Czechoslovak Mathematical Journal

Jonas Sauer

Maximal regularity of the spatially periodic Stokes operator and application to nematic liquid crystal flows

Czechoslovak Mathematical Journal, Vol. 66 (2016), No. 1, 41–55

Persistent URL: http://dml.cz/dmlcz/144878

Czechoslovak Mathematical Journal, 66 (141) (2016), 41–55

MAXIMAL REGULARITY OF THE SPATIALLY PERIODIC STOKES OPERATOR AND APPLICATION TO NEMATIC LIQUID CRYSTAL FLOWS

Jonas Sauer, Darmstadt

(Received November 10, 2014)

Abstract. We consider the dynamics of spatially periodic nematic liquid crystal flows in the whole space and prove existence and uniqueness of local-in-time strong solutions using maximal L^p-regularity of the periodic Laplace and Stokes operators and a local-in-time existence theorem for quasilinear parabolic equations à la Clément-Li (1993). Maximal regularity of the Laplace and the Stokes operator is obtained using an extrapolation theorem on the locally compact abelian group $G := \mathbb{R}^{n-1} \times \mathbb{R}/L\mathbb{Z}$ to obtain an $\mathcal{R}$-bound for the resolvent estimate. Then, Weis' theorem connecting $\mathcal{R}$-boundedness of the resolvent with maximal L^p regularity of a sectorial operator applies.

Keywords: Stokes operator; spatially periodic problem; maximal L^p regularity; nematic liquid crystal flow; quasilinear parabolic equations

MSC 2010: 35B10, 35K59, 35Q35, 76A15, 76D03

1. Introduction

Consider the spatially periodic problem

$$\text{(LCD)} \qquad \begin{cases} \partial_t u - \nu\Delta u + u \cdot \nabla u + \nabla \mathfrak{p} = -\kappa \operatorname{div}([\nabla d]^T [\nabla d]) & \text{in } (0,T) \times G, \\ \partial_t d + u \cdot \nabla d = \sigma(\Delta d + |\nabla d|^2 d) & \text{in } (0,T) \times G, \\ \operatorname{div} u = 0 & \text{in } (0,T) \times G, \\ |d| = 1 & \text{in } (0,T) \times G, \\ (u(0), d(0)) = (u_0, d_0) & \text{in } G, \end{cases}$$

where $\nu, \sigma, \kappa > 0$, $|d_0| = 1$ in G and $G := \mathbb{R}^{n-1} \times \mathbb{R}/L\mathbb{Z}$, $n \geqslant 2$. This is the simplified Ericksen-Leslie model describing the nematic liquid crystal flow in the whole space

The author has been supported by the International Research Training Group 1529.

which is periodic of length $L > 0$ in the last variable. Here, the function u denotes the velocity of the flow, $\mathfrak{p}$ the pressure and d the macroscopic molecular orientation. The constants ν, σ and κ represent the viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. This model bases on the continuum theory of liquid crystals developed by Ericksen and Leslie, see for example the survey article [10], and was considered for the first time by [15] and [16]. In [13], problem (LCD) was treated on bounded domains using quasilinear theory based upon the maximal L^p-regularity of the Stokes operator. We will adopt this ansatz and extend it to the periodic case by showing maximal L^p-regularity of the spatially periodic Laplace and Stokes operators in weighted spaces, see Theorems 3.2 and 3.5 below. Observe that G is a locally compact abelian group, and hence one can define the *Haar measure* μ (see [1], [6], [12], [23]) which is unique up to multiplication with a constant. We will choose the constant in such a way that

$$\int_G f \, \mathrm{d}\mu = \frac{1}{L} \int_0^L \int_{\mathbb{R}^{n-1}} f(x', x_n) \, \mathrm{d}x' \, \mathrm{d}x_n, \quad f \in C_0(G).$$

In [21], the theory for Muckenhoupt weights $\omega \in A_q(G)$ has been developed in the group setting. For $1 < q < \infty$, a nonnegative function $\omega \in L^1_{\mathrm{loc}}(G)$ is said to be in $A_q(G)$ if

$$\mathcal{A}_q(\omega) := \sup_{U \subset G} \left(\frac{1}{\mu(U)} \int_U \omega \, \mathrm{d}\mu \right) \left(\frac{1}{\mu(U)} \int_U \omega^{-q'/q} \, \mathrm{d}\mu \right)^{q/q'} < \infty,$$

where the supremum runs over all base sets $U \subset G$; see [21] for details. We call a constant $c = c(\omega) > 0$ that depends on $A_q(G)$-weights $A_q(G)$-*consistent*, if for each $d > 0$ we have

$$\sup\{c(\omega) \colon \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Moreover, we say that $\omega \in A_\infty(G)$ if there is $1 < q < \infty$ such that $\omega \in A_q(G)$. For $1 < q \leqslant \infty$ and $\omega \in A_q(G)$, let us denote by $L^q_\omega(G)$ the space of all measurable functions that are p-integrable with respect to the measure $\omega \, \mathrm{d}\mu$. If $\omega = 1$, we will omit the index ω in all function spaces. In [20] it was shown that for $1 < q \leqslant \infty$ and $\omega \in A_q(G)$ the weighted Sobolev space

$$W^{m,q}_\omega(G) := \{u \in L^q_\omega(G) \colon D^\alpha u \in L^q_\omega(G), \ \forall \, \alpha \in \mathbb{N}_0^n, \ |\alpha| \leqslant m\},$$
$$\|u\|_{W^{m,q}_\omega(G)} := \sum_{|\alpha| \leqslant m} \|D^\alpha u\|_{L^q_\omega(G)}$$

