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THE CENTRAL HEIGHTS OF STABILITY GROUPS OF SERIES IN VECTOR SPACES

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Abstract. We compute the central heights of the full stability groups S of ascending series and of descending series of subspaces in vector spaces over fields and division rings. The aim is to develop at least partial right analogues of results on left Engel elements and related nilpotent radicals in such S proved recently by Casolo & Puglisi, by Traustason and by the current author. Perhaps surprisingly, while there is an absolute bound on these central heights for descending series, for ascending series the central height can be any ordinal number.

Keywords: central height; linear group; stability group

MSC 2010: 20F19, 20H25, 20F45

1. INTRODUCTION

Let $\mathbf{L} = \{(\Lambda_\alpha, V_\alpha)\colon \alpha \in \mathbf{A}\}$ be a series of subspaces of a vector space V over the division ring D running from $\{0\}$ to V (see [2], [3] for the definition and basic properties of series in general). So in particular $\mathbf{A}$ is a linearly ordered set, the Λ_α/V_α are the jumps of the series, $V \setminus \{0\} = \bigcup\limits_{\alpha\in\mathbf{A}} \Lambda_\alpha \setminus V_\alpha$ and $\Lambda_\alpha \leqslant V_\beta$ whenever $\alpha < \beta$, except for descending series, when we usually order the series from the top rather than from the bottom.

Let $S = \mathrm{Stab}(\mathbf{L})$, the full stability group of $\mathbf{L}$ in $\mathrm{GL}(V)$; that is, we have $S = \bigcap\limits_{\alpha} C_{\mathrm{GL}(V)}(\Lambda_\alpha/V_\alpha)$. Casolo and Puglisi in [1] studied various nilpotent radicals of S, at least for complete series and D a field, a study extended in [4] and [5], the latter extending it to cover left and bounded left Engel elements of S. In particular Casolo and Puglisi showed that frequently the Hirsch-Plotkin radical $\mathrm{HP}(S)$, the Fitting subgroup $\mathrm{Fitt}(S)$, various other nilpotent radicals of S and a certain subset $F(\mathbf{L})$ of $\mathrm{Fitt}(S)$ depending on the completion $\mathbf{L}^*$ of $\mathbf{L}$ are all equal. For example they showed that this is the case if $\mathbf{L}$ is an ascending series or if V has countable dimension or if V has a special type of basis. They asked whether this is the case in general. A counterexample for descending series is given by Traustason in [4], where he also proves that $\mathrm{Fitt}(S)$ at least is always equal to $F(\mathbf{L})$.

Here we consider the right Engel analogues of some of these results. The right analogue of the Hirsch-Plotkin radical in this context is the hypercentre $\zeta(S)$ of S and of the Fitting subgroup is the ω-th term $\zeta_\omega(S)$ of the upper central series of S. Thus we might reasonably expect that frequently $\zeta_\omega(S)$ and $\zeta(S)$ are equal; that is, that the central height $\mathrm{ctht}(S)$ of S is at most ω. However this actually happens very infrequently for ascending series but does always hold for descending series. We show this by calculating precisely the central height of S for both ascending and descending series. These calculations are the main results of this paper. We do have the following for series in general, a result we make use of in both the ascending and the descending cases.

Theorem 1.1. *Let $\mathbf{L} = \{(\Lambda_\alpha, V_\alpha)\colon\ \alpha \in \mathbf{A}\}$ be a series of subspaces of the vector space V over the division ring D running from $\{0\}$ to V. Set $S = \mathrm{Stab}(\mathbf{L})$. If either $V \neq \Lambda_\alpha$ for all α in $\mathbf{A}$ with $\Lambda_\alpha > V_\alpha$, or $\{0\} \neq V_\alpha$ for all α in $\mathbf{A}$ with $V_\alpha < \Lambda_\alpha$, then $\zeta(S) = \langle 1 \rangle$; that is, S has central height 0.*

Theorem 1.2. *Let $\mathbf{L} = \{V_\alpha\colon\ 0 \leqslant \alpha \leqslant \lambda\}$ be a strictly ascending series of subspaces of the vector space V over the division ring D indexed by the ordinal numbers $\alpha \leqslant \lambda$, where $\{0\} = V_0$ and $V_\lambda = V$. Set $\lambda = \mu + n$, where μ is zero or a limit ordinal and n is a non-negative integer, and set $S = \mathrm{Stab}(\mathbf{L})$. Then S has central height exactly*

a) 0 *if* $n = 0$,

b) $n - 1$ *if* $\mu = 0 < n$ *and*

c) $\lambda - 1$ *if* $\mu > 0$ *and* $n > 0$.

