CE\text{-}id}(X^+)=0$ by (3). Thus $X^{++} \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$. Conversely, if $X^{++} \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$. According to the pure exact sequence $0 \to X \to X^{++}$, we can get X is a CE flat complex by [19], Lemma 2.9.

(4) $\Rightarrow$ (1). Let M be a flat R-module. Then $\underline{M}$ is CE flat, and so $\underline{M^{++}} \cong (\underline{M})^{++}$ is a CE flat complex by (4), which yields that M^{++} is a flat module and R is right coherent by [6], Theorem 1.

(1) $\Leftrightarrow$ (5) holds by [19], Theorem 2.10.

(1) $\Rightarrow$ (6). Assume that $\{F_\alpha\}_{\alpha\in\Lambda}$ is a family of CE flat complexes. Theorem 3.2.24 in [9] yields that every product of flat modules is flat, hence $\prod\limits_{\alpha\in\Lambda} F_\alpha$ is a CE flat complex by Proposition 2.4.

(6) $\Rightarrow$ (1). Let $\{N_\alpha\}_{\alpha\in\Lambda}$ be a family of flat R-modules. Then $\{\underline{N_\alpha}\}_{\alpha\in\Lambda}$ is a family of CE flat complexes. By (6), we have $\underline{\prod\limits_{\alpha\in\Lambda} N_\alpha} \cong \prod\limits_{\alpha\in\Lambda} \underline{N_\alpha}$ is CE flat. This implies that $\prod\limits_{\alpha\in\Lambda} N_\alpha$ is a flat module, hence R is right coherent by [9], Theorem 3.2.24. $\square$

Next we give some characterizations of left Noetherian rings.

Theorem 3.4. *Let X be an R-complex. The following conditions are equivalent for a ring R:*

(1) *R is left Noetherian.*

(2) $\mathrm{CE\text{-}fd}(X^+)=\mathrm{CE\text{-}id}(X)$.

(3) *$X \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$ if and only if $X^+ \in \mathrm{CE}(\mathcal{F}(\mathcal{R}))$.*

(4) *$X \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$ if and only if $X^{++} \in \mathrm{CE}(\mathcal{I}(\mathcal{R}))$.*

(5) *The class $\mathrm{CE}(\mathcal{I}(\mathcal{R}))$ is precovering.*

(6) *The class $\mathrm{CE}(\mathcal{I}(\mathcal{R}))$ is covering.*

(7) *Every direct sum of CE injective complexes is CE injective.*

P r o o f. (1) $\Rightarrow$ (2). By [12], Theorem 2.2, we have $\operatorname{id}(B_i(X)) = \operatorname{fd}((B_i(X))^+) = \operatorname{fd}(B_{-i-1}(X^+))$, $\operatorname{id}(H_i(X)) = \operatorname{fd}((H_i(X))^+) = \operatorname{fd}(H_{-i}(X^+))$ for all $i \in \mathbb{Z}$. From Remark 3.2 we know that $\operatorname{CE-fd}(X^+) = \operatorname{CE-id}(X)$.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). By analogy with the proof of (4) $\Rightarrow$ (1) in Theorem 3.3 by using [6], Theorem 2.

(1) $\Rightarrow$ (5). By [9], Theorem 5.4.1, every module has an injective precover, so there exist injective precovers $g_i\colon E_i \to Z_i(X)$ of $Z_i(X)$ and $f_i\colon I_i \to X_i$ of X_i for all $i \in \mathbb{Z}$. Thus there is a morphism of complexes

$$\begin{tikzcd}
G = \ldots \arrow[r] & E_{i+1} \oplus I_{i+2} \oplus I_{i+1} \arrow[r, "\delta_{i+1}^G"] \arrow[d, "h_{i+1}"] & E_i \oplus I_{i+1} \oplus I_i \arrow[r, "\delta_i^G"] \arrow[d, "h_i"] & E_{i-1} \oplus I_i \oplus I_{i-1} \arrow[r, "\delta_{i-1}^G"] \arrow[d, "h_{i-1}"] & \cdots \\
X = \ldots \arrow[r] & X_{i+1} \arrow[r, "\delta_{i+1}^X"] & X_i \arrow[r, "\delta_i^X"] & X_{i-1} \arrow[r, "\delta_{i-1}^X"] & \cdots
\end{tikzcd}$$