yields a Banach space for all $m \in \mathbb{N}_0$ and that the space $C_0^\infty(G)$ of smooth functions with compact support is dense in $W_\omega^{m,q}(G)$ whenever $1 < q < \infty$. Here, $D^\alpha u$ is to be understood as a tempered distribution in $\mathcal{S}'(G)$, the dual of the Schwartz-Bruhat space $\mathcal{S}(G)$, see [4]. Observe that using the canonical identification of G and $\mathbb{R}^{n-1} \times [0, L)$, we have $L_\omega^q(G) = L_\omega^q(\mathbb{R}^{n-1} \times [0, L))$ and

$$W_\omega^{m,q}(G) \subset W_\omega^{m,q}(\mathbb{R}^{n-1} \times [0, L)).$$

Hence, Sobolev embeddings known in the $\mathbb{R}^n$-setup carry over to our setting. In particular, for $\omega = 1$ the classical Sobolev embeddings are valid.

Moreover, for $1 < q < \infty$ and $\omega \in A_q(G)$ we introduce the space

$$L_{\omega,\sigma}^q(G) := \{u \in L_\omega^q(G)^n : \operatorname{div} u = 0\},$$
$$\|u\|_{L_{\omega,\sigma}^q(G)} := \|u\|_{L_\omega^q(G)^n},$$

where $\operatorname{div} u$ is to be understood again as a tempered distribution in $\mathcal{S}'(G)$. Note that $L_{\omega,\sigma}^q(G)$ is a closed subspace of $L_\omega^q(G)^n$ by Proposition 3.3 below.

Let $1 < p, q < \infty$ and define the spaces

$$X_0 := L_\sigma^q(G) \times L^q(G)^n,$$
$$X_1 := D(A_q) \times D(\Delta_q),$$

where $D(A_q) := W^{2,q}(G)^n \cap L_\sigma^q(G)$ and $D(\Delta_q) := W^{2,q}(G)^n$. Furthermore, we define the real interpolation space $X_\gamma := (X_0, X_1)_{1-1/p,p}$.

Theorem 1.1. *Let $n \geqslant 2$, $d_* \in \mathbb{R}^n$ with $|d_*| = 1$, and let $1 < p, q < \infty$ be such that $2/p + n/q < 1$. Assume furthermore $z_0 := (u_0, d_0 - d_*) \in X_\gamma$ is such that $|d_0| = 1$. Then there exists $T_0 > 0$ such that* (LCD) *admits a unique solution $(u, d, \mathfrak{p})$ on $J = [0, T_0]$ in the regularity class*

$$(u, d - d_*) \in W^{1,p}(J; X_0) \cap L^p(J; X_1) \hookrightarrow C(J; X_\gamma),$$
$$\nabla \mathfrak{p} \in L^p(J; L^q(G)).$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.

Remark 1.2. (1) As we are working in an L^q-setting over an unbounded domain, the condition $|d| = 1$ suggests to split the microscopic molecular orientation into a constant part $d_* \in \mathbb{R}^n$ and a deviation $d - d_*$ which can be investigated in terms of the L^q-techniques.

(2) Note that X_γ is the natural space of initial values, as it is the trace space of $W^{1,p}(J;X_0) \cap L^p(J;X_1)$ at $t=0$, see e.g. [17], Section 1.2.

(3) As pointed out by Wang [22] in the more general case of a compact Riemannian manifold N instead of the unit sphere $\mathbb{S}^{n-1}$, given an initial value $d_0\colon \mathbb{R}^n \to N$, the properties of the harmonic map guarantee that the condition $d(t,x) \in N$ is automatically satisfied for all times. Therefore, we may drop the condition $|d| = 1$ in the sequel.

This paper is organized as follows. In Section 2, we provide basic information about $\mathcal{R}$-boundedness and maximal L^p-regularity and show that the maximal regularity constant of a sectorial operator defined on a weighted L^q-space is $A_q(G)$-consistent if the $\mathcal{R}$-bound of the corresponding family of resolvent operators is $A_q(G)$-consistent. In Section 3 we establish maximal L^p-regularity of the spatially periodic Laplace and Stokes operators. Finally, in Section 4 we prove Theorem 1.1 using the theory of abstract quasilinear parabolic equations.

2. Maximal regularity on UMD spaces

The notion of $\mathcal{R}$-boundedness is known to be suitable if one is interested in maximal regularity of evolution equations due to an operator-valued multiplier theorem for Bochner spaces $L^p(\mathbb{R},X)$ by Weis [24], where X is a UMD space as defined in Definition 2.4 below. The $\mathcal{R}$-boundedness comes into play exactly at the level of this multiplier theorem, since the conditions on the multiplier symbols are given in terms of $\mathcal{R}$-bounds, see Theorem 2.8 below. We will not give proofs in full detail here, but rather refer to standard literature for maximal regularity, in particular [8] and [14], whenever they are applicable. As we will see in Remark 2.3 below, we are interested in $A_q(G)$-consistency of the maximal regularity constant in case of $X = L^q_\omega(G)^n$, where $1 < q < \infty$ and $\omega \in A_q(G)$. Therefore it will be necessary to follow the mainly well-known arguments in this section. Let us start by defining $\mathcal{R}$-boundedness and maximal L^p-regularity.

