Jinjun Li; Min Wu

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POINTS WITH MAXIMAL BIRKHOFF AVERAGE OSCILLATION

Jinjun Li, Zhangzhou, Min Wu, Guangzhou

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Abstract. Let $f\colon X \to X$ be a continuous map with the specification property on a compact metric space X. We introduce the notion of the maximal Birkhoff average oscillation, which is the "worst" divergence point for Birkhoff average. By constructing a kind of dynamical Moran subset, we prove that the set of points having maximal Birkhoff average oscillation is residual if it is not empty. As applications, we present the corresponding results for the Birkhoff averages for continuous functions on a repeller and locally maximal hyperbolic set.

Keywords: irregular set; maximal Birkhoff average oscillation; specification property; residual set

MSC 2010: 54H20, 54E52, 37C45

1. Introduction

The aim of the paper is to generalize our former work [18]. Let us begin with recalling some notation in [18]. Let $f\colon X \to X$ be a continuous map on a compact metric space (X, d). For $x \in X$ and $n \in \mathbb{N}$, let $f^n(x)$ denote the n-th iterate of x under f. That is, $f^n(x) = f(f^{n-1}(x))$ and $f^0(x) = x$. Let $\varphi : X \to \mathbb{R}$ be a continuous function. The Birkhoff average of φ, denoted by $B_n(\varphi, x)$, is defined by

$$B_n(\varphi, x) = \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)). \tag{1.1}$$

The set consisting of those points for which the limit $\lim\limits_{n\to\infty} B_n(\varphi, x)$ does not exist is called the *irregular set* (or the set of divergence points) for φ and it is denoted

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by $X_{\varphi,f}$. More precisely,

$$X_{\varphi,f} = \Big\{x \in X\colon \liminf_{n\to\infty} B_n(\varphi,x) < \limsup_{n\to\infty} B_n(\varphi,x)\Big\}.$$

The irregular set arises naturally in the context of multifractal analysis, where one decomposes the space X into the disjoint union

$$X = \bigcup_{\alpha\in\mathbb{R}} X_\varphi(\alpha) \cup X_{\varphi,f}.$$

Here $X_\varphi(\alpha)$ denotes the level set

$$X_\varphi(\alpha) = \Big\{x \in X\colon \lim_{n\to\infty} B_n(\varphi,x) = \alpha\Big\}. \tag{1.2}$$

It follows from the Birkhoff ergodic theorem that an irregular set has zero measure with respect to any invariant measure. Therefore, the set $X_{\varphi,f}$ is often ignored in ergodic theory. However, there is now an extensive literature showing that the irregular sets can be large from other points of view, such as from the points of view of topological entropy and Hausdorff dimension, see [4], [5], [8], [11], [13], [16], [21], [19], [17], [23], [26], [27], [32] and references therein. In particular, under the assumption that f satisfies the specification property, it was shown by Chen, Tassilo and Shu [8] that the irregular set $X_{\varphi,f}$ has full topological entropy and by Thompson [32] that it has full topological pressure if it is not empty. Recall that a map f is said to have the *specification property* if for each $\varepsilon > 0$ there exists an integer $m = m(\varepsilon)$ such that for any collection $\{I_j = [a_j, b_j]\colon\ a_j, b_j \in \mathbb{N},\ j = 1,\dots,k\}$ of finite intervals with $a_{j+1} - b_j \geqslant m(\varepsilon)$ for $j = 1,\dots,k-1$ and any $x_1,\dots,x_k$ in X, there exists a point $x \in X$ satisfying

$$d(f^{p+a_j}(x), f^p(x_j)) < \varepsilon \tag{1.3}$$

for all $p = 0,\dots,b_j - a_j$ and $j = 1,\dots,k$. The specification property was first introduced by Bowen [6] who required x to be periodic. We say that $f\colon X \to X$ satisfies the *Bowen specification property* if under the assumptions of the above definition and for every $p \geqslant b_k - a_1 + m(\varepsilon)$, there exists a p-periodic point $x \in X$ satisfying (1.3). It has turned out to be very useful in spite of its rather complicated appearance. The reader can refer to [7], [9], [15], [30] for results about the specification property (particularly the Bowen specification).

Recall that in a metric space X, a set R is called *residual* if its complement is of the first category. Moreover, in a complete metric space a set is residual if it contains a dense G_δ set, see [25]. We say that a set is large from the topological point of view if it is residual. Recently, some results show that certain irregular sets can also be large from the topological point of view. For example, Volkmann [33], Šalát [29], Albeverio, Pratsiovytyi and Torbin [1], Hyde et al. [14] and Olsen [22] proved that some kinds of irregular sets associated with integer expansion are residual. Baek and Olsen [2] discussed the set of extremely non-normal points of a self-similar set from the topological point of view. Li and Wu [20] proved that the set of divergence points of self-similar measure with the open set condition is either residual or empty. Barreira, Li and Valls [3] proved that the irregular set for a continuous function on a certain subshift is either residual or empty. More generally, Li and Wu [18] proved the following result.

Theorem 1.1 ([18])**.** *Let $f\colon X \to X$ be a continuous map with the specification property on a compact metric space X and let $\varphi\colon X \to \mathbb{R}$ be a continuous function. Then the set $X_{\varphi,f}$ is residual if it is not empty.*

In this paper, we continue to consider a class of refined subsets of $X_{\varphi,f}$ from the topological point of view.

Next we would like to present the motivation for this work by recalling one of results by Denker, Grillenberger and Sigmund [9]. For $x \in X$, let $V_f(x)$ denote the set of accumulation points of the sequence $n \to n^{-1} \sum_{i=0}^{n-1} \delta_{f^i x}$, where δ_x denotes the Dirac measure. Denoting by $\mathcal{M}_f$ the set of all f-invariant probability measures on X, a point $x \in X$ is said to have *maximal oscillation* if $V_f(x) = \mathcal{M}_f$.