Clearly Theorem 1.2 a) is immediate from Theorem 1.1 and Theorem 1.2 b) is well-known. The proof of Theorem 1.2 c) constitutes a major part of this paper and occupies most of Section 3 below. It is easy to construct for each ordinal γ and any division ring D an example as in Theorem 1.2 with S of central height γ, $\lambda = \gamma + 1$ and $\dim_D(V)$ the cardinality of λ. In particular we have examples with $\dim_D V$ countable for all countable ordinals γ. The following is immediate from Theorem 1.2.

Corollary 1.3. *With the notation and hypotheses of Theorem 1.2 we have that $\zeta_\omega(S) = \zeta(S)$ if and only if either $n = 0$ or $\lambda \leqslant \omega + 1$.*

With descending series the situation is in some ways much nicer.

Theorem 1.4. *Let* $\mathbf{L} = \{V_\alpha\colon 0 \leqslant \alpha \leqslant \lambda\}$ *be a strictly descending series of subspaces of the vector space* V *over the division ring* D *indexed by the ordinal numbers* $\alpha \leqslant \lambda$, *where* $\{0\} = V_\lambda$ *and* $V_0 = V$. *Set* $\lambda = \mu + n$, *where* μ *is zero or a limit ordinal and* n *is a non-negative integer, and set* $S = \mathrm{Stab}(\mathbf{L})$. *Then* S *has central height at most* ω. *Precisely* S *has central height exactly*

a) 0 *if* $n = 0$,

b) $n - 1$ *if* $\mu = 0 < n$ *and*

c) ω *if* $\mu > 0$ *and* $n > 0$.

Again Theorem 1.4 c) is the substantial part of the theorem and its proof occupies Section 4 below. The following is immediate.

Corollary 1.5. *With the notation and hypotheses of Theorem* 1.4 *always* $\zeta_\omega(S) = \zeta(S)$.

In our proofs below we make a number of uses of the following three lemmas from [5], where D is a division ring, V is a left vector space over D, $\mathbf{L}$ is a series of subspaces of V running from $\{0\}$ to V, $S = \mathrm{Stab}(\mathbf{L})$ and $\mathbf{L}^*$ is the completion of $\mathbf{L}$.

Lemma 1.6 ([5], Lemma 1.2). *Let* $W \in \mathbf{L}^*$, $u \in W$ *and* $v \in V \setminus W$ *with* $V = Dv \oplus U \oplus W$ *for some subspace* U. *Then there is an* x *in* S *with* $vx = u + v$, $V(x-1) = Du \leqslant W$, $W(x-1) = \{0\}$ *and* $U(x-1) = \{0\}$.

Lemma 1.7 ([5], Lemma 1.3). *Let* $g \in \mathrm{GL}(V)$, *let* $\xi \in \mathrm{End}_D V$ *with* $\xi^2 = 0$ *and set* $x = 1 + \xi$. *Then* $x \in \mathrm{GL}(V)$. *Working in* $\mathrm{End}_D V$, *if* $(g-1)\xi = 0$, *then* $[x, g] = 1 + \xi(g-1)$, *and if* $\xi(g-1) = 0$, *then* $[x, g] = 1 + (g^{-1} - 1)\xi$.

Lemma 1.8 ([5], Lemma 1.4). *Let* $m \geqslant 2$ *be an integer and set* $n = 2m$ *and* $r = [\log_2(m-1)]$. *For any division ring* D, *in* $\mathrm{GL}(n, D)$ *let* $x = 1 + \sum\limits_i e_{2i,2i+1}$ *and* $g = 1 + \sum\limits_i e_{2i-1,2i}$. *Then* $[x, {}_r g] \neq 1$.

2. General series

In this section D denotes a division ring, V a left vector space over D and $\mathbf{L} = \{(\Lambda_\alpha, V_\alpha)\colon \alpha \in \mathbf{A}\}$ a series of subspaces of V running from $\{0\}$ to V with all jumps nontrivial. Set $S = \mathrm{Stab}(\mathbf{L})$, the full stability subgroup of $\mathbf{L}$ in $\mathrm{GL}(V)$. Theorem 1.1 is immediate from the following.