Definition 2.1. Let X be a Banach space and let $1 < p < \infty$. A set $\mathcal{T} \subset \mathcal{L}(X)$ is called *$\mathcal{R}_p$-bounded*, if there is a constant $c > 0$ such that

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|_X^p \, \mathrm{d}t \leqslant c \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_X^p \, \mathrm{d}t \tag{2.1}$$

for all $T_1, \ldots, T_N \in \mathcal{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. Here, $(r_j)_{j\in\mathbb{N}}$ is the sequence of the Rademacher functions

$$r_j\colon\ [0,1] \to \{-1,1\},$$
$$r_j(t) := \operatorname{sgn}[\sin(2^{j-1}\pi t)].$$

The smallest constant $c > 0$ such that (2.1) holds is called the *$\mathcal{R}_p$-bound of $\mathcal{T}$* and is denoted by $\mathcal{R}_p(\mathcal{T})$.

It is worth noting that $\mathcal{R}_p$-boundedness is independent of $1 \leqslant p < \infty$, i.e., there is $k_{p,q} > 0$ depending only on $1 \leqslant p,\, q < \infty$ such that $k_{p,q}^{-1}\mathcal{R}_q(\mathcal{T}) \leqslant \mathcal{R}_p(\mathcal{T}) \leqslant k_{p,q}\mathcal{R}_q(\mathcal{T})$ holds for every Banach space X and every $\mathcal{T} \subset \mathcal{L}(X)$, see [9], Theorem 11.1. Therefore, we will talk about $\mathcal{R}$-boundedness instead of $\mathcal{R}_p$-boundedness.

Definition 2.2. Let $1 < p < \infty$, $0 < T \leqslant \infty$ and let $-A$ be the generator of a bounded analytic semigroup on a Banach space X with domain $D(A) \subset X$. Then A is said to admit *maximal L^p-regularity* on $[0,T)$, if for all $f \in L^p(0,T;X)$ there is a unique solution $u \in \mathbb{E} := W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ to the abstract Cauchy problem

$$\dot{u} + Au = f, \quad u(0) = 0. \tag{2.2}$$

Remark 2.3. Let $1 < p < \infty$, $0 < T \leqslant \infty$ and suppose that $-A$ is a generator of a bounded analytic semigroup on a Banach space X such that A admits maximal L^p-regularity on $[0,T)$.

(1) Let $u \in \mathbb{E}$ be the solution to (2.2) with $f \in L^p(0,T;X)$. By the closed graph theorem there is $c_A > 0$, called the *maximal regularity constant*, such that

$$\|u\|_{\mathbb{E}} := \|\dot{u}\|_{L^p(0,T;X)} + \|u\|_{L^p(0,T;D(A))} \leqslant c_A(\|f\|_{L^p(0,T;X)}).$$

(2) Let $(X, D(A))_{1-1/p,p}$ denote the real interpolation space of X and $D(A)$. Given $u_0 \in (X, D(A))_{1-1/p,p}$, consider the abstract Cauchy problem

$$\dot{u} + Au = f, \quad u(0) = u_0. \tag{2.3}$$

Since $(X, D(A))_{1-1/p,p}$ is the trace space of $\mathbb{E}$, it is easy to see that there is a unique solution $u \in \mathbb{E}$ to (2.3) satisfying

$$\|u\|_{\mathbb{E}} \leqslant c'_A(\|f\|_{L^p(0,T;X)} + \|u_0\|_{(X,D(A))_{1-1/p,p}}),$$

where $c'_A = 1 + c_A$.

In the context of weighted spaces, i.e., if $X = L^q_\omega(G)^n$ or $X = L^q_{\omega,\sigma}(G)$ for $1 < q < \infty$ and $\omega \in A_q(G)$, it is of interest whether the maximal regularity constant is $A_q(G)$-consistent, for example in order to apply extrapolation results, where uniform information in all weighted spaces is necessary. From the consideration in (ii) it is clear that c'_A is $A_q(G)$-consistent if c_A is $A_q(G)$-consistent, so we may concentrate on the latter.

Definition 2.4. A Banach space X is a UMD space if the Hilbert transform

$$Hf(t) := \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t-s}\,\mathrm{d}s, \quad f \in \mathcal{S}(\mathbb{R}, X)$$

extends to a bounded linear operator in $L^p(\mathbb{R}, X)$ for some $1 < p < \infty$. Here, $\mathcal{S}(\mathbb{R}, X)$ is the Schwartz space of rapidly decreasing X-valued functions.

It is known that if X is a UMD space, then the Hilbert transform is bounded in $L^p(\mathbb{R}, X)$ for all exponents $1 < p < \infty$, see [18], Theorem 1.3. Moreover, for all σ-finite measure spaces (Ω, μ_Ω), closed subspaces of $L^q(\Omega, \mu_\Omega)$ for $1 < q < \infty$ are UMD spaces, cf. [5]. In particular, the spaces $L^q_\omega(G)^n$ and $L^q_{\omega,\sigma}(G)$ are UMD spaces for all $1 < q < \infty$ and all $\omega \in A_q(G)$.

Definition 2.5. Let $\mathfrak{X}$ be a Banach space and $(x_k)_{k\in\mathbb{Z}} \subset \mathfrak{X}$. We call the series $\sum\limits_{k=-\infty}^{\infty} x_k$ *unconditionally convergent*, if $\sum\limits_{k=-\infty}^{\infty} x_{\pi(k)}$ is norm convergent for every permutation $\pi\colon \mathbb{Z} \to \mathbb{Z}$.

A sequence of projections $(\Delta_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(\mathfrak{X})$ is called a *Schauder decomposition* of $\mathfrak{X}$ if

$$\Delta_i \Delta_j = 0, \quad \forall\, i \neq j, \qquad \sum_{k=-\infty}^{\infty} \Delta_k x = x, \quad \forall\, x \in \mathfrak{X}.$$

A Schauder decomposition is called *unconditional* if the series $\sum\limits_{k=-\infty}^{\infty} \Delta_k x$ converges unconditionally.