Theorem 1.2 ([9])**.** *Let $f\colon X \to X$ be a continuous map with the Bowen specification property on a compact metric space X. Then the set of points having maximal oscillation is residual in X.*

In this paper, we are interested in the set consisting of the "worst" divergence points for the Birkhoff average. To state our main result, we need to introduce a notion which is inspired by the notion of maximal oscillation. Let us first introduce some notation. Write

$$\mathcal{L}_\varphi = \{\alpha \in \mathbb{R}\colon\ X_\varphi(\alpha) \neq \emptyset\}.$$

It follows from the fact that any invariant measure of the map with the specification property has a generic point (see, for example, [9]) that

$$\mathcal{L}_\varphi = \left\{ \int_X \varphi \, \mathrm{d}\mu\colon\ \mu \in \mathcal{M}_f \right\}.$$

Since $\mathcal{M}_f$ is compact and connected, and the map $\mu \mapsto \int_X \varphi \, \mathrm{d}\mu$ is continuous, the set $\mathcal{L}_\varphi$ is a closed interval when f has the specification property.

Let $A_\varphi(x)$ denote the set of accumulation points of the sequence $n \mapsto B_n(\varphi, x)$. Let us remark that the set $A_\varphi(x)$ is a closed interval for any $x \in X$, see, for example, [19] or [24]. That is,

$$A_\varphi(x) = \Big[\liminf_{n\to\infty} B_n(\varphi, x),\ \limsup_{n\to\infty} B_n(\varphi, x)\Big].$$

A point $x \in X$ is said to have *maximal Birkhoff average oscillation* if $A_\varphi(x) = \mathcal{L}_\varphi$. Let $X_{\max}$ denote the set of points having maximal Birkhoff average oscillation. That is,

$$X_{\max} = \{x \in X\colon\ A_\varphi(x) = \mathcal{L}_\varphi\}.$$

Intuitively, we feel that the set $X_{\max}$ shall be "small". However, under the hypothesis that f satisfies the specification property we will show that the set $X_{\max}$ is large from the topological point of view. More precisely, we will prove the following result.

Theorem 1.3. *Let $f\colon\ X \to X$ be a continuous map with the specification property on a compact metric space X and let $\varphi\colon\ X \to \mathbb{R}$ be a continuous function. Then the set $X_{\max}$ is residual if it is not empty.*

We end this section by giving some remarks on Theorem 1.3. Obviously, Theorem 1.1 follows readily from Theorem 1.3 since $X_{\max} \subset X_{\varphi,f}$. In the setting of frequencies of N-adic digits, several authors have studied "points of maximal oscillation" and obtained results similar to Theorem 1.3, see [2], [14], [22], [33].

2. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. To prove Theorem 1.3, it suffices to show that there exists a set $F \subset X$ with the following properties:

(1) $F \subset X_{\max}$;
(2) F is dense in X;
(3) F is a G_δ set.

2.1. The construction of the desired set. We first present a construction of a set of Moran type. The approach to the construction is inspired by the idea in [10], [12], [31], [18].

Fix $\varepsilon > 0$. Let $\{m_k\}_{k\geqslant 0}$ be the sequence of integers defined by $m_k = m(2^{-k}\varepsilon)$ which is the constant appearing in the definition of the specification property. Let $\{W_k\}_{k\geqslant 0}$ be a sequence of finite sets in X with $W_0 = \{x_0\} \subset X$, and $\{n_k\}_{k\geqslant 0}$ a sequence of positive integers. Assume that

$$d_{n_k}(x, y) \geqslant 8\varepsilon, \quad x, y \in W_k,\ x \neq y. \tag{2.1}$$

Let $\{N_k\}_{k\geqslant 0}$ be another sequence of positive integers with $N_0 = 1$. Using these data, we are going to construct a subset of Cantor type, which will be denoted by $F = F(\varepsilon, \{x_0\}, \{W_k\}, \{n_k\}, \{N_k\})$.

Denote

$$M_k = \#W_k,$$

where $\#A$ denotes the cardinality of the set A. Fix $k \geqslant 0$. For any N_k points $x_1, \dots, x_{N_k}$ in W_k, i.e., $(x_1, \dots, x_{N_k}) \in W_k^{N_k}$, we choose a point $y(x_1, \dots, x_{N_k}) \in X$ such that

$$d_{n_k}(x_j, f^{a_j} y(x_1, \dots, x_{N_k})) < \frac{\varepsilon}{2^k}, \quad j = 1, \dots, N_k, \tag{2.2}$$

where $a_j = (j-1)(n_k + m_k)$. Such a point $y(x_1, \dots, x_{N_k})$ exists because f has the specification property. We claim that for two distinct points $(x_1, \dots, x_{N_k})$ and $(\overline{x}_1, \dots, \overline{x}_{N_k})$ in $W_k^{N_k}$,

$$d_{t_k}(y(x_1, \dots, x_{N_k}), y(\overline{x}_1, \dots, \overline{x}_{N_k})) > 6\varepsilon, \tag{2.3}$$

where $t_k = a_{N_k} + n_k$, i.e.,

$$t_k = (N_k - 1)m_k + N_k n_k.$$

In fact, let $y = y(x_1, \dots, x_{N_k})$ and $\overline{y} = y(\overline{x}_1, \dots, \overline{x}_{N_k})$. Suppose $x_s \neq \overline{x}_s$ for some $s \in \{1, \dots, N_k\}$. Then

$$\begin{aligned} d_{t_k}(y, \overline{y}) &\geqslant d_{n_k}(f^{a_s}(y), f^{a_s}(\overline{y})) \\ &\geqslant d_{n_k}(x_s, \overline{x}_s) - d_{n_k}(x_s, f^{a_s}(y)) - d_{n_k}(\overline{x}_s, f^{a_s}(\overline{y})) \\ &\geqslant 8\varepsilon - \varepsilon - \varepsilon = 6\varepsilon. \end{aligned}$$

Let

$$D_0 = W_0, \quad D_k = \{y(x_1, \dots, x_{N_k})\colon\ (x_1, \dots, x_{N_k}) \in W_k^{N_k}\}, \quad k \geqslant 1.$$

Now we will define recursively L_k and l_k as follows. Put

$$L_0 = D_0 = W_0, \quad l_0 = n_0.$$

Suppose we have already defined the set L_k, now we present a construction of L_{k+1}. Let

$$l_{k+1} = l_k + m_{k+1} + t_{k+1} = N_0 n_0 + \sum_{i=1}^{k+1} N_i(n_i + m_i).$$