Lemma 2.1. *If the centre $Z = \zeta_1(S)$ of S is nontrivial, then $\mathbf{L}$ has a maximal jump (V, V_α) and a minimal jump $(\Lambda_\beta, \{0\})$. Also $[V, Z] \leqslant \Lambda_\beta$ and $[V_\alpha, Z] = \{0\}$.*

P r o o f. Let $z \in Z \setminus \langle 1 \rangle$. There exists $v \in V$ with $u = v(z-1) \neq 0$. Then $v \in \Lambda_\alpha \setminus V_\alpha$ and $u \in \Lambda_\beta \setminus V_\beta$ for some $\alpha > \beta$ in $\mathbf{A}$. Suppose there exists $\gamma \in \mathbf{A}$ with $\beta > \gamma$. Pick $w \in \Lambda_\gamma \setminus V_\gamma$. By Lemma 1.6 there exists $x \in S$ with $vx = v$ and $ux = u + w$. Then $vxz = vz = u + v$ and $vzx = (u+v)x = v + u + w$. But $w \neq 0$, so $xz \neq zx$, contradicting $z \in Z$. Consequently no such γ exists, β is the minimal member of $\mathbf{A}$ and $V_\beta = \{0\}$.

Now v above is any v in V with $v(z-1) \neq 0$. Thus $V(z-1) \leqslant \Lambda_\beta = U$, say. Suppose there exists $\delta \in \mathbf{A}$ with $\delta > \alpha$. Pick $w \in \Lambda_\delta \setminus V_\delta$. By Lemma 1.6 again there exists $y \in S$ with $wy = w+v$ and $U(y-1) = \{0\}$. Then $wyz = (w+v)z = w+w(z-1)+v+u$ and $wzy = (w + w(z-1))y = w + v + w(z-1)$, since $w(z-1) \in U$. But $u \neq 0$, so $yz \neq zy$, a contradiction. Thus no such δ exists, α is the maximal member of $\mathbf{A}$ and $\Lambda_\alpha = V$. If $v \in V$ with $v(z-1) \neq 0$, then $v \in V \setminus V_\alpha$ and hence $[V_\alpha, Z] = \{0\}$. □

Let $\mathbf{L}^*$ denote the completion of $\mathbf{L}$ (again see [2], [3] for definitions). Then $S = \mathrm{Stab}(\mathbf{L}) = \mathrm{Stab}(\mathbf{L}^*)$. Now $\mathbf{L}^*$ contains an element L such that $\{X \in \mathbf{L}^* \colon X \leqslant L\}$ is an ascending series of L and $\{\alpha \in \mathbf{A} \colon V_\alpha \geqslant L\}$ has no minimal member. Then Theorem 1.2 gives us the central height of $S/C_S(L)$ and Theorem 1.1 the central height of $S/C_S(V/L)$. Thus we have the central height of $S/(C_S(L) \cap C_S(V/L))$. On the other hand $\mathbf{L}^*$ also contains an element U such that $\{V \in \mathbf{L}^* \colon X \geqslant U\}$ modulo U is a descending series of V/U and $\{\alpha \in \mathbf{A} \colon \Lambda_\alpha \leqslant U\}$ has no maximal member. Hence Theorem 1.4 and Theorem 1.1 give us the central height of $S/(C_S(V/U) \cap C_S(U))$. All this gives us some, admittedly very weak, information about the central height of S for a general series.

3. Ascending series

In this section we assume throughout the following notation. Let $\lambda = \mu + n$ be an infinite ordinal, where μ is a limit ordinal and n is a positive integer (so λ is not a limit ordinal). Let V be a left vector space over the division ring D and $\mathbf{L}$ the strictly ascending series

$$\{0\} = V_0 < V_1 < \ldots < V_\alpha < \ldots < V_\lambda = V$$

of subspaces of V of length λ. Let S denote the full stability group of $\mathbf{L}$ in $\mathrm{GL}(V)$.

Lemma 3.1. *The group S has central height exactly $\lambda - 1$.*

Proof. Set $C = C_S(V/V_\mu)$ and $T = C_C(V_\mu)$. Then S/C is nilpotent of class $n-1$ and $S/C_S(V_\mu)$ has central height 0 by Theorem 1.1. Also $T \cong H = \mathrm{Hom}_D(V/V_\mu, V_\mu)$ as S-module, where S acts on H diagonally (meaning $v\theta^x = vx^{-1}\theta x$ for θ in H, x in S and v in V) and where we regard H as a subset of $\mathrm{Hom}_D(V, V_\mu)$ in the obvious way. We begin our proof by considering the action of S on H.