Proposition 2.6. *If $(\Delta_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(\mathfrak{X})$ is an unconditional Schauder decomposition of $\mathfrak{X}$, then for each $1 \leqslant r < \infty$ there is a constant $c_\Delta = c_\Delta(r, \mathfrak{X}) > 0$ such that for all $x \in \mathfrak{X}$ the Paley-Littlewood estimate*

$$c_\Delta^{-1} \Big\| \sum_{k=i}^{j} \Delta_k x \Big\|_{\mathfrak{X}} \leqslant \Big(\int_0^1 \Big\| \sum_{k=i}^{j} r_k(t) \Delta_k x \Big\|_{\mathfrak{X}}^r \mathrm{d}t \Big)^{1/r} \leqslant c_\Delta \Big\| \sum_{k=i}^{j} \Delta_k x \Big\|_{\mathfrak{X}} \tag{2.4}$$

holds for all $i, j \in \mathbb{Z}$, $i \leqslant j$.

P r o o f. See [14], Proposition 3.10, or [8], Section 3. $\square$

It is important to note that for every Banach space $\mathfrak{X} = L^p(\mathbb{R}, X)$, where $1 < p < \infty$ and X is a UMD space, there *is* an unconditional Schauder decomposition, namely $\Delta^0 := (\Delta_k^0)_{k\in\mathbb{Z}}$ with $\Delta_k^0 := \mathcal{F}_{\mathbb{R}}^{-1}\chi_{[2^k,2^{k+1})}\mathcal{F}_{\mathbb{R}}$. This is a result essentially due to Bourgain [3], who proved this for $\mathfrak{X} = L^p([0, 2\pi], X)$, and Zimmermann [25], who extended this result to the real line. Moreover, Δ^0 is $\mathcal{R}$-bounded in $\mathfrak{X} = L^p(\mathbb{R}, X)$, which follows from [8], Corollary 3.7.

Remark 2.7. If we choose the Schauder decomposition $\Delta = \Delta^0$, then in the particular case that $X = L^q_\omega(\mathbb{R}^n)^n$ or $X = L^q_{\omega,\sigma}(\mathbb{R}^n)$ for some $1 < q < \infty$ and some Muckenhoupt weight $\omega \in A_q(\mathbb{R}^n)$, the constant c_Δ appearing in (2.4) can be chosen independently of ω by an argument of Farwig and Ri [11], Remark 5.7. Furthermore, Farwig and Ri showed that the $\mathcal{R}$-bound of a family $\mathcal{T} \subset \mathcal{L}(X)$ does not depend on the Muckenhoupt weight ω.

Note that their arguments carry over to $L^p(\mathbb{R}, L^q_\omega(G)^n)$ and $L^p(\mathbb{R}, L^q_{\omega,\sigma}(G))$ without any changes.

Let us recall the operator valued Mikhlin multiplier theorem in one variable.

Theorem 2.8. *Let X be a* UMD *space and suppose $1 < p < \infty$. Let furthermore $M \in C^1(\mathbb{R}\setminus\{0\}, \mathcal{L}(X))$ and assume that there are constants $c_1, c_2 > 0$ such that*
(1) $\mathcal{R}_p(\{M(t)\colon\ t \in \mathbb{R}\setminus\{0\}\}) = c_1$,
(2) $\mathcal{R}_p(\{tM'(t)\colon\ t \in \mathbb{R}\setminus\{0\}\}) = c_2$.
Then the operator $T_M := \mathcal{F}_{\mathbb{R}}^{-1} M \mathcal{F}_{\mathbb{R}}$ is bounded in $L^p(\mathbb{R}, X)$ with norm

$$\|T_M\|_{\mathcal{L}(L^p(\mathbb{R},X))} \leqslant c^2_{\Delta^0}\mathcal{R}_p(\Delta^0)\cdot(c_1 + c_2) =: c. \tag{2.5}$$

P r o o f. See [8], Theorem 3.19, or [14], Theorem 3.12. □

The following well-known result due to Weis [24] connects the notion of maximal regularity with the $\mathcal{R}$-boundedness of the corresponding resolvent operator.

Proposition 2.9. *Let $1 < p < \infty$ and $0 < T \leqslant \infty$. Assume furthermore that $-A$ is the generator of a bounded analytic semigroup in a* UMD *space X. Then A has maximal L^p-regularity on $[0, T)$ if and only if the operator family*

$$\{\mathrm{i}t(\mathrm{i}t + A)^{-1}\colon\ t \in \mathbb{R},\ t \neq 0\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.

Remark 2.10. Since $\mathcal{R}$-boundedness is independent of p and T, so is the property of A admitting maximal L^p-regularity on $[0, T)$.

Theorem 2.11. *Let $1 < q < \infty$ and $\omega \in A_q(G)$ and suppose that $-A$ is the generator of a bounded analytic semigroup on $L^q_\omega(G)^n$ or $L^q_{\omega,\sigma}(G)$ such that the $\mathcal{R}$-bound of $\{\mathrm{i}t(\mathrm{i}t + A)^{-1}\colon\ t \in \mathbb{R},\ t \neq 0\}$ is $A_q(G)$-consistent. Then the maximal regularity constant c_A is $A_q(G)$-consistent as well.*

Proof. Following [14], Section 1.5, one sees that the maximal regularity constant c_A is exactly $2c + 1$, where c is the constant appearing in (2.5), if we apply Theorem 2.8 with

$$\begin{aligned} M(t) &:= \mathrm{i}t(\mathrm{i}t + A)^{-1} - I, \\ tM'(t) &:= (t(\mathrm{i}t + A)^{-1})^2 + \mathrm{i}t(\mathrm{i}t + A)^{-1}. \end{aligned}$$

Now, since c_1 and c_2 are $A_q(G)$-consistent by assumption, the assertion follows from Remark 2.7. □

3. Spatially periodic Laplace and Stokes operators

The arguments in this section are based upon the following extrapolation theorem.

