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Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 1, 111–117

Persistent URL: `http://dml.cz/dmlcz/144885`

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ON R-CONJUGATE-PERMUTABILITY OF SYLOW SUBGROUPS

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(Received January 18, 2015)

Abstract. A subgroup H of a finite group G is said to be conjugate-permutable if $HH^g = H^gH$ for all $g \in G$. More generaly, if we limit the element g to a subgroup R of G, then we say that the subgroup H is R-conjugate-permutable. By means of the R-conjugate-permutable subgroups, we investigate the relationship between the nilpotence of G and the R-conjugate-permutability of the Sylow subgroups of A and B under the condition that $G = AB$, where A and B are subgroups of G. Some results known in the literature are improved and generalized in the paper.

Keywords: R-conjugate-permutable subgroup; nilpotent group; quasinilpotent group; Sylow subgroup

MSC 2010: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. Let the group $G = AB$ be the product of two subgroups A and B. The relationship between the structure of G and the properties of the subgroups A and B has been extensively studied by a number of authors, with many interesting results available. For example, Kegel and Wielandt stated the solvability of G under the condition that A and B are nilpotent (see [5], [8]), Huppert in [4] showed the supersolvability of G when A and B are cyclic, etc. Besides, the book "Products of Finite Groups" (see [2]) has described the structure of the groups which are products of some subgroups by using the properties of the corresponding subgroups.

Based on these results, this paper is aimed at describing the structure of G by employing conjugate-permutability of A and B.

The research has been supported by the NSFC (Grant no. 11501176, U1504101, U1204101, 11471266), Fundamental Research Funds for the Central Universities (no. XDJK2014C163) and the Major Project of Education Department of Henan Province (no. 13B110085).

We begin with the definition of a conjugate-permutable subgroup:

Definition 1.1 ([3]). A subgroup H of a group G is a conjugate-permutable subgroup of G if $HH^g = H^gH$ for all $g \in G$.

Let R be a subgroup of G. In Definition 1.1, if we limit the element g to R, then we get the following:

Definition 1.2 ([6]). Let R be a subgroup of a group G. A subgroup H of the group G is an *R-conjugate-permutable subgroup of* G if $HH^g = H^gH$ for all $g \in R$.

The results in [6] have described the structure of G by employing R-conjugate-permutability of some subgroups of G. In this paper, we investigate the structure of G ($G = AB$) by the R-conjugate-permutability of the Sylow subgroups of the factors A and B. (See Theorems 3.1–3.3.)

In order to prove Theorem 3.1 we need the following definition and the following property (see [4], §13):

Definition 1.3 ([4]). A group G is called *quasinilpotent* if given any chief factor X of G, any element of G induces an inner automorphism on X.

Property 1.1. The generalized Fitting subgroup $F^*(G)$ is quasinilpotent, and every subnormal quasinilpotent subgroup of G is contained in $F^*(G)$.

Throughout this paper, we use $Z_\infty(G)$ to denote the hypercenter of a group G. Apart from this, all unexplained notations and terminology are standard and taken from [7].

2. Preliminaries

We first list some lemmas which will be useful for the proof of our main results.

Lemma 2.1 ([6], Lemma 2.2). *Let H and R be subgroups of a group G. If H is R-conjugate-permutable, and $RH = HR$, then $H \lhd\lhd HR$.*

Lemma 2.2. *Let H and R be subgroups of a group G and N a normal subgroup of G. If H is R-conjugate-permutable, then HN/N is RN/N-conjugate-permutable.*

P r o o f. For any $\overline{s} \in RN/N$ there exists $r \in R$ such that $\overline{s} = rN$. We have

$$\begin{aligned}(HN/N)^{\overline{s}} \cdot (HN/N) &= (HN/N)^{rN} \cdot HN/N\\ &= H^rN/N \cdot HN/N\\ &= H^rN \cdot HN/N\\ &= H^rHN/N\\ &= HH^rN/N \quad (H \text{ is } R\text{-conjugate-permutable})\\ &= HN/N \cdot H^rN/N\\ &= HN/N \cdot (HN/N)^{\overline{s}}.\end{aligned}$$

Therefore HN/N is RN/N-conjugate-permutable. □

Lemma 2.3. *Let G be a group. If $\Phi(G) = 1$, then $Z_\infty(G) = Z(G)$.*

P r o o f. Let the upper central series of G be

$$1 = Z_0(G) \leqslant Z_1(G) \leqslant \ldots \leqslant Z_n(G) \leqslant \ldots$$