For every $x \in L_k$ and $y \in D_{k+1}$ let $z = z(x,y)$ be a point such that

$$d_{l_k}(x,z) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{t_{k+1}}(y, f^{l_k+m_{k+1}}(z)) < \frac{\varepsilon}{2^{k+1}}. \tag{2.4}$$

Such a point exists due to the specification property of f. Collect all these points into the set

$$L_{k+1} = \{z = z(x,y)\colon\ x \in L_k,\ y \in D_{k+1}\}.$$

For any $x \in L_k$ and $y, \overline{y} \in D_{k+1}$ with $y \neq \overline{y}$, it follows from (2.3) and (2.4) that

$$d_{l_k}(z(x,y), z(x,\overline{y})) < \frac{\varepsilon}{2^k} \quad \text{and} \quad d_{l_{k+1}}(z(x,y), z(x,\overline{y})) > 5\varepsilon. \tag{2.5}$$

For every $k \geqslant 0$, put

$$F_k = \bigcup_{x \in L_k} \widetilde{B}_{l_k}\Big(x, \frac{\varepsilon}{2^k}\Big), \tag{2.6}$$

where

$$\widetilde{B}_{l_k}(x,\delta) = \{y \in X\colon\ d(f^i(x), f^i(y)) < \delta,\ d(f^j(x), f^j(y)) \leqslant \delta,$$
$$i = 0, 1, \dots, l_{k-1} - 1,\ j = l_{k-1}, \dots, l_k - 1\}.$$

By (2.5) one can prove that for any $x, \overline{x} \in L_k$ with $x \neq \overline{x}$, the sets $\widetilde{B}_{l_k}(x, \varepsilon/2^k)$, $\widetilde{B}_{l_k}(\overline{x}, \varepsilon/2^k)$ are disjoint and $F_{k+1} \subset F_k$.

Finally, define

$$F(\varepsilon, \{x_0\}) := F(\varepsilon, \{x_0\}, \{W_k\}, \{n_k\}, \{N_k\}) = \bigcap_{k=1}^{\infty} F_k.$$

Remark 2.1. It is not difficult to check that $d(x_0, y) < \varepsilon$ for any $y \in F(\varepsilon, \{x_0\})$.

Next, we will prepare specific data $\{W_k\}$, $\{n_k\}$ and $\{N_k\}$, then use the approach described above to construct a dense G_δ subset $F \subset X$ such that $F \subset X_{\max}$.

Let $k \in \mathbb{N}$. Choose $\alpha_{k,1}, \dots, \alpha_{k,q_k} \in \mathcal{L}_\varphi$ such that

$$\mathcal{L}_\varphi \subset \bigcup_{i=1}^{q_k} B\Big(\alpha_{k,i}, \frac{1}{k}\Big), \quad |\alpha_{k,i+1} - \alpha_{k,i}| < \frac{1}{k} \quad \text{for all } i, \quad \text{and}$$
$$|\alpha_{k,q_k} - \alpha_{k+1,1}| < \frac{1}{k}. \tag{2.7}$$

Fix $\varepsilon > 0$. Let $\{m_k\}_{k\geqslant 0}$ be the sequence of integers defined by $m_k = m(2^{-k}\varepsilon)$, which is the constant appearing in the definition of the specification property.

For $\alpha \in \mathcal{L}_\varphi$, $\delta > 0$ and $n \in \mathbb{N}$ put

$$P(\alpha, \delta, n) = \{x \in X\colon\ |B_n(\varphi, x) - \alpha| < \delta\}.$$

Given $\delta > 0$, we have $P(\alpha, \delta, n) \neq \emptyset$ for each $\alpha \in \mathcal{L}_\varphi$ and any sufficiently large n. Now choose sequences $\{\delta_{k,i}\}_{k\in\mathbb{N},\, i=1,\ldots,q_k}$ and $\{n_{k,i}\}_{k\in\mathbb{N},\, i=1,\ldots,q_k}$ with

$$\delta_{1,1} > \delta_{1,2} > \ldots > \delta_{1,q_1} > \delta_{2,1} > \delta_{2,2} > \ldots > \delta_{2,q_2} > \ldots,$$

$$n_{1,1} < n_{1,2} < \ldots < n_{1,q_1} < n_{2,1} < n_{2,2} < \ldots < n_{2,q_2} < \ldots$$

such that for any $k \in \mathbb{N}$ and $i = 1, \ldots, q_k$

$$P(\alpha_{k,i}, \delta_{k,i}, n_{k,i}) \neq \emptyset. \tag{2.8}$$

Let $D = \{d_1, d_2, \ldots, d_v, \ldots\} \subset X$ be a countable dense set (note that X is compact). Fix $d_v \in D$ and let $W_0^v = \{d_v\}$. For $k \in \mathbb{N}$ and $i \in \{1, \ldots, q_k\}$, let

$$W_{k,i}^v = \{x_j^{k,i}\colon\ j = 1, \ldots, M_{k,i}^v\}$$

be one of those maximal $(n_{k,i}, 8\varepsilon)$-separated sets in $P(\alpha_{k,i}, \delta_{k,i}, n_{k,i})$.

Choose a sequence of integers $\{N_{k,i}^v\}_{k\in\mathbb{N},\, i=1,\ldots,q_k}$ such that the following conditions are satisfied:

(i) $N_{k,i}^v \geqslant 2^{n_{k,i+1}+m_k}$ for $k \geqslant 1, 1 \leqslant i \leqslant q_k - 1$ and $N_{k,q_k}^v \geqslant 2^{n_{k+1,1}+m_{k+1}}$ for $k \geqslant 1$;

(ii) $N_{k,i+1}^v \geqslant 2^{N_{1,1}^v n_{1,1} + N_{1,2}^v(n_{1,2}+m_1)+\ldots+N_{k,i}^v(n_{k,i}+m_k)}$ for $k \geqslant 1$, $1 \leqslant i \leqslant q_k - 1$ and $N_{k+1,1}^v \geqslant 2^{N_{1,1}^v n_{1,1} + N_{1,2}^v(n_{1,2}+m_1)+\ldots+N_{k,q_k}^v(n_{k,q_k}+m_k)}$ for $k \geqslant 1$.