Theorem 3.1. *Let $1 < r, q < \infty$, $v \in A_r(G)$ and let $\mathcal{T}$ be a family of linear operators such that for all $\omega \in A_q(G)$ there is an $A_q(G)$-consistent constant $c_q = c_q(\omega) > 0$ with*

$$\|Tf\|_{L^q_\omega(G)} \leqslant c_q \|f\|_{L^q_\omega(G)}$$

for all $f \in L^q_\omega(G)$ and all $T \in \mathcal{T}$. Then every $T \in \mathcal{T}$ extends to $L^r_v(G)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_v(G))$ with an $A_r(G)$-consistent $\mathcal{R}$-bound c_r.

Proof. See [21], Theorem 1.5. □

Let $1 < q < \infty$, and $\omega \in A_q(G)$. Then we may define the *spatially periodic Laplace operator* $\Delta_{q,\omega}$ by

$$\begin{aligned} D(\Delta_{q,\omega}) &:= W^{2,q}_\omega(G)^n, \\ \Delta_{q,\omega} u &:= \Delta u. \end{aligned} \tag{3.1}$$

Theorem 3.2. *Let $n \geqslant 2$, $1 < p, q < \infty$ and $\omega \in A_q(G)$. Then the spatially periodic Laplace operator $-\Delta_{q,\omega}$ admits maximal L^p-regularity in $L^q_\omega(G)^n$ and the maximal regularity constant is $A_q(G)$-consistent.*

P r o o f. In virtue of theorem [20], Theorem 1, we know that for every $\theta \in (0, \pi/2)$, the sector $\Sigma_\theta := \{\lambda \in \mathbb{C}\colon |\arg\lambda| < \pi/2 + \theta,\ \lambda \neq 0\}$ is contained in the resolvent set $\varrho(\Delta_{q,\omega})$ and the estimate

$$\|\lambda(\lambda - \Delta_{q,\omega})^{-1}\|_{\mathcal{L}(L^q_\omega(G)^n)} \leqslant c, \quad \lambda \in \Sigma_\theta, \tag{3.2}$$

holds with an $A_q(G)$-consistent constant $c = c(\omega, n, q, \theta)$. Thus, since $\varrho(\Delta_{q,\omega})$ is not empty, $\Delta_{q,\omega}$ is a densely defined closed operator, and (3.2) yields that $\Delta_{q,\omega}$ is sectorial of the angle $\varphi_{\Delta_{q,\omega}} = 0$, see [8]. With help of the Extrapolation Theorem 3.1 we see that $\lambda(\lambda - \Delta_{q,\omega})^{-1}$, $\lambda \in \Sigma_\theta$, is even $\mathcal{R}$-bounded in $\mathcal{L}(L^q_{\omega,\sigma}(G)^n)$ with an $A_q(G)$-consistent bound. Proposition 2.9 shows in turn maximal L^p-regularity of $-\Delta_{q,\omega}$ if we choose $\theta \in (0, \pi/2)$. The $A_q(G)$-consistency of the maximal regularity constant is a consequence of Theorem 2.11. □

Let us now define the *Stokes operator* $A_{q,\omega}$ via

$$D(A_{q,\omega}) := W^{2,q}_\omega(G)^n \cap L^q_{\omega,\sigma}(G),$$
$$A_{q,\omega} u := -P_{q,\omega}\Delta u,$$

where

$$P_{q,\omega}\colon\ \mathcal{S}'(G)^n \to \mathcal{S}'(G)^n,$$
$$P_{q,\omega} f := \mathcal{F}^{-1}\Big(\Big(I - \frac{\eta \otimes \eta}{|\eta|^2}\Big)\hat{f}\Big)$$

is the Helmholtz projection on G. Here, $\mathcal{F}$ denotes the Fourier transform on the locally compact abelian group G, see [19] for details.

Proposition 3.3. *Let $1 < q < \infty$ and $\omega \in A_q(G)$. Then the Helmholtz projection $P_{q,\omega}\colon L^q_\omega(G)^n \to L^q_{\omega,\sigma}(G)$ is a bounded and surjective projection with an $A_q(G)$-consistent bound. In particular, $L^q_{\omega,\sigma}(G)$ is a closed subspace of $L^q_\omega(G)^n$ and hence a* UMD *space.*

P r o o f. In virtue of [20], Theorem 2, we obtain for every $f \in L^q_\omega(G)^n$ a unique solution $(u, \mathfrak{p}) \in W^{2,q}_\omega(G)^n \times \widehat{W}^{1,q}_\omega(G)$ to the system

$$\begin{cases} u - \Delta u + \nabla\mathfrak{p} = f & \text{in } G, \\ \operatorname{div} u = 0 & \text{in } G, \end{cases} \tag{3.3}$$

satisfying the a priori estimate

$$\|u, \nabla^2 u, \nabla\mathfrak{p}\|_{L^q_\omega(G)} \leqslant c\|f\|_{L^q_\omega(G)}, \tag{3.4}$$

where $c = c(\omega, n, q) > 0$ is an $A_q(G)$-consistent constant. Since $P_{q,\omega} f = f - \nabla \mathfrak{p}$, we obtain $P_{q,\omega} f \in L^q_{\omega,\sigma}(G)$ and

$$\|P_{q,\omega} f\|_{L^q_{\omega,\sigma}(G)} \leqslant (c+1)\|f\|_{L^q_\omega(G)}.$$

