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THE INDEX FOR FREDHOLM ELEMENTS IN A BANACH ALGEBRA VIA A TRACE II

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Abstract. We show that the index defined via a trace for Fredholm elements in a Banach algebra has the property that an index zero Fredholm element can be decomposed as the sum of an invertible element and an element in the socle. We identify the set of index zero Fredholm elements as an upper semiregularity with the Jacobson property. The Weyl spectrum is then characterized in terms of the index.

Keywords: trace; index; Fredholm element

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1. Introduction

In [4] we defined, via a trace, an index for Fredholm elements relative to the socle in a semisimple Banach algebra. We then succeeded in developing a fairly complete theory for this trace index. One important result that evaded us was the decomposition result for index zero Fredholm elements, stating that if the index of a Fredholm element a is zero, then $a = x + y$ with x invertible and y an element of the socle.

This result is one of the fundamental results in Fredholm theory and holds for the usual index for Fredholm operators and also for the index defined by Kraljević, Suljagić and Veselić in [6] for Fredholm elements in a Banach algebra. We also provided in [4] an alternative, simple proof of their result, but were not able to prove it for the trace index. The aim of this short note is to fill this gap. The main tool used in the proof is the Sinclair version of Jacobson's density theorem (see [1], Corollary 4.2.6).

2. Preliminaries and notation

Throughout this paper we will consider A to be a complex Banach algebra with identity 1. We denote by A^{-1} the group of invertible elements in A and by $\sigma(a, A)$ or simply $\sigma(a)$ the spectrum of $a \in A$. By an ideal in A we mean a two sided ideal in A. An ideal I in A is said to be *inessential* [1], page 106, if for every element a in I its spectrum is either a finite set or a sequence converging to zero.

The radical of A will be denoted by $\operatorname{Rad} A$ and A is said to be *semisimple* if $\operatorname{Rad} A = \{0\}$. The set $\operatorname{kh}(I)$ is defined by $\operatorname{kh}(I) := \{x \in A;\ x + I \in \operatorname{Rad}\ A/I\}$. An element $a \neq 0$ in a semisimple Banach algebra A is called *rank one* if there exists a linear functional f_a on A such that $axa = f_a(x)a$ for all $x \in A$. For properties of these elements we refer to [10].

A *minimal idempotent* in A is a nonzero idempotent p such that pAp is a division algebra. Minimal idempotents are examples of rank one elements ([3], Proposition 31.3), and conversely if a is a rank one element, then $p = f_a(1)^{-1}a$ is a minimal idempotent. The set of *finite rank elements* of A, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$ of the form $a = \sum\limits_{i=1}^{n} a_i$ with each a_i a rank one element. In the case of a semiprime Banach algebra $\mathcal{F}$ coincides with the socle of A (denoted by $\operatorname{Soc} A$), which is defined to be the sum of the minimal ideals in A. By [10], Lemma 2.7, $\mathcal{F}(A)$ is an ideal in A.

Let I be an ideal in a Banach algebra A. A function $\tau\colon I \to \mathbb{C}$ is called a *trace* on I if

(TN) $\tau(p) = 1$ for every rank one idempotent $p \in I$,
(TA) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in I$,
(TH) $\tau(\alpha a) = \alpha\tau(a)$ for all $\alpha \in \mathbb{C}$ and $a \in I$,
(TC) $\tau(ba) = \tau(ab)$ for all $a \in I$ and $b \in A$.

We shall refer to an ideal on which a trace is defined as a *trace ideal.*

Definition 2.1. Let I be an ideal in a Banach algebra A. We call an element $a \in A$ a *Fredholm element relative to* I if there exists an element $a_0 \in A$ such that

(i) $aa_0 - 1 \in I$;
(ii) $a_0a - 1 \in I$.

The set of all Fredholm elements relative to I is denoted by $\Phi(A, I)$.

Clearly, $a \in \Phi(A, I)$ if and only if $\bar{a} = a + I$ is invertible in the quotient algebra A/I. Also, $A^{-1} \subset \Phi(A, I)$ and $\Phi(A, I)$ is a multiplicative semi-group.