Since $Z_n(G)/Z_{n-1}(G) = Z(G/Z_{n-1}(G))$, we are required to prove equality $Z_2(G) = Z(G)$. If $Z(G) = 1$, obviously, $Z_2(G) = Z(G)$. If $Z(G) > 1$, notice that $\Phi(G) = 1$, so there exists a complement H for $Z(G)$ in G. This means that $G = Z(G)H$ and $Z(G) \cap H = 1$. Also $Z_2(G)/Z(G) = Z(Z(G)H/Z(G)) \cong Z(H/Z(G) \cap H) = Z(H)$ and $Z(H) \leqslant Z(G) \cap H = 1$, from which we have $Z_2(G) = Z(G)$. Therefore $Z_\infty(G) = Z(G)$. □

Lemma 2.4 ([1], Corollary 3)**.** *The hypercenter $Z_\infty(G)$ is the intersection of the normalizers of all Sylow subgroups of a group G.*

Lemma 2.5 ([4], Theorem 13.6)**.** *A group G is quasinilpotent if and only if $G/Z_\infty(G)$ is quasinilpotent.*

3. Main results

Now we are equipped to prove the main results.

Theorem 3.1. *Suppose that A and B are subgroups of a group G such that $G = AB$. If every Sylow subgroup of A is $BF^*(G)$-conjugate-permutable, and every Sylow subgroup of B is $AF^*(G)$-conjugate-permutable, then G is nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order.

Let p be any prime factor of $|G|$ and let $P \in \mathrm{Syl}_p(G)$, then there exist $P_A \in \mathrm{Syl}_p(A)$ and $P_B \in \mathrm{Syl}_p(B)$ such that $P = P_A P_B$. Keeping in mind that P_A is $BF^*(G)$-conjugate-permutable, it follows that P_A is $P_B F^*(G)$-conjugate-permutable. By Lemma 2.1, we have $P_A \lhd\lhd P_A P_B F^*(G)$, that is to say $P_A \lhd\lhd PF^*(G)$. Similarly, we have $P_B \lhd\lhd PF^*(G)$. Applying $P = \langle P_A, P_B \rangle$, we conclude that $P \lhd\lhd PF^*(G)$. Notice that $P \in \mathrm{Syl}_p(PF^*(G))$, we have $P \trianglelefteq PF^*(G)$, hence $F^*(G) \subseteq N_G(P)$. By Lemma 2.4 and the arbitrariness of p, we have $F^*(G) \subseteq Z_\infty(G)$, hence $F(G) = F^*(G) = Z_\infty(G)$.

We shall complete the proof with regard to whether $\Phi(G) = 1$ or not.

Case 1: $\Phi(G) \neq 1$. Let us consider the quotient group $G/\Phi(G)$.

Let $H/\Phi(G) = F^*(G/\Phi(G))$. We aim next to show that $H/\Phi(G) = F^*(G)/\Phi(G)$. It is clear that $F^*(G)/\Phi(G) \subseteq F^*(G/\Phi(G)) = H/\Phi(G)$, thus we are required to prove and $H/\Phi(G) \leqslant F^*(G)/\Phi(G)$. If not, we have $(H/\Phi(G))/(F^*(G)/\Phi(G)) > 1$. Notice that both $H/\Phi(G)$ and $F^*(G)/\Phi(G)$ are quasinilpotent, that also

$$(H/\Phi(G))/(F^*(G)/\Phi(G)) \cong H/F^*(G),$$

therefore $H/F^*(G)$ is quasinilpotent, which would imply that $H/Z_\infty(G)$ is quasinilpotent by $F^*(G) = Z_\infty(G)$. Note that

$$(H/Z_\infty(G))/(Z_\infty(H)/Z_\infty(G)) \cong H/Z_\infty(H),$$

therefore $H/Z_\infty(H)$ is quasinilpotent. We deduce from Lemma 2.5 the following fact that H is a normal quasinilpotent subgroup of G, so $H \subseteq F^*(G)$, in contradiction to $H/F^*(G) > 1$. Thus $F^*(G/\Phi(G)) = F^*(G)/\Phi(G)$.