Moreover, $P_{q,\omega} f = f$ for all $f \in \mathcal{S}'(G)^n$ with $\operatorname{div} f = \mathcal{F}^{-1}(\mathrm{i}\eta \cdot \hat{f}) = 0$, and so $P_{q,\omega} \colon L^q_\omega(G)^n \to L^q_{\omega,\sigma}(G)$ is a surjective projection. □

Lemma 3.4. *Let $1 < q < \infty$ and $\omega \in A_q(G)$. Then $D(A_{q,\omega})$ is dense in $L^q_{\omega,\sigma}(G)$.*

P r o o f. Note that $P_{q,\omega}$ is given via a Fourier multiplier and hence commutes on the level of tempered distributions with differential operators. Thus, it preserves regularity, yielding

$$P_{q,\omega}(W^{2,q}_\omega(G)^n) \subset D(A_{q,\omega}).$$

Since $C_0^\infty(G)^n$ (and consequently also $W^{2,q}_\omega(G)^n$) is dense in $L^q_\omega(G)^n$, we conclude by the boundedness of the Helmholtz projection that $P_{q,\omega}(W^{2,q}_\omega(G)^n)$ is dense in $P_{q,\omega}(L^q_\omega(G)^n) = L^q_{\omega,\sigma}(G)$. This shows the assertion. □

Theorem 3.5. *Let $n \geqslant 2$, $1 < p$, $q < \infty$ and $\omega \in A_q(G)$. Then the Stokes operator $A_{q,\omega}$ has maximal L^p-regularity in $L^q_{\omega,\sigma}(G)$ and the maximal regularity constant is $A_q(G)$-consistent.*

P r o o f. Note that $A_{q,\omega}$ is densely defined by Lemma 3.4. Theorem 2 in [20] and Proposition 3.3 imply that for every $\theta \in (0, \pi/2)$ the sector Σ_θ is contained in the resolvent set $\varrho(-A_{q,\omega})$ and that the estimate

$$\|\lambda(\lambda + A_{q,\omega})^{-1}\|_{\mathcal{L}(L^q_{\omega,\sigma}(G))} \leqslant c, \quad \lambda \in \Sigma_\theta,$$

holds with an $A_q(G)$-consistent constant $c = c(n, q, \theta, \omega)$. Note that [20], Theorem 2, is formulated only for $n \geqslant 3$, but a revision of the proof shows that $n = 2$ is admissible as well. Thus, we obtain that $-A_{q,\omega}$ is sectorial of the angle $\varphi_{-A_{q,\omega}} = 0$ and Theorem 3.1 yields that $\lambda(\lambda + A_{q,\omega})^{-1}$, $\lambda \in \Sigma_\theta$, is $\mathcal{R}$-bounded in $\mathcal{L}(L^q_{\omega,\sigma}(G))$ with an $A_q(G)$-consistent bound. Hence, choosing $\theta \in (0, \pi/2)$, Proposition 2.9 and Theorem 2.11 imply the assertion. □

In order to prove Theorem 1.1, we want to employ the theory of quasilinear evolution equations. Note that quasilinear evolution equations have been studied in more general contexts than that presented here, see Amann [2] and the references therein.

Let Y_0 and Y_1 be Banach spaces such that $Y_1 \overset{d}{\hookrightarrow} Y_0$, i.e., Y_1 is continuously and densely embedded in Y_0. Assume $1 < p < \infty$ and $0 < T \leqslant \infty$. By a *quasilinear autonomous parabolic evolution equation* we understand an equation of the form

$$\text{(QL)} \qquad \dot{z}(t) + A(z(t))z(t) = F(z(t)), \quad t \in (0,T), \quad z(0) = z_0,$$

where A is a mapping from the real interpolation space $Y_\gamma := (Y_0, Y_1)_{1-1/p,p}$ into $\mathcal{L}(Y_1, Y_0)$ and F maps from Y_γ into Y_0. Let us assume the following regularity assumptions on A and F:
(A) $A\colon Y_\gamma \to \mathcal{L}(Y_1, Y_0)$ is locally Lipschitz,
(F) $F\colon Y_\gamma \to Y_0$ is locally Lipschitz.

Then we have the following local-in-time existence result.

Proposition 4.1. *Let $1 < p < \infty$, $z_0 \in Y_\gamma$, and suppose that the assumptions* (A) *and* (F) *are satisfied. Furthermore assume that $A(z^*)$ has the property of maximal L^p-regularity on $[0,T)$ for all $z^* \in Y_\gamma$. Then there exists $T_0 > 0$ such that* (QL) *admits a unique solution z on $J = [0, T_0]$ in the regularity class*

$$z \in W^{1,p}(J; Y_0) \cap L^p(J; Y_1) \hookrightarrow C(J; Y_\gamma).$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.