Now, let

$$\begin{aligned}
&(n_0, n_1, n_2, n_3, \ldots) = (1, n_{1,1}, n_{1,2}, \ldots, n_{1,q_1}, n_{2,1}, n_{2,2}, \ldots);\\
&(W_0, W_1, W_2, W_3, \ldots) = (\{d_v\}, W_{1,1}^v, W_{1,2}^v, \ldots, W_{1,q_1}^v, W_{2,1}^v, W_{2,2}^v, \ldots);\\
&(N_0, N_1, N_2, N_3, \ldots) = (1, N_{1,1}^v, N_{1,2}^v, \ldots, N_{1,q_1}^v, N_{2,1}^v, N_{2,2}^v, \ldots).
\end{aligned}$$

By these data and the construction approach presented in the former paragraph, we obtain the set

$$F(\varepsilon, \{d_v\}) = F(\varepsilon, \{d_v\}, \{W_k^v\}, \{n_k\}, \{N_k^v\}) = \bigcap_{k=1}^{\infty} \bigcap_{i=1}^{q_k} \bigcup_{x \in L_{k,i}^v} \widetilde{B}_{l_{k,i}}\Bigl(x, \frac{\varepsilon}{2^{r_{k-1}+i}}\Bigr).$$

Here, and in the sequel, $r_k = q_1 + \ldots + q_k$ and $r_0 = 0$.

Write

$$F(\varepsilon)=\bigcup_{v=1}^{\infty}F(\varepsilon,\{d_v\}).$$

Finally, let

$$F'=\bigcup_{j=1}^{\infty}F\Big(\frac{1}{j}\Big)=\bigcup_{j=1}^{\infty}\bigcup_{v=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcap_{i=1}^{q_k}\bigcup_{x\in L_{k,i}^v}\widetilde{B}_{l_{k,i}}\Big(x,\frac{1}{j2^{r_{k-1}+i}}\Big)$$

and

$$F=\bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcup_{v=1}^{\infty}\bigcap_{i=1}^{q_k}\bigcup_{x\in L_{k,i}^v}\widetilde{B}_{l_{k,i}}\Big(x,\frac{1}{j2^{r_{k-1}+i}}\Big).$$

In the next subsection we will show that the set F is the desired set, and complete the proof of Theorem 1.3.

2.2. The proof of Theorem 1.3. Theorem 1.3 follows from the Propositions 2.1, 2.2 and 2.3.

Proposition 2.1. *We have $F\subset X_{\max}$.*

Proof. Fix $x\in F$. For any $k\geqslant 1$, there exist integers $j,v\in\mathbb{N}$ and $z\in L_{k,i+1}^v$ such that

$$d_{l_{k,i+1}^v}(x,z)<\frac{1}{j2^k},\quad i=1,\dots,q_k. \tag{2.9}$$

In order to prove $F\subset X_{\max}$ we must show that

$$\mathcal{L}_\varphi\subset A_\varphi(x) \tag{2.10}$$

and

$$A_\varphi(x)\subset\mathcal{L}_\varphi. \tag{2.11}$$

Proof of (2.10). For $\alpha\in\mathcal{L}_\varphi\subset\bigcup_{i=1}^{q_k}B(\alpha_{k,i},1/k)$ there exists $i_k\in\{2,\dots,q_k-1\}$ such that $\alpha\in B(\alpha_{k,i_k},1/k)$. Let us remark that if $i_k=1$ or q_k we also obtain the desired result. However, in order to avoid tedious discussion we suppose $i_k\neq 1$, q_k without loss of generality.

Write

$$s_k=1+\sum_{i=1}^{k-1}\sum_{j=1}^{q_i}N_{i,j}^v(n_{i,j}+m_i)+\sum_{j=1}^{i_k}N_{k,j}^v(n_{k,j}+m_k).$$

Note that s_k is nothing but the quantity l^v_{k,i_k} which appeared in the construction of F.

Now we will show that

$$\left|\frac{1}{s_k}\sum_{p=0}^{s_k-1}\varphi(f^p(x))-\alpha_{k,i_k}\right|\to 0 \quad \text{as } k\to\infty. \tag{2.12}$$

If (2.12) holds then we have

$$\begin{aligned}\left|\frac{1}{s_k}\sum_{p=0}^{s_k-1}\varphi(f^p(x))-\alpha\right| &\leqslant \left|\frac{1}{s_k}\sum_{p=0}^{s_k-1}\varphi(f^p(x))-\alpha_{k,i_k}\right|+|\alpha_{k,i_k}-\alpha|\\ &\leqslant \left|\frac{1}{s_k}\sum_{p=0}^{s_k-1}\varphi(f^p(x))-\alpha_{k,i_k}\right|+\frac{1}{k}\to 0,\end{aligned}$$

which implies that $\alpha\in A_\varphi(x)$ and therefore (2.10) holds.

To prove (2.12), we need the following lemma.

Lemma 2.1. *For $k,j,v\in\mathbb{N}$ and $i\in\{1,\dots,q_k\}$, define*

$$R^v_{k,i}:=\max_{z\in L^v_{k,i}}\left|\sum_{p=0}^{l^v_{k,i}-1}\varphi(f^p(z))-l^v_{k,i}\alpha_{k,i}\right|.$$

Then

$$\frac{R^v_{k,i}}{l^v_{k,i}}\to 0 \quad \textit{as } k\to\infty.$$

P r o o f. For any $c>0$ put

$$\operatorname{Var}(\varphi,c)=\sup\{|\varphi(x)-\varphi(y)|\colon\ d(x,y)<c\}.$$

Since X is compact, $\operatorname{Var}(\varphi,c)\to 0$ as $c\to 0$ for any continuous function φ. Clearly, if $d_n(x,y)<c$, then

$$\left|\sum_{i=0}^{n-1}\varphi(f^i(x))-\sum_{i=0}^{n-1}\varphi(f^i(y))\right|\leqslant\sum_{i=0}^{n-1}|\varphi(f^i(x))-\varphi(f^i(y))|\leqslant n\operatorname{Var}(\varphi,c).$$