where $\delta_i^G\colon E_i \oplus I_{i+1} \oplus I_i \to E_{i-1} \oplus I_i \oplus I_{i-1}$ via $\delta_i^G(x,y,z) = (0,z,0)$, for all $(x,y,z) \in E_i \oplus I_{i+1} \oplus I_i$, and $h_i\colon G_i \to X_i$ via $h_i(x,y,z) = g_i(x) + \delta_{i+1}^X f_{i+1}(y) + f_i(z)$, for all $(x,y,z) \in E_i \oplus I_{i+1} \oplus I_i$. It is easy to check that G is a CE injective complex. We have to prove that $h\colon G \to X$ is a CE injective precover. Since every CE injective complex can be written as $\Bigl(\bigoplus\limits_{i\in\mathbb{Z}} \Sigma^i \overline{D_i}\Bigr) \oplus \Bigl(\bigoplus\limits_{i\in\mathbb{Z}} \Sigma^i \underline{E_i}\Bigr)$ by Proposition 2.8, where D_i, E_i are injective modules, we only need to prove $\operatorname{Hom}(D, G_{i+1}) \to \operatorname{Hom}(D, X_{i+1}) \to 0$ and $\operatorname{Hom}(D, Z_i(G)) \to \operatorname{Hom}(D, Z_i(X)) \to 0$ are exact for any $D \in \mathcal{I}(R)$ and $i \in \mathbb{Z}$ by [7], Proposition 2.1. Suppose that $\alpha \in \operatorname{Hom}(D, X_{i+1})$, then there exists $\beta \in \operatorname{Hom}(D, I_{i+1})$ such that $\alpha = f_{i+1}\beta$ since $f_{i+1}\colon I_{i+1} \to X_{i+1}$ is an injective precover of X_{i+1}. Define $\widetilde{\beta} = (0,0,\beta)\colon D \to E_{i+1} \oplus I_{i+2} \oplus I_{i+1}$, then it is easy to see that $\alpha = h_{i+1}\widetilde{\beta}$. That is, the diagram

$$\begin{tikzcd}
 & D \arrow[dl, "\widetilde{\beta}"'] \arrow[d, "\alpha"] \\
E_{i+1} \oplus I_{i+2} \oplus I_{i+1} \arrow[r, "h_{i+1}"] & X_{i+1}
\end{tikzcd}$$

commutes. Hence $\operatorname{Hom}(D, G_{i+1}) \to \operatorname{Hom}(D, X_{i+1}) \to 0$ is exact. Define $\varphi_i\colon Z_i(G) = E_i \oplus I_{i+1} \to Z_i(X)$ via $\varphi_i(x,y) = g_i(x) + \delta_{i+1}^X f_{i+1}(y)$, for all $(x,y) \in E_i \oplus I_{i+1}$. Suppose that $\gamma \in \operatorname{Hom}(D, Z_i(X))$, then there exists $\eta \in \operatorname{Hom}(D, E_i)$ such that $\gamma = g_i\eta$ since $g_i\colon E_i \to Z_i(X)$ is an injective precover of $Z_i(X)$. Define

$\widetilde{\eta} = (\eta, 0)\colon D \to E_i \oplus I_{i+1}$. It is easy to see that $\gamma = \varphi_i \widetilde{\eta}$. That is, the diagram

$$\begin{tikzcd} & D \arrow[dl, "\widetilde{\eta}"'] \arrow[d, "\gamma"] \\ E_i \oplus I_{i+1} \arrow[r, "\varphi_i"] & Z_i(X) \end{tikzcd}$$

commutes. Hence $\mathrm{Hom}(D, Z_i(G)) \to \mathrm{Hom}(D, Z_i(X)) \to 0$ is exact.