For $\alpha \leqslant \mu$ set $H_\alpha = \mathrm{Hom}_D(V/V_\mu, V_\alpha)$. Then

$$L_\alpha = H_{\alpha+1} \setminus H_\alpha \ {}_D{\cong}_S \ \mathrm{Hom}_D(V/V_\mu, V_{\alpha+1}/V_\alpha).$$

Thus L_α is centralized by C. If $L_{\alpha,i} = \mathrm{Hom}_D(V/V_{\mu+i}, V_{\alpha+1}/V_\alpha)$ then

$$L_\alpha = L_{\alpha,0} \geqslant L_{\alpha,1} \geqslant \ldots \geqslant L_{\alpha,n} = \{0\}$$

and $L_{\alpha,i}/L_{\alpha,i+1} \ {}_D{\cong}_S \ \mathrm{Hom}_D(V_{\mu+i+1}/V_{\mu+i}, V_{\alpha+1}/V_\alpha)$. The latter is clearly S-central. We now have the following.

Lemma 3.2. *For each $\alpha < \mu$ the space $H_{\alpha+1}/H_\alpha$ is C-central and S-hypercentral with S-central height at most n.*

The problem now is what happens at the limit ordinals. Suppose $\nu \leqslant \mu$ is a limit ordinal and set $H_{\nu-} = \bigcup\limits_{\alpha<\nu} H_\alpha$. Consider, if possible, some θ in $H_\nu \setminus H_{\nu-}$. Then for each $\alpha < \nu$ there is some v_α in $V \setminus V_\mu$ with $v_\alpha\theta$ in $V_\nu \setminus V_{\alpha+1}$. Then there exists α' with $\alpha < \alpha' < \nu$ and $v_\alpha\theta \in V_{\alpha'+1} \setminus V_{\alpha'}$. This is for all $\alpha < \nu$, so $\bigcup\limits_{\alpha<\nu} \alpha' = \nu$ and consequently $\{\alpha' \colon \alpha < \nu\}$ contains a strictly ascending sequence $\{\alpha(\beta) \colon \beta < \nu' \leqslant \mu\}$ for some limit ordinal ν' with $\nu = \bigcup\limits_{\beta} \alpha(\beta)$ and with $\alpha(\beta) + 1 < \alpha(\beta+1)$ (e.g. choose any $\alpha < \nu$, set $\alpha(0) = \alpha$, set $\alpha(\beta+1) = \alpha(\beta)''$ and for limit ordinals δ set $\alpha(\delta) = \bigcup\limits_{\beta<\delta} \alpha(\beta)$).

Changing notation slightly we have an element v_β of $V \setminus V_\mu$ with $v_\beta\theta$ lying in $V_{\alpha(\beta)+1} \setminus V_{\alpha(\beta)}$. Choose for each β an element u_β of $V_{\alpha(\beta+1)} \setminus V_{\alpha(\beta)+1}$. There exists an $\mathbf{L}$-basis $\mathbf{B}$ of V containing all the $v_\beta\theta$ and all the u_β and then there exists an element x of C given by $v_{\beta+1}\theta(x-1) = u_\beta$ for each $\beta < \nu'$ and $bx = b$ for all other b in $\mathbf{B}$. Then $v_{\beta+1}\theta(x-1) \notin V_{\alpha(\beta)+1}$ and thus $\theta(x-1) \notin H_{\alpha(\beta)+1}$ for all $\beta < \nu'$. Hence $[\theta, x] = \theta(x-1) \notin H_{\nu-}$ and we have proved the following.

Lemma 3.3. *For each limit ordinal $\nu \leqslant \mu$ the space $H_\nu/H_{\nu-}$ contains no non-identity C-fixed points and hence no non-identity S-fixed points.*

Let Z denote the S-hypercentre of H. Then Lemma 3.3 implies that $H_\nu \cap Z = H_{\nu-} \cap Z$ for all limit ordinals $\nu \leqslant \mu$. Consequently Lemma 3.2 yields that $\{H_\alpha \cap Z\colon \alpha \leqslant \mu\}$ is a C-hypercentral series of Z of length at most μ which refines to an S-hypercentral series of Z, also of length at most μ. We have proved the following.