First we let $y\in D^v_{k,i}$ and estimate $\left|\sum\limits_{p=0}^{t^v_{k,i}-1}\varphi(f^p(y))-t^v_{k,i}\alpha_{k,i}\right|$. By the definition of $D^v_{k,i}$, there exist $(x^{k,i}_1,\dots,x^{k,i}_{N_{k,i}})\in(W^v_{k,i})^{N^v_{k,i}}$ such that

$$d_{n_{k,i}}(x^{k,i}_j,f^{a_j}(y))<\frac{1}{j2^k},$$

where $a_j = (j-1)(n_{k,i}+m_k)$, $j = 1, \ldots, N^v_{k,i}$. (We remark that the above inequality and some of the following ones in this subsection are not optimal. However, they simplify the notation and are sufficient for our estimates.) Hence,

$$\left| \sum_{p=0}^{n_{k,i}-1} \varphi(f^p(x_j^{k,i})) - \sum_{p=0}^{n_{k,i}-1} \varphi(f^{a_j+p}(y)) \right| \leqslant n_{k,i} \operatorname{Var}\Big(\varphi, \frac{1}{j2^k}\Big).$$

It follows from $x_j^{k,i} \in W^v_{k,i} \subset P(\alpha_{k,i}, \delta_{k,i}, n_{k,i})$ that

$$\left| \sum_{p=0}^{n_{k,i}-1} \varphi(f^{a_j+p}(y)) - n_{k,i}\alpha_{k,i} \right| \leqslant n_{k,i}\Big(\operatorname{Var}\Big(\varphi, \frac{1}{j2^k}\Big) + \delta_{k,i}\Big). \tag{2.13}$$

Decompose the interval $[0, t^v_{k,i} - 1]$ into small intervals:

$$[0, t^v_{k,i} - 1] = \bigcup_{j=1}^{N^v_{k,i}} [a_j, a_j + n_{k,i} - 1] \cup \bigcup_{j=1}^{N^v_{k,i}-1} [a_j + n_{k,i}, a_j + n_{k,i} + m_k - 1].$$

Write $\|\varphi\| = \max\limits_{x\in X} |\varphi|$. On the intervals $[a_j, a_j + n_{k,i} - 1]$ we will use the estimate (2.13), and on the intervals $[a_j + n_{k,i}, a_j + n_{k,i} + m_k - 1]$, since $\alpha_{k,i} \in [-\|\varphi\|, \|\varphi\|]$ we use the estimate

$$\left| \sum_{p=0}^{m_k-1} \varphi(f^{a_j+n_{k,i}+p}(y)) - m_k\alpha_{k,i} \right| \leqslant m_k(\|\varphi\| + |\alpha_{k,i}|) \leqslant 2m_k\|\varphi\|.$$

Therefore,

$$\begin{aligned} \left| \sum_{p=0}^{t^v_{k,i}-1} \varphi(f^p(y)) - t^v_{k,i}\alpha_{k,i} \right| &\leqslant N^v_{k,i} n_{k,i_k} \Big(\operatorname{Var}\Big(\varphi, \frac{1}{j2^k}\Big) + \delta_{k,i}\Big) \\ &\quad + 2(N^v_{k,i} - 1)m_k\|\varphi\|. \end{aligned} \tag{2.14}$$

On the other hand, it follows from the definition of $L^v_{k,i}$ that for every $z \in L^v_{k,i}$ there exist $x \in L^v_{k,i-1}$ and $y \in D^v_{k,i}$ such that

$$d_{l^v_{k,i-1}}(x, z) < \frac{1}{j2^k}, \quad d_{t^v_{k,i}}(y, f^{l^v_{k,i-1}+m_k}(z)) < \frac{1}{j2^k}. \tag{2.15}$$

Hence,

$$\left| \sum_{p=0}^{l^v_{k,i}-1} \varphi(f^p(z)) - l^v_{k,i}\alpha_{k,i} \right| \leqslant S_1(k) + S_2(k) + S_3(k),$$

where

$$S_1(k) = \Big|\sum_{p=0}^{l^v_{k,i-1}-1} \varphi(f^p(z)) - l^v_{k,i-1}\alpha_{k,i}\Big|,$$

$$S_2(k) = \Big|\sum_{p=l^v_{k,i-1}}^{l^v_{k,i-1}+m_k-1} \varphi(f^p(z)) - m_k\alpha_{k,i}\Big|,$$

$$S_3(k) = \Big|\sum_{p=l^v_{k,i-1}+m_k}^{t^v_{k,i}-1} \varphi(f^p(z)) - t^v_{k,i}\alpha_{k,i}\Big|.$$

Clearly, $S_1(k) \leqslant 2l^v_{k,i-1}\|\varphi\|$ and $S_2(k) \leqslant 2m_k\|\varphi\|$. It follows from (2.14) and (2.15) that

$$\begin{aligned} S_3(k) &\leqslant \Big|\sum_{p=l^v_{k,i-1}+m_k}^{t^v_{k,i}-1} \varphi(f^p(z)) - t^v_{k,i}\alpha_{k,i}\Big| \\ &\leqslant \Big|\sum_{p=0}^{t^v_{k,i}-1} \varphi(f^{l^v_{k,i-1}+m_k+p}(z)) - \sum_{p=0}^{t^v_{k,i}-1} \varphi(f^p(y))\Big| + \Big|\sum_{p=0}^{t^v_{k,i}-1} \varphi(f^p(y)) - t^v_{k,i}\alpha_{k,i}\Big| \\ &\leqslant t^v_{k,i}\mathrm{Var}\Big(\varphi, \frac{1}{j2^k}\Big) + N^v_{k,i}n_{k,i}\Big(\mathrm{Var}\Big(\varphi, \frac{1}{j2^k}\Big) + \delta_{k,i}\Big) + 2(N^v_{k,i}-1)m_k\|\varphi\|. \end{aligned}$$