(5) $\Rightarrow$ (1). Suppose M is an R-module. Then there exists a CE injective precover $f\colon I \to \overline{M}$ of $\overline{M}$ in $C(R\text{-Mod})$. Let E be an injective module. Then $\mathrm{Hom}(\Sigma^{-1}\overline{E}, I) \to \mathrm{Hom}(\Sigma^{-1}\overline{E}, \overline{M}) \to 0$ is exact. According to [7], Proposition 2.1, we get $\mathrm{Hom}(E, I_0) \to \mathrm{Hom}(E, M) \to 0$ is exact, which implies $f_0\colon I_0 \to M$ is an injective precover of M in R-Mod. Thus R is a left Noetherian ring by [9], Theorem 5.4.1.

(5) $\Rightarrow$ (6). If $\mathrm{CE}(\mathcal{I}(\mathcal{R}))$ is precovering, then R is left Noetherian by the above and so $\mathcal{I}(\mathcal{R})$ is closed under direct limits, see [9], Theorem 3.1.17. Thus $\mathrm{CE}(\mathcal{I}(\mathcal{R}))$ is closed under direct limits by Proposition 2.3. Therefore (6) holds by [14], Proposition 1.

(6) $\Rightarrow$ (5) is obvious.

(1) $\Leftrightarrow$ (7). The proof is similar to (1) $\Leftrightarrow$ (6) in Theorem 3.3 due to [9], Theorem 3.1.17.

Recall that a complex C is finitely generated if, in case $C = \sum\limits_{\lambda\in\Lambda} C_\lambda$ with C_λ subcomplexes of C, there exists a finite subset $F \subseteq \Lambda$ such that $C = \sum\limits_{\lambda\in F} C_\lambda$. A complex G is finitely presented if G is finitely generated and there exists an exact sequence $0 \to K \to L \to G \to 0$ with L a finitely generated free complex, K also finitely generated. In fact, a complex C is finitely generated or presented if and only if C is bounded and each C_i is finitely generated or presented, respectively. $\square$

Theorem 3.5. *The following conditions are equivalent for a ring R:*

(1) *R is a von Neumann regular ring.*
(2) *Every R-complex is* CE *flat.*
(3) *Every finitely presented R-complex is* CE *projective.*
(4) *Every* CE *cotorsion R-complex is* CE *flat.*
(5) *Every nonzero R-complex contains a nonzero* CE *flat subcomplex.*
(6) *Every* CE *cotorsion R-complex is* CE *injective.*
(7) *Every* CE *pure injective R-complex is* CE *injective.*

P r o o f. (2) $\Rightarrow$ (4), (2) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) are clear.

(1) $\Rightarrow$ (2). Since every R-module is flat by (1), (2) holds by Definition 1.1.

(2) $\Rightarrow$ (3). Let X be a finitely presented complex. Then X is CE flat, so X_i is finitely presented flat and $X_i/B_i(X)$ is flat for each $i \in \mathbb{Z}$ by Lemma 2.1. Thus X_i is finitely generated projective. Consider the exact sequence $0 \to B_i(X) \to X_i \to X_i/B_i(X) \to 0$, where $B_i(X)$ is finitely generated. Then $X_i/B_i(X)$ is finitely presented and so $X_i/B_i(X)$ is projective. Hence the desired result follows from Lemma 2.1.

(3) $\Rightarrow$ (1). Let M be a finitely presented R-module. Then $\underline{M}$ is CE projective, which gives that M is a projective R-module. Thus, R is a von Neumann regular ring, see [1], Exercise 20.14.

(4) $\Rightarrow$ (1). Let $\mathcal{D}(\mathcal{R})$ denote the class of cotorsion R-modules. By [7], Theorem 9.4, and [19], Theorem 3.8, we have that $(\mathrm{CE}(\mathcal{F}(\mathcal{R})), \mathrm{CE}(\mathcal{D}(\mathcal{R})))$ is a complete and hereditary cotorsion pair in $C(R\text{-Mod})$ relative to $\overline{\mathrm{Ext}}^i(-,-)$. For any R-module M, there exists a CE exact sequence $0 \to \underline{M} \to C \to F \to 0$, where C is a CE cotorsion and F is CE flat. Thus $\underline{M}$ is CE flat by Proposition 2.2, which gives that M is flat and (1) follows.