Assume that p is any prime factor of $|A\Phi(G)/\Phi(G)|$, and $S/\Phi(G)$ is any Sylow p-subgroup of $A\Phi(G)/\Phi(G)$, thus there exists a Sylow p-subgroup P_A of A such that $P_A\Phi(G)/\Phi(G) = S/\Phi(G)$. By Lemma 2.2 we have that $P_A\Phi(G)/\Phi(G)$ is $BF^*(G)/\Phi(G)$-conjugate-permutable. Observe that $F^*(G)/\Phi(G) = F^*(G/\Phi(G))$, therefore

$$B \cdot F^*(G)/\Phi(G) = B\Phi(G)/\Phi(G) \cdot F^*(G)/\Phi(G) = B\Phi(G)/\Phi(G) \cdot F^*(G/\Phi(G)).$$

Hence $P_A\Phi(G)/\Phi(G)$ $(= S/\Phi(G))$ is $B\Phi(G)/\Phi(G) \cdot F^*(G/\Phi(G))$-conjugate-permutable, namely, every Sylow subgroup $S/\Phi(G)$ of $A\Phi(G)/\Phi(G)$ is $B\Phi(G)/\Phi(G) \times F^*(G/\Phi(G))$-conjugate-permutable. Similarly we deduce that every Sylow subgroup of $B\Phi(G)/\Phi(G)$ is $A\Phi(G)/\Phi(G) \cdot F^*(G/\Phi(G))$-conjugate-permutable. So the hypothesis of the theorem is inherited by $G/\Phi(G)$, and we have that $G/\Phi(G)$ is nilpotent by the minimality of G, hence G is nilpotent, a contradiction.

Case 2: $\Phi(G) = 1$. According to Lemma 2.3, we have $Z_\infty(G) = Z(G)$. Applying the result of the second paragraph, we conclude that $F^*(G) = Z(G)$, which shows that $G = C_G(F^*(G))$. Note that $C_G(F^*(G)) \subseteq F^*(G)$. It means that $G \subseteq F^*(G) = F(G)$, therefore G is nilpotent, a contradiction. $\square$

Remark 3.1. When G is a solvable group, we have $F^*(G) = F(G)$. So we deduce from Theorem 3.1 the following fact:

Corollary 3.1. *Let A and B be such subgroups of a solvable group G that $G = AB$. If every Sylow subgroup of A is $BF(G)$-conjugate-permutable, and every Sylow subgroup of B is $AF(G)$-conjugate-permutable, then G is nilpotent.*

Remark 3.2. In Theorem 3.1, let $A = 1$. We deduce the following fact:

Corollary 3.2. *If every Sylow subgroup of a group G is $F^*(G)$-conjugate-permutable, then G is nilpotent.*

Theorem 3.2. *Suppose that A and B are such subgroups of a solvable group G that $G = AB$, and P is a normal Sylow p-subgroup of G. If every Sylow subgroup of A is BP-conjugate-permutable, and every Sylow subgroup of B is AP-conjugate-permutable, then G is p-nilpotent.*

Proof. Assume that the theorem is false, let a counterexample G of smallest order be chosen. It is clear that $P \nleqslant \Phi(G)$. We show next that $\Phi(G) = 1$. If not, we have $P\Phi(G)/\Phi(G) \trianglelefteq G/\Phi(G)$ by $P \trianglelefteq G$, and $G/\Phi(G) = A\Phi(G)/\Phi(G) \times B\Phi(G)/\Phi(G)$. From Lemma 2.2 we deduce that the hypothesis of the theorem is inherited by $G/\Phi(G)$, and we have that $G/\Phi(G)$ is p-nilpotent by the minimality of G, hence G is p-nilpotent, a contradiction. Therefore $\Phi(G) = 1$.

Assume that q $(q \neq p)$ is any prime factor of $|G|$ and put $Q \in \mathrm{Syl}_q(G)$, then there exist $Q_A \in \mathrm{Syl}_q(A)$ and $Q_B \in \mathrm{Syl}_q(B)$ such that $Q = Q_AQ_B$. Keeping in mind that Q_A is BP-conjugate-permutable, it follows that Q_A is Q_BP-conjugate-permutable. We have $Q_A \lhd\lhd Q_AQ_BP$ by Lemma 2.1, that is to say $Q_A \lhd\lhd QP$. Similarly, we have $Q_B \lhd\lhd QP$. Applying $Q = \langle Q_A, Q_B\rangle$, we conclude that $Q \lhd\lhd QP$. Notice that Q is Sylow q-subgroup of QP, we have $Q \trianglelefteq QP$, then $P \subseteq N_G(Q)$. By the arbitrariness of q and Lemma 2.4 we have $P \subseteq Z_\infty(G)$. From $\Phi(G) = 1$ we deduce that $Z_\infty(G) = Z(G)$, which shows that $P \leqslant Z(G)$, therefore G is p-nilpotent, a contradiction. $\square$