It follows from the above argument that

$$R^v_{k,i} \leqslant 2(l^v_{k,i-1}+N^v_{k,i}m_k)\|\varphi\| + (t^v_{k,i}+N^v_{k,i}n_{k,i_k})\mathrm{Var}\Big(\varphi, \frac{1}{j2^k}\Big) + N^v_{k,i}n_{k,i}\delta_{k,i}. \tag{2.16}$$

By the choice of $\{N^v_{k,i}\}$, we have $l^v_{k,i} \geqslant 2^{l^v_{k,i-1}}$. Hence,

$$\frac{R^v_{k,i}}{l^v_{k,i}} \to 0 \quad \text{as } k \to \infty.$$

□

Now with help of Lemma 2.1 we are going to prove (2.12). Since $z \in L^v_{k,i+1}$ (z is defined as in (2.9)) there exist $\overline{x} \in L^v_{k,i}$ and $y \in D^v_{k,i+1}$ such that

$$d_{l^v_{k,i}}(\overline{x}, z) < \frac{1}{j2^k}, \quad d_{t^v_{k,i+1}}(y, f^{l^v_{k,i}+m_k}(z)) < \frac{1}{j2^k}.$$

Therefore,

$$d_{l^v_{k,i}}(\overline{x}, x) < \frac{1}{j2^{k-1}}, \quad d_{t^v_{k,i+1}}(y, f^{l^v_{k,i}+m_k}(x)) < \frac{1}{j2^{k-1}},$$

and

$$
\begin{aligned}
(2.17)\quad \Big|\sum_{p=0}^{s_k-1}\varphi(f^p(x))-s_k\alpha_{k,i_k}\Big| &= \Big|\sum_{p=0}^{l^v_{k,i_k}-1}\varphi(f^p(x))-l^v_{k,i_k}\alpha_{k,i_k}\Big| \\
&\leqslant \Big|\sum_{p=0}^{l^v_{k,i_k}-1}\varphi(f^p(x))-\sum_{p=0}^{l^v_{k,i_k}-1}\varphi(f^p(\overline{x}))\Big|+\Big|\sum_{p=0}^{l^v_{k,i}-1}\varphi(f^p(\overline{x}))-l^v_{k,i_k}\alpha_{k,i_k}\Big| \\
&\leqslant l^v_{k,i_k}\operatorname{Var}\Big(\varphi,\frac{1}{j2^{k-1}}\Big)+R^v_{k,i_k}.
\end{aligned}
$$

Clearly, (2.17) and Lemma 2.1 imply (2.12). The proof of (2.12) and therefore the proof of (2.10) is completed. □

Proof of (2.11). Let $n\in\mathbb{N}$ and $n>l^v_{1,1}$. Then there exist k, i_k and j with $i_k\in\{1,\dots,q_k-1\}$ (similarly to the above, we can suppose that $i_k\neq q_k$), $0\leqslant j\leqslant N^v_{k,i+1}-1$ such that

$$l^v_{k,i_k}+j(n_{k,i_k+1}+m_k)<n\leqslant l^v_{k,i_k}+(j+1)(n_{k,i_k+1}+m_k).$$

We have the following estimate.

Lemma 2.2.

$$\Big|\frac{1}{n}\sum_{p=0}^{n-1}\varphi(f^p(x))-\alpha_{k,i_k}\Big|\to 0\quad \text{as } k\to\infty.$$

Proof. Since $z\in L^v_{k,i_k}$ (z is defined as in (2.9)) there exist $\overline{x}\in L^v_{k,i_k}$ and $y\in D^v_{k,i_k}$ such that

$$d_{l^v_{k,i_k}}(\overline{x},z)<\frac{1}{j2^k},\quad d_{t_{k,i_k+1}}(y,f^{l^v_{k,i_k}+m_k}(z))<\frac{1}{j2^k}.$$

Therefore

$$d_{l^v_{k,i_k}}(\overline{x},x)<\frac{1}{j2^{k-1}},\quad d_{t_{k,i_k+1}}(y,f^{l^v_{k,i_k}+m_k}(x))<\frac{1}{j2^{k-1}}.$$

Moreover, if $j>0$, it follows from the definition of D^v_{k,i_k+1} that there exist points $x_1^{k,i_k+1},\dots,x_j^{k,i_k+1}\in W^v_{k,i_k+1}$ such that

$$d_{n_{k,i_k+1}}(x_t^{k,i_k+1},f^{b_t}(y))<\frac{1}{j2^k},$$

where $b_t = (t-1)(n_{k,i_k+1} + m_k)$, $t = 1, \dots, j$, and hence

$$(2.18) \qquad d_{n_{k,i_k+1}}(x_t^{k,i_k+1}, f^{l^v_{k,i_k}+m_k+b_t}(x)) < \frac{1}{j2^{k-2}}.$$

We represent $[0, n-1]$ as the union

$$\begin{aligned}[0, l^v_{k,i_k} - 1] \cup \bigcup_{t=1}^{j} [l^v_{k,i_k} + (t-1)(m_k + n_{k,i_k+1}),\ l^v_{k,i_k} + t(m_k + n_{k,i_k+1}) - 1] \\ \cup [l^v_{k,i_k} + j(m_k + n_{k,i_k+1}),\ n-1].\end{aligned}$$

On the interval $[0, l^v_{k,i_k} - 1]$ we have the estimate (2.17). On each of the intervals $[c_t, c_t + (m_k + n_{k,i_k+1}) - 1]$, where $c_t = l^v_{k,i_k} + (t-1)(m_k + n_{k,i_k+1})$, we estimate

$$\begin{aligned}(2.19) \qquad & \left| \sum_{p=c_t}^{c_t+(m_k+n_{k,i_k+1})-1} \varphi(f^p(x)) - (m_k + n_{k,i_k+1})\alpha_{k,i_k} \right| \\ & \leqslant \left| \sum_{p=c_t}^{c_t+m_k-1} \varphi(f^p(x)) - m_k\alpha_{k,i_k} \right| \\ & \quad + \left| \sum_{p=c_t+m_k}^{c_t+m_k+n_{k,i_k+1}-1} \varphi(f^p(x)) - n_{k,i_k+1}\alpha_{k,i_k} \right| \\ & \leqslant 2m_k\|\varphi\| + n_{k,i_k+1}\delta_{k,i_k+1} \\ & \quad + n_{k,i_k+1}\operatorname{Var}\Big(\varphi, \frac{1}{j2^{k-2}}\Big) \quad \text{(by (2.18))}.\end{aligned}$$