Lemma 3.4. *The group H has S-central height at most μ.*

Let $K = \{\theta \in H\colon \dim_D V\theta \text{ is finite}\}$ and set $K_\alpha = H_\alpha \cap K$ for all $\alpha \leqslant \mu$. Note that $K_\nu = \bigcup_{\alpha<\nu} K_\alpha$ whenever $\nu \leqslant \mu$ is a limit ordinal. Also Lemma 3.2 implies that $[K_{\alpha+1}, C] \leqslant K_\alpha$ for all $\alpha < \mu$ and that K is S-hypercentral with S-central height at most μ. Let $\{Z_\alpha\}$ denote the upper C-central series of K. Then $Z_\alpha \geqslant K_\alpha$ for all α. Suppose $Z_\alpha = K_\alpha$ for some $\alpha < \mu$ (as it clearly does for $\alpha = 0$). Suppose $\theta \in Z_{\alpha+1} \setminus K_{\alpha+1}$. There exists $v \in V \setminus V_\mu$ with $v\theta \notin V_{\alpha+1}$. Pick $w \in V_{\alpha+1} \setminus V_\alpha$. There exists $x \in C$ with $v\theta(x-1) = w$ and clearly $\dim_D V\theta(x-1) \leqslant \dim_D V\theta$, which is finite. Thus $\theta(x-1) \in K \setminus K_\alpha$. But $[\theta, x] \in Z_\alpha = K_\alpha$. This contradiction yields that $Z_{\alpha+1} = K_{\alpha+1}$. If $\nu \leqslant \mu$ is a limit ordinal with $Z_\alpha = K_\alpha$ for all $\alpha < \nu$, then $Z_\nu = \bigcup_{\alpha<\nu} Z_\alpha = \bigcup_{\alpha<\nu} K_\alpha = K_\nu$. We have proved the following.

Lemma 3.5. *The set $\{K_\alpha\colon \alpha \leqslant \mu\}$ is the upper C-central series of K.*

Thus the C-central height of K is μ and hence the S-central height of K is also at least μ; consequently the C-central height and the S-central height of H are at least μ. Therefore by Lemma 3.4 we have the following.

Lemma 3.6. *The S-module H has S-central height exactly μ.*

We had above a canonical isomorphism of T onto H. Let T_K denote the inverse image of K in T. The above shows that $T_K \leqslant \zeta_\mu(S)$ and that T_K and $\zeta_\mu(S)$ have S-central heights exactly μ. Let T_α denote the inverse image of H_α in T (including $T_{\nu-}$ the inverse image of $H_{\nu-}$ for limit ordinals ν). Further $\{T_\alpha \cap T_K\colon \alpha \leqslant \mu\}$ is the upper C-series of T_K by Lemma 3.5 and $T \cap \zeta(S) \leqslant \zeta_\mu(S)$ by Lemma 3.6.

The following structure for S was critical for the paper [5]. Choose a subspace W of V with $V = W \oplus V_\mu$. Let S_1 be the stability group of the series $\{L \cap V_\mu\colon L \in \mathbf{L}\}$ in V_μ with its action extended to V by letting S_1 centralize W. Similarly let S_2 be the stability group of the series $\{L \cap W\colon L \in \mathbf{L}\}$ with its action extended to V by letting S_2 centralize V_μ. Then $[S_1, S_2] = \langle 1\rangle$, $S = S_1 T S_2$, $C = S_1 T$ and $C_S(V_\mu) = T S_2$. Now S_1 has central height 0 by Theorem 1.1, so S/T has central height $n-1$, $[\zeta(S), {}_{n-1}S] \leqslant T \cap \zeta(S) \leqslant \zeta_\mu(S)$. Hence

Lemma 3.7. *The group S has central height at most $\mu + n - 1$.*

We need the following lemma.

Lemma 3.8. *If $z \in S_2 \backslash \langle 1 \rangle$ and if $\alpha < \mu$, then there exists θ in H with $[\theta, z] \notin H_\alpha$.*

To see Lemma 3.8, note that since $z \neq 1$, there exists $w \in W$ with $wz \neq w$. If $wz \in Dw$, then z normalizes Dw and also stabilizes a proper series in Dw. The latter can only be $\langle 1 \rangle < Dw$. Since $wz \neq w$ this yields that w and wz are linearly independent elements of V. Choose $u \in V_{\alpha+1} \setminus V_\alpha$. There exists some θ in H with $w\theta = u$ and $wz\theta = 0$. Now $z \in S_2$, which centralizes V_μ. Hence $wz[\theta, z] = w\theta z - wz\theta = w\theta \notin V_\alpha$. Consequently $[\theta, z] \notin H_\alpha$, confirming the truth of Lemma 3.8.

For $x \in S_2$ set $V_x = \{v \in V\colon\ v(x-1) \in V_\mu\}$ and set $S_3 = \{x \in S_2\colon\ \dim_D(V/V_x) = \dim_D V(x-1)$ is finite$\}$. Clearly S_3 is a normal subgroup of S_2 and, like S_2, is nilpotent of class $n-1$ (both stabilize a series of length n in W and S_3 contains a copy of the unitriangular group $\mathrm{Tr}_1(n, D)$).