The following definition of an index function with the aid of a trace was suggested by Grobler in [4]. We refer the reader to [4], Example 3.2, which motivated this definition in the case of Fredholm operators on a Banach space.

Definition 2.2. Let τ be a trace on the ideal I in A. We define the *index function* $\iota\colon \Phi(A,I) \to \mathbb{C}$ by

$$\iota(a) := \tau(aa_0 - a_0a) = \tau([a, a_0]) \quad \text{for all } a \in \Phi(A,I)$$

where $a_0 \in A$ satisfies $aa_0 - 1 \in I$ and $a_0a - 1 \in I$.

That this function has all the required properties of an index was shown in [4].

Let $a \in A$. An idempotent p is called a *left Barnes idempotent* for $a \in A$ if $aA = (1-p)A$, and a *right Barnes idempotent* for $a \in A$ if $Aa = A(1-p)$. Also, for $a \in A$ we define the *right* and *left* annihilators, respectively, as the sets

$$N_r(a) := \{x \in A;\ ax = 0\} \quad \text{and} \quad N_l(a) := \{x \in A;\ xa = 0\}.$$

In general, a Barnes idempotent belonging to a given element $a \in A$ is not unique (see for instance the remark preceding [4], Corollary 3.15).

Definition 2.3 ([6], Definition 1.2)**.** A nonempty subset R of A is called a *regularity* if

(1) $a \in A$ and $n \in \mathbb{N}$ then $a \in R \iff a^n \in R$;
(2) a, b, c, d are commuting elements of A and $ac+bd = 1$, then $ab \in R \iff a \in R$ and $b \in R$.

A subset $R \subset A$ is a regularity if it satisfies the condition

(P1) $\quad ab \in R \iff a \in R$ and $b \in R$ for all commuting elements $a, b \in A$.

For examples and unexplained notions on regularities we refer the reader to [5], [7].

The notion of a regularity can be weakened in the following way:

Definition 2.4 ([9], Definition 10)**.** A nonempty subset $R \subset A$ is called an *upper semiregularity* if

(1) $a \in A$ and $n \in \mathbb{N}$ then $a \in R \Longrightarrow a^n \in R$,
(2) a, b, c, d are commuting elements of A and $ac + bd = 1$, then $a \in R$ and $b \in R$ implies $ab \in R$,
(3) R contains a neighbourhood of the identity element 1.

A semigroup $R \subset A$ satisfies conditions 1 and 2 of Definition 2.4 and so any semigroup containing a neighbourhood of 1 is an upper semiregularity.

Every regularity (semiregularity) $R \subset A$ defines for every $a \in A$ a spectrum $\sigma_R(a) \subset \mathbb{C}$ by

$$\sigma_R(a) := \{\lambda \in \mathbb{C};\ a - \lambda \notin R\}.$$

$\sigma_R(a)$ is called the spectrum of a corresponding to R. Properties of semiregularities can be found in [9] or [8], Section 23.

Finally, we will say that a regularity (semiregularity) R in A has the *Jacobson property* if for every $a, b \in A$ and nonzero $\lambda \in \mathbb{C}$ we have $\lambda - ab \in R$ if and only if $\lambda - ba \in R$. We can say equivalently that the spectrum σ_R has the *Jacobson property* if $\sigma_R(ab) \setminus \{0\} = \sigma_R(ba) \setminus \{0\}$ for all $a, b \in A$.

3. Main results

Let A be a semisimple Banach algebra and let I be a trace ideal in A satisfying $\operatorname{Soc} A \subset I \subset \operatorname{kh} \operatorname{Soc} A$. We denote by $\Phi_0(A, I) \subset \Phi(A, I)$ the set of Fredholm elements of index zero. As stated in the introduction our main result is that $\Phi_0(A, I) = A^{-1} + \operatorname{Soc} A$, that is, every Fredholm element of index zero can be written as the sum of an invertible element and an element in the socle of A.