(5) $\Rightarrow$ (1). Let M be a nonzero R-module. Then $\underline{M}$ contains a nonzero CE flat subcomplex F, and so $F_0 \neq 0$ is a flat submodule of M. That is, M contains a nonzero flat submodule. Hence, R is von Neumann regular.

(2) $\Rightarrow$ (6). Let C be a CE cotorsion R-complex and G an R-complex. Then G is CE flat and $\overline{\mathrm{Ext}}^1(G, C) = 0$ by [7], Theorem 9.4. Thus C is CE injective by Proposition 2.8.

(7) $\Rightarrow$ (1). The proof is similar to (3) $\Rightarrow$ (1) by using [18], Theorem 3.3.2. □

In [1], Theorem 31.9, it was proved that a ring R is QF if and only if every projective R-module is injective if and only if every injective R-module is projective. By Definition 1.1, we have the following lemma.

Lemma 3.6. *The following conditions are equivalent for a ring R:*

(1) *R is a* QF *ring.*

(2) *Every* CE *projective R-complex is* CE *injective.*

(3) *Every* CE *injective R-complex is* CE *projective.*

It is well known that R is a semisimple ring if and only if every R-module is projective if and only if every R-module is injective (see [15], Theorem 4.13). Hence we have:

Theorem 3.7. *The following conditions are equivalent for a ring R:*

(1) *R is a semisimple ring.*

(2) *Every R-complex is* CE *projective.*

(3) *Every R-complex is* CE *injective.*

(4) *Every* CE *Gorenstein injective R-complex is* CE *projective.*
(5) *Every* CE *Gorenstein projective R-complex is* CE *injective.*

P r o o f. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) hold by [15], Theorem 4.13, and Definition 1.1.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) are obvious.

(4) $\Rightarrow$ (1). Let M be any R-module. Then R is a QF ring since every CE injective R-complex is CE projective by (4) and Lemma 3.6. Then $\underline{M}$ is CE Gorenstein injective by [3], Proposition 2.6, and the dual version of [20], Proposition 2.15. By (4) again, $\underline{M}$ is CE projective, which gives that M is projective. Consequently, R is semisimple by [15], Theorem 4.13.

(5) $\Rightarrow$ (1). The proof is similar to (4) $\Rightarrow$ (1). □

It is well known that a ring R is left hereditary if and only if every submodule of a projective R-module is projective if and only if every quotient of an injective R-module is injective (see [15], Theorem 4.23). We have the following result.

Theorem 3.8. *The following conditions are equivalent for a ring R:*
(1) *R is left hereditary.*
(2) *For a* CE *exact sequence $0 \to A \to B \to C \to 0$ with B a* CE *projective R-complex, A is also* CE *projective.*
(3) *For a* CE *exact sequence $0 \to A \to B \to C \to 0$ with B a* CE *injective R-complex, C is also* CE *injective.*
(4) *For any R-complex C,* $\mathrm{CE}\text{-}\mathrm{pd}(C) \leqslant 1$.
(5) *For any R-complex C,* $\mathrm{CE}\text{-}\mathrm{id}(C) \leqslant 1$.
(6) *For any R-complex C and G, $\overline{\mathrm{Ext}}^i(C,G) = 0$ for all $i \geqslant 2$.*

P r o o f. (1) $\Rightarrow$ (2). For each $i \in \mathbb{Z}$, consider the exact sequences $0 \to H_i(A) \to H_i(B) \to H_i(C) \to 0$ and $0 \to B_i(A) \to B_i(B) \to B_i(C) \to 0$. Since $H_i(B)$, $B_i(B)$ are projective, $H_i(A)$, $B_i(A)$ are also projective by [15], Theorem 4.23. Therefore A is CE projective by Lemma 2.1.

(2) $\Rightarrow$ (1). Let M be a projective R-module and $N \subseteq M$ a submodule. Then there is a short CE exact sequence $0 \to \underline{N} \to \underline{M} \to \underline{M/N} \to 0$. Since $\underline{M}$ is CE projective, $\underline{N}$ is CE projective, which means that N is a projective R-module and R is left hereditary.

(1) $\Leftrightarrow$ (3). The proof is similar to (1) $\Leftrightarrow$ (2).