If $x \in S_3$, $\theta \in H$ and $v \in V_x$, then $v[\theta, x] = vx^{-1}\theta x - v\theta = v\theta - v\theta = 0$, where we have used that $vx^{-1}\theta = v\theta$ since $v \in V_x$ and $V_\mu\theta = \{0\}$, and $\theta x = \theta$ since $x \in S_2 \leqslant C_S(V_\mu)$. Thus $V_x[\theta, x] = \{0\}$, so $\dim_D V[\theta, x] \leqslant \dim_D(V/V_x)$, which is finite. Consequently $[\theta, x] \in K$. Hence $[H, S_3] \leqslant K$ and therefore $[T, S_3] \leqslant T_K \leqslant \zeta_\mu(S)$, the latter by Lemma 3.6. But $S = S_1 T S_2$, so $[S_3, S] \leqslant T_K[S_3, S_2]$. Also S_2 is nilpotent of class $n-1$, so $[S_3, {}_{n-1}S] \leqslant T_K$. Consequently

Lemma 3.9. $T_K S_3 \leqslant \zeta_{\mu+n-1}(S) = \zeta_{\lambda-1}(S)$.

Let $z \in S_3 \cap \zeta_\mu(S)$. Then $z \in \zeta_\alpha(S)$ for some $\alpha < \mu$ and $[T, z] \leqslant T \cap \zeta_\alpha(S) \leqslant T \cap \zeta_\alpha(C) \leqslant T_\alpha$ by Lemma 3.5 again. But then $[H, z] \leqslant H_\alpha$, which contradicts Lemma 3.8 unless $z = 1$. Hence $S_3 \cap \zeta_\mu(S) = \langle 1 \rangle$. Since $S_3 \leqslant \zeta(S)$ by Lemma 3.9, so $\zeta_\mu(S) S_3$ has S-central height at least $\mu + n - 1 = \lambda - 1$. Consequently S has central height at least $\lambda - 1$ and therefore exactly $\lambda - 1$ by Lemma 3.7. This completes the proofs of Lemma 3.1 and of Theorem 1.2. $\square$

Remark 3.10. Note that $C_S(V_\mu)$ is nilpotent of class n, since $C_S(V_\mu)$ is just the stability group of the series $\langle 1 \rangle = V_0 < V_\mu < V_{\mu+1} < \ldots < V_\lambda = V$. Trivially, therefore, $[T, S_2] = [T, C_S(V_\mu)] < T$.

Suppose $\dim_D(V/V_\mu)$ is finite. Then $H = K$, $T = T_K$, $S_3 = S_2$, $\zeta(S) = TS_2$ and $\zeta_\mu(S) = T$. Also $[T, S_1] = T$, so $[T, C] = T = [T, S]$. To see this a simple induction on α produces a basis $\mathbf{B}$ of V with $\mathbf{B} \cap (V_{\alpha+1} \setminus V_\alpha)$ a basis of $V_{\alpha+1}$ modulo V_α for each $\alpha < \lambda$ and with $\mathbf{B}_1 = \mathbf{B} \cap W$ a basis of W. Let $\theta \in H$. Then $\theta \in H_\alpha$ for some $\alpha < \mu$ and also $V_\mu\theta = \{0\}$, so θ is determined by the $b\theta$ for $b \in \mathbf{B}_1$. Now $\mathbf{B}_1$ is finite, while $\mathbf{B}_2 = \mathbf{B} \cap (V_\mu \setminus V_\alpha)$ is infinite. Hence we can choose for each $b \in \mathbf{B}_1$ a b' in $\mathbf{B}_2$

such that the b' are distinct for distinct b. Define $\varphi \in H$ by $b\varphi = b'$ for all b in $\mathbf{B}_1$ (and with $V_\mu \varphi = \{0\}$ of course). Define $x \in \mathrm{GL}(V)$ by $b'x = b' + b\theta$ for all $b \in \mathbf{B}_1$ and $cx = c$ for all other c in $\mathbf{B}$. Note that x does indeed lie in $\mathrm{GL}(V)$ since $b' \notin V_\alpha$ and $b\theta \in V_\alpha$, this for all b in $\mathbf{B}_1$. Clearly, $x \in S_1$. Now

$$b[\varphi, x] = b(x^{-1}\varphi x - \varphi) = b\varphi x - b\varphi = b'x - b' = b\theta,$$

for all b in $\mathbf{B}_1$. Consequently $[\varphi, x] = \theta$. It follows that $[H, S_1] = H$ and hence that $[T, S_1] = T$, as claimed. (More generally if $\dim_D(V/V_\mu)$ is countable and if for each $\alpha < \mu$ there exists β with $\alpha < \beta < \mu$ and with $\dim_D(V_\beta \setminus V_\alpha)$ infinite, the above argument also shows that $T_K = [T_K, S_1] = [T_K, C] = [T_K, S]$.)