Finally, on $[l^v_{k,i_k} + j(m_k + n_{k,i_k+1}), n-1]$ we have

$$\begin{aligned}(2.20) \qquad & \left| \sum_{p=l^v_{k,i_k}+j(m_k+n_{k,i_k+1})}^{n-1} \varphi(f^p(x)) - (n - l^v_{k,i_k} - j(m_k + n_{k,i_k+1}))\alpha_{k,i_k} \right| \\ & \leqslant 2(n - l^v_{k,i_k} - j(m_k + n_{k,i_k+1}))\|\varphi\| \leqslant 2(n_{k,i_k+1} + m_k)\|\varphi\|.\end{aligned}$$

Collecting the estimates (2.17), (2.19) and (2.20) we have

$$\begin{aligned}\left| \sum_{p=0}^{n-1} \varphi(f^p(x)) - n\alpha_{k,i_k} \right| \leqslant R^v_{k,i_k} + (l^v_{k,i_k} + jn_{k,i_k+1})\operatorname{Var}\Big(\varphi, \frac{1}{j2^{k-2}}\Big) \\ + 2(n_{k,i_k+1} + (j+1)m_k)\|\varphi\| + jn_{k,i_k+1}\delta_{k,i_k+1}.\end{aligned}$$

Clearly, we have

$$\Big|\frac{1}{n}\sum_{p=0}^{n-1}\varphi(f^p(x))-\alpha_{k,i_k}\Big|<\frac{R^v_{k,i_k}}{l^v_{k,i_k}}+\mathrm{Var}\Big(\varphi,\frac{1}{j2^{k-2}}\Big)$$
$$+2\Big(\frac{n_{k,i_k+1}+m_k}{N^v_{k,i_k}}+\frac{m_k}{n_{k,i_k+1}}\Big)\|\varphi\|+\delta_{k,i_k+1}.$$

By Lemma 2.1 and the choice of $\{N^v_{k,i}\}$ we claim that the right-hand side tends to zero as $k\to\infty$. The proof of Lemma 2.2 is completed. □

Now we use Lemma 2.2 to prove (2.11). Fix $x\in F$. For any sufficiently large n, by (2.7) and Lemma 2.2 we have

$$\mathrm{dist}(B_n(\varphi,x),\mathcal{L}_\varphi)\leqslant|B_n(\varphi,x)-\alpha_{k,i_k}|+\mathrm{dist}(\alpha_{k,i_k},\mathcal{L}_\varphi)\to 0,$$

which implies that $\mathrm{dist}(B_n(\varphi,x),\mathcal{L}_\varphi)\to 0$ as $n\to\infty$. Note that since $\mathcal{L}_\varphi$ is closed, we have $A_\varphi(x)\subset\mathcal{L}_\varphi$. This completes the proof of (2.11). □

Proposition 2.2. *The set F is dense in X.*

Proof. Note that since $F'\subset F$, it suffices to show that $F'\cap B(x,r)\neq\emptyset$ for every $x\in X$ and $r>0$. Given $x\in X$ and $r>0$, there exist $j\in\mathbb{N}$ with $2/j<r$ and $d_v\in D$ such that $d(x,d_v)<1/j$.

For any $y\in F(1/j,\{d_v\})\subset F'$, it follows from Remark 2.1 that $d(y,d_v)<1/j$. Therefore,

$$d(x,y)\leqslant d(x,d_v)+d(d_v,y)<\frac{2}{j}<r.$$

This implies that $F'\cap B(x,r)\neq\emptyset$. □

Proposition 2.3. *The set F is a G_δ set.*

Proof. Fix $j\in\mathbb{N}$ and $d_v\in D$. For $k\geqslant 1$ and $i\in\{1,\dots,q_k\}$, let

$$G^v_{k,i}=\bigcup_{x\in L^v_{k,i}}B_{l^v_{k,i}}\Big(x,\frac{\varepsilon}{2^{r_{k-1}+i}}\Big),$$

where

$$B_{l^v_{k,i}}\Big(x,\frac{1}{j2^{r_{k-1}+i}}\Big)=\Big\{y\in X\colon\ d(f^j(x),f^j(x))<\frac{1}{j2^{r_{k-1}+i}},\quad j=0,\dots,l^v_{k,i}-1\Big\}.$$

Clearly, the sets $G^v_{k,i}$ are open sets. Observe that $G^v_{k,i} \subset F^v_{k,i}$ for any $k \geqslant 1$ and $i = 1, \ldots, q_k$, where $F^v_{k,i}$ are defined as in (2.6). That is,

$$F^v_{k,i} = \bigcup_{x \in L^v_{k,i}} \widetilde{B}_{l^v_{k,i}}\Big(x, \frac{1}{j2^{r_{k-1}+i}}\Big).$$

On the other hand, we claim that

$$F^v_{k,i+1} \subset G^v_{k,i}, \quad F^v_{k+1,1} \subset G^v_{k,q_k} \tag{2.21}$$

for any $k \geqslant 1$ and $i = 1, \ldots, q_k - 1$. Therefore,

$$\bigcup_{j=1}^{\infty} \bigcup_{v=1}^{\infty} \bigcap_{i=1}^{q_k} \bigcup_{x \in L^v_{k,i}} \widetilde{B}_{l^v_{k,i}}\Big(x, \frac{1}{j2^{r_{k-1}+i}}\Big) = \bigcup_{j=1}^{\infty} \bigcup_{v=1}^{\infty} \bigcap_{i=1}^{q_k} \bigcup_{x \in L^v_{k,i}} B^v_{l_{k,i}}\Big(x, \frac{1}{j2^{r_{k-1}+i}}\Big),$$

which implies that F is a G_δ set.