We recall that two elements a and b in A are called similar if there exists an element $u \in A^{-1}$ such that $a = u^{-1}bu$. As a first result we prove a condition enabling one to find a decomposition for a Fredholm element. Both in this proof and the next one, the reader will find similarities with the proof of [4], Theorem 3.19,

Theorem 3.1. *Let A be a semisimple Banach algebra and let I be a trace ideal in A satisfying $\operatorname{Soc} A \subset I \subset \operatorname{kh} \operatorname{Soc} A$. Let $a \in \Phi(A, I)$ and let p and q be left and right Barnes idempotents for a, respectively. If p and q are similar, then $a = x + y$ with $x \in A^{-1}$ and $y \in \operatorname{Soc} A$.*

Proof. Let $a \in \Phi(A, I)$. Since p and q are Barnes idempotents for a, there exists an element $a_0 \in A$ such that $aa_0 = 1 - p$ and $a_0 a = 1 - q$ ([4], Theorem 3.11). If p and q are similar, there exists an element $u \in A^{-1}$ such that $up = qu$. Put $\bar{u} = up = qu$ and $\bar{v} = pu^{-1} = u^{-1}q$. Then $\bar{v}\bar{u} = p$ and $\bar{u}\bar{v} = q$. It follows from [4], Proposition 3.10, that $\bar{u} \in N_l(a) \cap N_r(a)$ and that $\bar{v} \in N_l(a_0) \cap N_r(a_0)$. We therefore get

$$(a + \bar{v})(a_0 + \bar{u}) = 1 - p + p = 1 \quad \text{and} \quad (a_0 + \bar{u})(a + \bar{v}) = 1 - q + q = 1.$$

Thus, $a = (a + \bar{v}) - \bar{v}$ with $a + \bar{v} \in A^{-1}$ and $\bar{v} = pu^{-1} \in \operatorname{Soc} A$, since $\operatorname{Soc} A$ is an ideal and $p \in \operatorname{Soc} A$. This completes the proof. □

Corollary 3.2. *If $a \in \Phi(A, I)$ is such that it has similar associated Barnes idempotents, then $a \in \Phi_0(A, I)$.*

Proof. Let $a = x + y \in A^{-1} + \operatorname{Soc} A$. Then

$$\iota(a) = \iota(x + y) = \iota(x) = 0.$$

□

A result of Zemánek [2], Lemma 2.5, states that if the Barnes idempotents p and q satisfy $\|p-q\|<1$, then they are similar. Now $\|p-q\|<1$ if and only if $\|aa_0-a_0a\|<1$ and so this condition implies by Corollary 3.2 that $\iota(a)=0$.

For a closed ideal $I\subset A$ we call an element a, a *Weyl element* with respect to I if $a=x+y\in A^{-1}+I$. We denote by $\mathcal{W}(A,I)$ the set of Weyl elements with respect to I. We then know from Theorem 3.1 that all elements in $\Phi(A,I)$ with similar Barnes idempotents are Weyl elements and the proof of the corollary above can be repeated to get the following corollary.

Corollary 3.3. *Let I be a closed trace ideal in a semisimple Banach algebra A such that $\operatorname{Soc}A\subset I\subset \operatorname{kh}\operatorname{Soc}A$. Then $\mathcal{W}(A,I)\subset\Phi_0(A,I)$.*

Our aim now is to reverse these inclusions. This brings us to our main theorem.