Proof. See [7] for a proof in a more general non-autonomous case. □

In virtue of Proposition 4.1, we want to reformulate problem (LCD) equivalently as an abstract quasilinear autonomous parabolic evolution equation with variable $z = (u, \delta) := (u, d - d_*)$. Therefore, assume $1 < q, p < \infty$ and let X_0, X_1 and X_γ be defined as in the introductory part. Then by Lemma 3.4 we see that the embedding $X_1 \hookrightarrow X_0$ is dense. We define a linear operator $L \in \mathcal{L}(X_1, X_0)$ via

$$L := \begin{pmatrix} \nu A_q & 0 \\ 0 & -\sigma \Delta_q \end{pmatrix},$$

with the Laplace operator $\Delta_q := \Delta_{q,1}$ and the Stokes operator $A_q := A_{q,1}$ as defined in Section 3. Observe that Theorem 3.2 and Theorem 3.5 yield maximal L^p-regularity for the operator L. Furthermore, for every $z^* = (u^*, \delta^*) \in X_\gamma$, we define an operator $B_q(\delta^*)$ on $D(\Delta_q)$ via

$$(B_q(\delta^*)h)_i := \partial_i \delta_j^* \Delta h_j + \partial_k \delta_j^* \partial_k \partial_i h_j, \quad 1 \leqslant i \leqslant n,$$

where we have used Einstein's sum convention and summed over the indices $1 \leqslant j, k \leqslant n$. In particular, $B_q(\delta^*)\delta^* = \operatorname{div}([\nabla \delta^*]^T [\nabla \delta^*])$ for $z^* = (u^*, \delta^*) \in X_1$. We thus introduce the quasilinear part

$$S(z^*) := \begin{pmatrix} 0 & \kappa P_q B_q(\delta^*) \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.2. *Let $1 < p,\, q < \infty$ be such that $2/p + n/q < 1$. Assume $0 < T \leqslant \infty$ and $z^* \in X_\gamma$. The operator $L + S(z^*)\colon X_1 \to X_0$ admits maximal L^p-regularity on $[0, T)$.*

Proof. Recall that we have the usual Sobolev embeddings at our disposal. Hence, the result follows by arguments similar to those given in [13], Section 3:

By definition, we have to prove that the problem $\dot{z} + (L + S(z^*))z = f$, $z(0) = z_0$ has a unique solution $z \in W^{1,p}(0, T; X_0) \cap L^p(0, T; X_1)$ for all $f \in L^p(0, T; X_0)$ and $z_0 \in X_\gamma$ with the corresponding a priori estimate. Recall that with the notation $z = (u, \delta)$, the operator $L + S(z^*)$ defines an upper triangular matrix operator. Hence, we can first solve for δ. More precisely, writing $f = (f_u, f_\delta)$, we obtain in virtue of Theorem 3.2 a solution

$$\delta \in W^{1,p}(0, T; L^q(G)) \cap L^p(0, T; D(\Delta_q))$$

such that

$$\|\dot{\delta}\|_{L^p(0,T;L^q(G))} + \|\delta\|_{L^p(0,T;D(\Delta_q))} \leqslant c \|f_\delta\|_{L^p(0,T;L^q(G))}, \tag{4.1}$$

where $c = c(p, q, n) > 0$. Since $2/p + n/q < 1$, we have the continuous embedding $X_\gamma \hookrightarrow W^{1,\infty}(G)$. By boundedness of the Helmholtz projection P_q we thus obtain

$$\begin{aligned} \|P_q B_q(\delta^*)\delta\|_{L^p(0,T;L^q(G))} &\leqslant c \|\nabla \delta^*\|_\infty \|\delta\|_{L^p(0,T;D(\Delta_q))} \\ &\leqslant c \|\nabla \delta^*\|_\infty \|f_\delta\|_{L^p(0,T;L^q_\omega(G))}. \end{aligned}$$

Therefore, $f_u - P_q B_q(\delta^*)\delta$ is an admissible right-hand side for the Stokes problem $\dot{u} + A_q u = f$, $u(0) = u_0$, and Theorem 3.5 yields a solution

$$u \in W^{1,p}(0, T; L^q_\sigma(G)) \cap L^p(0, T; D(A_q))$$

such that

$$
\begin{aligned}
(4.2)\qquad & \|\dot{u}\|_{L^p(0,T;L^q_\sigma(G))} + \|u\|_{L^p(0,T;D(A_q))} \\
&\quad \leqslant c(\|f_u\|_{L^p(0,T;L^q_\sigma(G))} + \|\nabla\delta^*\|_\infty \|f_\delta\|_{L^p(0,T;L^q(G))}) \\
&\quad \leqslant c\|\nabla\delta^*\|_\infty \|f\|_{L^p(0,T;X_0)}.
\end{aligned}
$$

Adding estimates (4.1) and (4.2) yields the claim. □

We define a right-hand side F via

$$F(z(t)) := (-P_q(u(t)\cdot\nabla u(t)), -u(t)\cdot\nabla\delta(t) + \sigma|\nabla\delta(t)|^2(\delta(t)+d_*))^T,$$

and rewrite the system (LCD) as

$$\dot{z}(t) + (L + S(z(t))z(t) = F(z(t)), \quad t\in(0,T), \quad z(0) = z_0. \tag{4.3}$$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Note that for all $z^*\in X_\gamma$ we have

$$A(z^*) := L + S(z^*) \in \mathcal{L}(X_1;X_0) \text{ and } F(z^*)\in X_0,$$

since the Helmholtz projection $P_q\colon L^q(G)^n \to L^q_\sigma(G)$ is bounded and since the embedding $X_\gamma \hookrightarrow W^{1,\infty}(G)^{2n}$ holds for $2/p + n/q < 1$. Furthermore, A and F are easily seen to be Fréchet differentiable. In particular, they are locally Lipschitz and hence conditions (A) and (F) are fulfilled.

Moreover, the operator $A(z^*)\colon X_1 \to X_0$ admits maximal L^p-regularity for all $z^*\in X_\gamma$ by Lemma 4.2. We thus can apply Proposition 4.1 to obtain a positive $T_0 > 0$ and a unique solution z of (4.3) on $J = [0,T_0]$ in the regularity class

$$z\in W^{1,p}(J;X_0)\cap L^p(J;X_1) \hookrightarrow C(J;X_\gamma).$$

Moreover, the solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.