Now we proceed to prove (2.21). We only prove that $F^v_{k,i+1} \subset G^v_{k,i}$ for any $k \geqslant 1$ and $i \in \{1, \ldots, q_k - 1\}$ since the proof of $F^v_{k+1,1} \subset G^v_{k,q_k}$ is analogous. Given $y \in F^v_{k,i+1}$, there exists $z \in L^v_{k,i+1}$ such that $y \in \widetilde{B}_{l^v_{k,i+1}}(z, (1/j)/2^{(r_{k-1}+i+1)})$. By the construction of the set $L^v_{k,i+1}$, there exists $x \in L^v_{k,i}$ such that $d_{l^v_{k,i}}(x, z) < (1/j)/2^{(r_{k-1}+i+1)}$. Therefore,

$$d_{l^v_{k,i}}(y, x) \leqslant d_{l^v_{k,i}}(y, z) + d_{l^v_{k,i}}(z, x) < \frac{1}{j2^{(r_{k-1}+i+1)}} + \frac{1}{j2^{(r_{k-1}+i+1)}} = \frac{1}{j2^{r_{k-1}+i}},$$

which implies that $y \in G^v_{k,i}$. The proof of (2.21) is completed. □

3. Applications

In this section we give some applications of Theorem 1.3. It is well known that any factor of a topologically mixing subshift of finite type has the specification property (see, for example, Proposition 21.4 in [9]) and thus our main theorem applies. In particular, we give the corresponding results for the Birkhoff averages for continuous functions on a repeller and locally maximal hyperbolic set.

3.1. Result for the Manneville-Pomeau map. Let $I = [0, 1]$. Given a number $s \in (0, 1)$, the Manneville-Pomeau map $f\colon I \to I$ is defined by

$$f(x) = x + x^{1+s} \bmod 1.$$

The map f has the specification property since it is topologically conjugate to a full one-sided shift on two symbols, see [31]. Let $\varphi(x) = \log|f'(x)|$. With such a choice, we have

$$B_n(\varphi, x) = \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)) = \frac{1}{n}\log|(f^n)'(x)|.$$

That is, the irregular set

$$I_{\varphi,f} = \left\{x \in I\colon \liminf_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)| < \limsup_{n\to\infty}\frac{1}{n}\log|(f^n)'(x)|\right\}$$

is the set of points for which the Lyapunov exponent does not exist. It is worth pointing out that Pollicott and Weiss [27] proved that the set $I_{\varphi,f}$ has Hausdorff dimension equal to 1. Moreover, the set $\mathcal{L}_\varphi$ can be represented as $[0,\alpha]$ with some $\alpha > 0$. It follows from Theorem 1.3 that the set

$$I_{\max} = \{x \in I\colon\ A_\varphi(x) = [0,\alpha]\}$$

is residual if it is not empty.

3.2. Result for the repeller. Let $f\colon\ M \to M$ be a C^1 map on a smooth manifold and let $J \subset M$ be a compact f-invariant set. We say that f is *expanding* on J and that J is a *repeller* for f if there exist $c > 0$ and $\tau > 1$ such that

$$\|d_x f^n v\| \geqslant c\tau^n\|v\|$$

for $x \in J$, $v \in T_xM$ and $n \in \mathbb{N}$. Given a continuous function $\varphi\colon\ J \to \mathbb{R}$, we consider the subset of irregular sets

$$J_{\max} = \{x \in J\colon\ A_\varphi(x) = \mathcal{L}_\varphi\}.$$

It is well known that the map $f\colon\ J \to J$ is a factor of a topologically mixing one-sided subshift of finite type, see [28]. Therefore the map f has the specification property and the following result is a version of Theorem 1.3 for the Birkhoff averages of a continuous function on repeller.

Theorem 3.1. *Let J be a repeller for a topologically mixing C^1 map f and let $\varphi\colon\ J \to \mathbb{R}$ be a continuous function. Then the irregular set $J_{\max}$ is residual if it is not empty.*

3.3. Result for the hyperbolic set. Let $f\colon M \to M$ be a C^1 diffeomorphism on a smooth manifold M and let $\Lambda \subset M$ be a compact f-invariant set. We say that f is a *hyperbolic set* for f if there exists $\tau \in (0,1)$, $c > 0$ and a decomposition

$$T_x M = E^s(x) \oplus E^u(x)$$

for each $x \in \Lambda$ such that

$$\begin{gathered} d_x f E^s(x) = E^s(f(x)), \quad d_x f E^u(x) = E^u(f(x)), \\ \|d_x f^n v\| \leqslant c\tau^n \|v\| \quad \text{whenever } v \in E^s(x), \end{gathered}$$

and

$$\|d_x f^{-n} v\| \leqslant c\tau^n \|v\| \quad \text{whenever } v \in E^u(x),$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$. We say that f is a *locally maximal hyperbolic set* if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

Given a continuous function $\varphi\colon \Lambda \to \mathbb{R}$, we consider the subset of irregular sets

$$\Lambda_{\max} = \{x \in \Lambda\colon\ B_\varphi(x) = \mathcal{L}_\varphi\},$$

where $B_\varphi(x)$ denotes the set of accumulation points of the sequence $n \to B_n(\varphi, x)$ with

$$B_n(\varphi, x) = \frac{1}{2n-1} \sum_{i=-(n-1)}^{n-1} \varphi(f^i(x)).$$

The map $f\colon \Lambda \to \Lambda$ is a factor of a topologically mixing two-sided subshift of finite type, and thus satisfies the specification property. The following result is a version of Theorem 1.3 for the Birkhoff averages on a locally maximal hyperbolic set.

Theorem 3.2. *Let Λ be a locally maximal hyperbolic set for a topologically mixing C^1 diffeomorphism f. Then the set $\Lambda_{\max}$ is residual if it is not empty.*

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Authors' addresses: Jinjun Li, School of Mathematics and Statistics, Minnan Normal University, 36 Xian-qian-zhi Street, Zhangzhou, 363000, Fujian, P. R. China, e-mail: li-jinjun@163.com, Min Wu, Department of Mathematics, South China University of Technology, Wushan Road, Guangzhou 510641, Tianhe, Guandong, P. R. China, e-mail: wumin@scut.edu.cn.