Remark 3.3. In Theorem 3.2, let $A = 1$. We deduce the following fact:

Corollary 3.3. *Let P be a normal Sylow p-subgroup. If every Sylow subgroup of G is P-conjugate-permutable, then G is p-nilpotent.*

Theorem 3.3. *Let A and B be subgroups of a solvable group G such that $G = AB$, N be such a normal subgroup of G that G/N is nilpotent. If every Sylow subgroup of A is BN-conjugate-permutable, and every Sylow subgroup of B is AN-conjugate-permutable, then G is nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample of minimal order. It is clear that $1 < N \nleqslant \Phi(G)$. We show next that $\Phi(G) = 1$. If not, $1 < N\Phi(G)/\Phi(G) \trianglelefteq G/\Phi(G)$ by $N \trianglelefteq G$. Notice that

$$(G/\Phi(G))/(N\Phi(G)/\Phi(G)) \cong (G/N)/(N\Phi(G)/N),$$

we have that $(G/\Phi(G))/(N\Phi(G)/\Phi(G))$ is nilpotent since G/N is nilpotent. From Lemma 2.2 we deduce that the hypothesis of the theorem is inherited by $G/\Phi(G)$, so we may assume that $\Phi(G) = 1$ by the choice of G.

Assume that p is any prime factor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. Then there exist $P_A \in \mathrm{Syl}_p(A)$ and $P_B \in \mathrm{Syl}_p(B)$ such that $P = P_A P_B$. Keeping in mind that P_A is BN-conjugate-permutable, we have that P_A is $P_B N$-conjugate-permutable. So $P_A \lhd\lhd P_A P_B N$ by Lemma 2.1, that is to say $P_A \lhd\lhd PN$. Similarly, we have $P_B \lhd\lhd PN$. Applying $P = \langle P_A, P_B \rangle$, we conclude that $P \lhd\lhd PN$. Notice that P is Sylow p-subgroup of PN, we have $P \trianglelefteq PN$, which implies that $N \subseteq N_G(P)$. By the arbitrariness of p and Lemma 2.4 we have $N \subseteq Z_\infty(G)$. From $\Phi(G) = 1$ we deduce that $Z_\infty(G) = Z(G)$, so $N \leqslant Z(G)$. Observe that G/N is nilpotent, therefore G is nilpotent, a contradiction. □

Remark 3.4. In Theorem 3.3, let $N = G$. We deduce the following fact:

Corollary 3.4. *Let A and B be such subgroups of a solvable group G that $G = AB$. If every Sylow subgroup of A and B is conjugate-permutable in G, then G is nilpotent.*

Remark 3.5. In Theorem 3.3, let $A = 1$. We deduce the following fact:

Corollary 3.5. *Suppose that N is a normal subgroup of G such that G/N is nilpotent. If every Sylow subgroup of G is N-conjugate-permutable, then G is nilpotent.*

Remark 3.6. Setting $N = F^*(N)$ in Corollary 3.5 and combining it with Corollary 3.2, we get the following fact:

Corollary 3.6. *Suppose that N is a normal subgroup of G such that G/N is nilpotent. If every Sylow subgroup of G is $F^*(N)$-conjugate-permutable, then G is nilpotent.*

P r o o f. Assume that p is any prime factor of $|N|$, and P_0 is any Sylow p-subgroup of N. Then there exists $P \in \mathrm{Syl}_p(G)$ such that $P_0 = P \cap N$. Also P is $F^*(N)$-conjugate-permutable, so we have $P \lhd\lhd PF^*(N)$ by Lemma 2.1. Notice that P is a Sylow p-subgroup of $PF^*(N)$, thus $P \trianglelefteq PF^*(N)$, which shows that $F^*(N) \leqslant N_G(P)$. Also $N \trianglelefteq G$ and $P_0 = P \cap N$, so $F^*(N) \leqslant N_G(P_0)$, therefore P_0 is $F^*(N)$-conjugate-permutable. Applying Corollary 3.2 we conclude that N is nilpotent, and so $N = F^*(N)$. By Corollary 3.5 we have G is nilpotent. □

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