(2) $\Rightarrow$ (4). Let C be an R-complex. There exists a short CE exact sequence $0 \to K \to P \to C \to 0$ where P is CE projective, and so K is also CE projective by (2), which means that $\mathrm{CE}\text{-}\mathrm{pd}(C) \leqslant 1$.

(4) $\Rightarrow$ (6). For any R-complex C, there exists a short CE exact sequence $0 \to P_1 \to P_0 \to C \to 0$ where P_0 and P_1 are CE projective R-complexes. Let G be an

R-complex. We can get $\overline{\mathrm{Ext}}^{j+1}(C,G) \cong \overline{\mathrm{Ext}}^{j}(P_1,G) = 0$ for all $j \geqslant 1$ by dimension shift. Thus (6) holds.

(6) $\Rightarrow$ (2). Consider the CE exact sequence $0 \to A \to B \to C \to 0$ with B a CE projective R-complex. Let G be an R-complex. We can get an exact sequence $\mathrm{Ext}^1(B,G) = 0 \to \overline{\mathrm{Ext}}^1(A,G) \to \overline{\mathrm{Ext}}^2(C,G) = 0$, which means that $\overline{\mathrm{Ext}}^1(A,G) = 0$. Therefore A is CE projective by Proposition 2.7.

The proofs of (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) are similar to (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6). □

A ring R is left perfect if and only if every R-module has a projective cover. There are many characterizations of left perfect rings (see [1], Theorem 28.4, [2], Theorem P, [9], Theorem 5.3.2).

Theorem 3.9. *The following conditions are equivalent for a ring R:*

(1) *R is a left perfect ring.*
(2) *Every* CE *flat R-complex is* CE *projective.*
(3) *Every* CE *projective precover is a* CE *flat precover.*
(4) *The class* $\mathrm{CE}(\mathcal{P}(\mathcal{R}))$ *is covering.*

P r o o f. (1) $\Rightarrow$ (2). It follows from the fact that every flat R-module is projective over a left perfect ring.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (2). Let F be a CE flat R-complex and $P \to F$ its CE projective precover. Then $P \to F$ is also a CE flat precover by assumption. But CE projective precovers are surjective by [7], Proposition 5.4. So the diagram

$$\begin{tikzcd} & F \arrow[dl] \arrow[d, equal] & \\ P \arrow[r] & F \arrow[r] & 0 \end{tikzcd}$$

can be completed to a commutative diagram. Hence F is a summand of P, that F is CE projective by Proposition 2.4.

(2) $\Rightarrow$ (4). Every R-complex has a CE flat cover by [7], Proposition 7.3. So every R-complex has a CE projective cover.

(4) $\Rightarrow$ (1). Suppose M is an R-module. There exists a CE projective cover $f\colon P \to \underline{M}$ of $\underline{M}$ in $C(R\text{-Mod})$. First we show that $P = \underline{P_0}$. By analogy with the proof of (5) $\Rightarrow$ (1) in Theorem 3.4, we get that $f_0\colon P_0 \to M$ is a projective precover of M in R-Mod. It is easy to check that $\underline{P_0} \to \underline{M}$ is a CE projective precover of $\underline{M}$. We get that P is a summand of $\underline{P_0}$ by the complex version of [9], Proposition 5.1.2, which implies that $P = \underline{P_0}$ since $\underline{P_0}$ is a summand of P. Next we

prove that $f_0\colon P_0 \to M$ is a projective cover. Let $g\colon P_0 \to P_0$ be a homomorphism such that $f_0 g = f_0$. Then we have the commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "\overline{g}"'] \arrow[d, "f"] \\ P \arrow[r, "f"] & \underline{M} \end{tikzcd}$$

where $\overline{g}$ is a morphism of complexes:

$$\begin{tikzcd} P = \ldots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & P_0 \arrow[r] \arrow[d, "g"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ P = \ldots \arrow[r] & 0 \arrow[r] & P_0 \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

Since the class $\mathrm{CE}(\mathcal{P}(\mathcal{R}))$ is covering, we get that $\overline{g} = 1_P$, which gives that $g = 1_{P_0}$. Therefore, the class $P(R)$ is covering in R-Mod. Thus R is a left perfect ring by [9], Theorem 5.3.2. $\square$

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