4. Descending series

In this section we assume throughout the following notation. Let $\lambda = \mu + n$ be an infinite ordinal, where μ is a limit ordinal and n is a positive integer (so λ is not a limit ordinal). Let V be a left vector space over the division ring D and $\mathbf{L}$ the strictly descending series

$$V = V_0 > V_1 > \ldots > V_\alpha > \ldots > V_\lambda = \{0\}$$

of subspaces of V of length λ. Let S denote the full stability group of $\mathbf{L}$ in $\mathrm{GL}(V)$. The following confirms Part c) of Theorem 1.4 and completes the proof of this theorem.

Lemma 4.1. *The group S has central height exactly ω.*

Set $H = \mathrm{Hom}_D(V/V_\mu, V_\mu)$. Regard H as a right S-module via the diagonal action, as in Section 3. Let $K = \{\theta \in H\colon V_i\theta = \{0\}$ for some $i < \omega\}$. Clearly K is a $D - S$ bisubmodule of H. We need the following.

Lemma 4.2. *The set K is the S-hypercentre of H and K has S-central height exactly ω.*

P r o o f of Lemma 4.2. Let $K_i = \mathrm{Hom}_D(V/V_i, V_\mu)$ for $0 \leqslant i < \omega$, regarded as a subset of H in the obvious way. Then $K_i \leqslant K_{i+1}$ for each i and $\bigcup\limits_{i<\omega} K_i = K$. Now K_{i+1}/K_i is isomorphic to $\mathrm{Hom}(V_i/V_{i+1}, V_\mu)$ as $D - S$ bimodule and the diagonal action of S on this Hom module is just right multipication given by the action of S on V_μ. Hence $[K_{i+1}/K_i, {}_nS] = \{0\}$, $[K_{i+1}, {}_nS] \leqslant K_i$ for each i and K is S-hypercentral with S-central height at most ω.

Let $v_i \in V_i \setminus V_{i+1}$ for each i and let $u \in V_\mu \setminus \{0\}$. For $1 \leqslant i < \omega$ there exists $x \in S$ with $v_j x = v_j + v_{j+1}$ for each $j \leqslant i$ and with $v(x-1) = 0$ for all v in V_{i+1}. Also there exists $\theta_i \in K_{i+1}$ with $v_i\theta_i = u$. Now $w[\theta_i, {}_i x^{-1}] = w(x-1)^i\theta$ for any w in V and $v_j(x-1) = v_{j+1}$ for each $j \leqslant i$. Thus $v_0[\theta_i, {}_i x^{-1}] = v_i\theta_i = u \neq 0$. Therefore $[K, {}_i S] \neq \{0\}$ for all $i \geqslant 1$ and consequently the S-central height of K is exactly ω.

If H/K has no nontrivial elements fixed by S, then K is the S-hypercentre of H. Thus suppose $\theta \in H \setminus K$ with $[\theta, S] \subseteq K$. Suppose first that $V_\omega\theta \neq \{0\}$, say $v\theta \neq 0$ for $v \in V_\omega$. If possible pick such a v with $v\theta \notin V_{\lambda-1}$ (so for the moment we are assuming $n \geqslant 2$). If $w \neq 0$ lies in $V_{\lambda-1}$, there exists $x \in S$ with $V(x-1) \leqslant V_\mu$ and $v\theta x = v\theta + w$, see Lemma 1.6. Then $v \in V_\omega = \bigcap_{i<\omega} V_i$ and

$$v(x^{-1}\theta x - \theta) = v\theta x - v\theta = v\theta + w - v\theta = w \neq 0.$$

Hence $[\theta, x] \notin K$, a contradiction. Thus $v\theta \in V_{\lambda-1}$; that is, $V\theta \leqslant V_{\lambda-1}$, not only for $n \geqslant 2$, but trivially also for $n = 1$. For each $i \geqslant 1$ pick $e_i \in V_i \setminus V_{i+1}$. There exists $y \in S$ with $V_\omega(y-1) = \{0\}$ and $e_i y = e_i + v$. (The e_i are linearly independent modulo V_ω so V has a basis $\mathbf{B}$ containing all the e_i such that $\mathbf{B} \cap (V \setminus V_\omega)$ is a basis of V modulo V_ω and $\mathbf{B} \cap V_\omega$ is a basis of V_ω. Let $\xi\colon V \to V_\omega$ be the linear map given by $e_i\xi = v$ for each i and $b\xi = 0$ for all other b in $\mathbf{B}$. Then set $y = 1 + \xi$.) Then

$$e_i[\theta, y^{-1}] = e_i y\theta - e_i\theta = (e_i + v)\theta - e_i\theta = v\theta \neq 0 \quad \text{for each } i.$$

Consequently $[\theta, y^{-1}] \notin K$, a contradiction.