Theorem 3.4. *Let A be a semisimple Banach algebra and let I be a trace ideal in A with $\operatorname{Soc}A\subset I\subset\operatorname{kh}\operatorname{Soc}A$. If $a\in\Phi(A,I)$ with $\iota(a)=0$, then $a=x+y$ with $x\in A^{-1}$ and $y\in\operatorname{Soc}A$, that is, $\Phi_0(A,I)=A^{-1}+\operatorname{Soc}A$.*

Proof. Let $a\in\Phi(A,I)$ with $\iota(a)=0$. By [4], Corollary 3.13, Theorem 3.14, $\iota(a)=\tau(p)-\tau(q)$ where $\tau(p)$ is equal to the cardinality of a maximal set of orthogonal minimal idempotents in $N_l(a)=Ap$ and similarly, $\tau(q)$ is equal to the cardinality of a maximal set of orthogonal minimal idempotents in $N_r(a)=qA$. Let k be the common cardinality and let $\{e_1,e_2,\dots,e_k\}$ be a maximal subset of orthogonal minimal idempotents in $N_l(a)$.

Then $N_l(a)=\sum\limits_{i=1}^{k}Ae_i$ and we can write $p=x_1e_1+x_2e_2+\ldots+x_ke_k$. Likewise, let $\{f_1,f_2,\dots,f_k\}$ be a maximal subset of orthogonal minimal idempotents in $N_r(a)$. Again, $N_r(a)=\sum\limits_{i=1}^{k}f_iA$ and we can write $q=f_1y_1+f_2y_2+\ldots+f_ky_k$. Since the sets $\{x_1e_1,x_2e_2,\dots,x_ke_k\}$ and $\{f_1y_1,f_2y_2,\dots,f_ky_k\}$ are linearly independent, the Sinclair version of Jacobson's density theorem ([1], Corollary 4.2.6), implies that there exists an element $u\in A^{-1}$ such that $ux_ie_i=f_iy_i$ for $i=1,\dots,k$ and consequently, $up=q$.

Put $\bar u=up=q$ and $\overline{v}=p=u^{-1}q$. Then $\bar u\overline{v}=up^2=up=q$ and $\overline{v}\bar u=u^{-1}q^2=u^{-1}q=p$. In view of [4], Proposition 3.10, we have $\bar u\in N_l(a)\cap N_r(a)$ and $\overline{v}\in N_l(a_0)\cap N_r(a_0)$. It follows that

$$(a+\overline{v})(a_0+\bar u)=1-p+p=1\quad\text{and}\quad(a_0+\bar u)(a+\overline{v})=1-q+q=1.$$

Hence, $a=(a+\overline{v})-\overline{v}$ with $a+\overline{v}\in A^{-1}$ and $\overline{v}=p\in\operatorname{Soc}A$. This completes the proof. $\square$

Corollary 3.5. *Let I be a closed trace ideal in a semisimple Banach algebra A such that* $\operatorname{Soc} A \subset I \subset \operatorname{kh} \operatorname{Soc} A$. *Then* $\mathcal{W}(A, I) = \Phi_0(A, I)$.

We now link our result to the spectral theory:

Theorem 3.6. *Let I be a closed trace ideal in a semisimple Banach algebra A such that* $\operatorname{Soc} A \subset I \subset \operatorname{kh} \operatorname{Soc} A$. *Then* $\mathcal{W}(A, I) = \Phi_0(A, I)$ *is an upper semiregularity with the Jacobson property.*

Proof. Since $\Phi_0(A, I)$ is an open semigroup containing 1 (see [4], Proposition 3.5 and Proposition 3.7), it is an upper semiregularity. Let $0 \neq \lambda \in \mathbb{C}$ and $a, b \in A$ with $\lambda - ab \in \Phi_0(A, I)$. By [4], Theorem 3.20, we have

$$0 = \iota(\lambda - ab) = \iota(\lambda - ba)$$

and so $\lambda - ba \in \Phi_0(A, I)$.

The converse follows by symmetry and we are done. □

A characterization of the Weyl spectrum follows immediately for the case we consider.

Corollary 3.7. *Let I be a closed trace ideal in a semisimple Banach algebra A such that* $\operatorname{Soc} A \subset I \subset \operatorname{kh} \operatorname{Soc} A$. *Then for every $a \in A$*

$$\sigma_{\mathcal{W}}(a) = \{\lambda \in \mathbb{C};\ \lambda - a \text{ is not Fredholm or } \iota(\lambda - a) \neq 0\}.$$

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