Recovering the pressure via the Helmholtz projection P_q we obtain Theorem 1.1. □

References

[1] *E. M. Alfsen*: A simplified constructive proof of the existence and uniqueness of Haar measure. Math. Scand. *12* (1963), 106–116.

[2] *H. Amann*: Quasilinear evolution equations and parabolic systems. Trans. Am. Math. Soc. *293* (1986), 191–227.

[3] *J. Bourgain*: Vector-valued singular integrals and the H^1-BMO duality. Probability Theory and Harmonic Analysis. Papers from the Mini-Conf. on Probability and Harmonic Analysis, Cleveland, 1983 (W. A. Woyczyński, ed.). Pure Appl. Math. 98, Marcel Dekker, New York, 1986, pp. 1–19.

[4] *F. Bruhat*: Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques. Bull. Soc. Math. Fr. *89* (1961), 43–75. (In French.)

[5] *D. L. Burkholder*: A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. Conf. on Harmonic Analysis in Honor of Antoni Zygmund, (W. Beckner et al., eds.). vol. 1, Chicago, Ill., 1981, The Wadsworth Math. Ser., Wadsworth, Belmont, 1983, pp. 270–286.

[6] *H. Cartan*: Sur la mesure de Haar. C. R. Acad. Sci., Paris *211* (1940), 759–762. (In French.)

[7] *P. Clément, S. Li*: Abstract parabolic quasilinear equations and application to a groundwater flow problem. Adv. Math. Sci. Appl. *3* (1993/1994), 17–32.

[8] *R. Denk, M. Hieber, J. Prüss*: $\mathscr{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Am. Math. Soc. *166* (2003), 114 pages.

[9] *J. Diestel, H. Jarchow, A. Tonge*: Absolutely Summing Operators. Cambridge Studies in Advanced Mathematics 43, Cambridge Univ. Press, Cambridge, 1995.

[10] *J. L. Ericksen, D. Kinderlehrer*: Theory and Applications of Liquid Crystals. The IMA Volumes in Mathematics and Its Applications Vol. 5, Papers from the IMA workshop, Minneapolis, Institute for Mathematics and Its Applications, University of Minnesota, Springer, New York, 1987.

[11] *R. Farwig, M.-H. Ri*: Resolvent estimates and maximal regularity in weighted L^q-spaces of the Stokes operator in an infinite cylinder. J. Math. Fluid Mech. *10* (2008), 352–387.

[12] *A. Haar*: Der Massbegriff in der Theorie der kontinuierlichen Gruppen. Ann. Math. (2) *34* (1933), 147–169. (In German.)

[13] *M. Hieber, M. Nesensohn, J. Prüß, K. Schade*: Dynamics of nematic liquid crystal flows. The quasilinear approach. (2014), 11 pages. ArXiv:1302.4596 [math.AP].

[14] *P. C. Kunstmann, L. Weis*: Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. Functional Analytic Methods for Evolution Equations. Autumn School on Evolution Equations and Semigroups, Levico Terme, Trento, Italy, 2001 (M. Iannelli, et al., eds.). Lecture Notes in Mathematics 1855, Springer, Berlin, 2004, pp. 65–311.

[15] *F.-H. Lin*: Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena. Commun. Pure Appl. Math. *42* (1989), 789–814.

[16] *F.-H. Lin, C. Liu*: Nonparabolic dissipative systems modeling the flow of liquid crystals. Commun. Pure Appl. Math. *48* (1995), 501–537.

[17] *A. Lunardi*: Interpolation Theory. Appunti. Scuola Normale Superiore di Pisa 9. Lecture Notes. Scuola Normale Superiore di Pisa, Edizioni della Normale, Pisa, 2009.

[18] *J. L. Rubio de Francia, F. J. Ruiz, J. L. Torrea*: Calderón-Zygmund theory for operator-valued kernels. Adv. Math. *62* (1986), 7–48.

[19] *W. Rudin*: Fourier Analysis on Groups. Interscience Tracts in Pure and Applied Mathematics, Vol. 12, Interscience Publishers, John Wiley, New York, 1962.

[20] *J. Sauer*: Weighted resolvent estimates for the spatially periodic Stokes equations. Ann. Univ. Ferrara Sez. VII Sci. Mat. *61* (2015), 333–354. DOI 10.1007/s11565-014-0221-4.
[21] *J. Sauer*: An extrapolation theorem in non-Euclidean geometries and its application to partial differential equations. To appear in J. Elliptic Parabol. Equ.
[22] *C. Wang*: Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data. Arch. Ration. Mech. Anal. *200* (2011), 1–19.
[23] *A. Weil*: L'intégration Dans les Groupes Topologiques et Ses Applications. Actualités Scientifiques et Industrielles 869, Hermann & Cie., Paris, 1940. (In French.)
[24] *L. Weis*: A new approach to maximal L_p-regularity. Evolution Equations and Their Applications in Physical and Life Sciences. Proc. Bad Herrenalb Conf., Karlsruhe, 1999 (G. Lumer et al., eds.). Lect. Notes in Pure and Appl. Math. 215, Marcel Dekker, New York, 2001, pp. 195–214.
[25] *F. Zimmermann*: On vector-valued Fourier multiplier theorems. Stud. Math. *93* (1989), 201–222.

Author's address: Jonas Sauer, Fachbereich Mathematik, Technische Universität Darmstadt, Schloßgartenstraße 7, 64283 Darmstadt, Germany, e-mail: `jsauer@mathematik.tu-darmstadt.de`.