The above shows that $V_\omega\theta = \{0\}$. Since $\theta \notin K$ we have $(V_i \setminus V_{i+1})\theta \neq \{0\}$ for infinitely many $i < \omega$. Suppose there exists m and $v_j \in V_{i(j)} \setminus V_{i(j)+1}$ with $v_j\theta \neq 0$ for $j = 1, 2, \ldots$ such that $0 \leqslant m < n$, $m < i(1)$, $i(j) + 1 < i(j+1)$ for each j and $V\theta \leqslant V_{\mu+m}$. Pick $u_j \in V_{i(j)-1} \setminus V_{i(j)}$ for each j. Then there exists $z \in S$ with $V_\omega(z-1) = \{0\}$ and $u_j z = u_j + v_j$ for each j. (Notice that since $i(j) + 1 < i(j+1)$ for each j, so $z^{-1} \in S$.) Then

$$u_j[\theta, z^{-1}] = u_j(z\theta - \theta) = v_j\theta \neq 0.$$

Therefore $[\theta, z^{-1}] \notin K$ and consequently we cannot choose m and the v_j with the above properties.

If possible pick $v_1 \in V \setminus V_\omega$ with $v_1\theta \in V_\mu \setminus V_{\mu+1}$. If $v_1 \in V_{i(1)} \setminus V_{i(1)+1}$ choose if possible $v_2 \in V_{i(2)} \setminus V_{i(2)+1}$ with $i(1) + 1 < i(2)$ and $v_2\theta \in V_\mu \setminus V_{\mu+1}$. Keep going. By the above, this process will stop after a finite number of steps with $V_h\theta \leqslant V_{\mu+1}$ for some $h < \omega$. Repeat the above process, but now picking the v_j in V_h with the $v_j\theta$ in $V_{\mu+1} \setminus V_{\mu+2}$. Again the process halts when we find a $k \geqslant h$ with $k < \omega$ and

$V_k\theta \leqslant V_{\mu+2}$. Once more we repeat and after a finite number of repeats we eventually arrive at an $l < \omega$ with $V_l\theta \leqslant V_{\mu+n} = \{0\}$. But then $\theta \in K$, contrary to our choice of θ. This final contradiction completes the proof of Lemma 4.2. $\square$

Proof of Lemma 4.1. We continue with the notation of the proof above of Lemma 4.2. Set $T = C_S(V/V_\mu) \cap C_S(V_\mu)$ and choose the subspace W with $V = W \oplus V_\mu$. As in Section 3 define $S_1 \leqslant \mathrm{GL}(V_\mu) \cap C_S(W)$ and $S_2 \leqslant \mathrm{GL}(W) \cap C_S(V_\mu)$ so that we have $S = S_1TS_2$, $S_1S_2 = S_1 \times S_2$, $C_S(V/V_\mu) = S_1T$ and $C_S(V_\mu) = TS_2$. The difference from Section 3 is that here S_1 is nilpotent of class $n-1$ and S_2 has central height 0 and not the other way round. Let T_K denote the inverse image of K in T under the canonical isomorphism of T onto H. The above and Lemma 4.2 show that $\zeta(S) \leqslant S_1T$ and $\zeta(S) \cap T = T_K \leqslant \zeta_\omega(S)$. In particular S has central height at least ω (and at most $\omega + n - 1$).

Let $x \in S_1$ and $t \in T$ with $xt \in \zeta(S)$. If also $t' \in T$, then $[t', x] = [t', xt] \in \zeta(S) \cap T = T_K$. Therefore $\theta(x-1) = [\theta, x] \in K$ for all θ in H. Suppose $x \neq 1$. Then $U = V_\mu \cap C_V(x) \neq V_\mu$. Pick $v_i \in V_{i-1} \setminus V_i$ for each $i \geqslant 1$ and pick $u \in V_\mu \setminus U$. Now the v_i are linearly independent modulo V_μ. Hence there exists θ in H with $v_i\theta = u$ for all i. Then $v_i[\theta, x] = u(x-1) \neq 0$ for all i and hence $[\theta, x] \notin K$. This contradiction shows that $x = 1$ and that $\zeta(S) \leqslant T$. Consequently $\zeta(S) = T_K$ and S has central height exactly ω. $\square$

Remark 4.3. Analogously to the case of ascending series, see Remarks in Section 3, here we have $C_S(V/V_\mu)$ nilpotent of class n, instead of $C_S(V_\mu)$.

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