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VOLUME COMPARISON THEOREMS FOR MANIFOLDS WITH RADIAL CURVATURE BOUNDED

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Abstract. In this paper, for complete Riemannian manifolds with radial Ricci or sectional curvature bounded from below or above, respectively, with respect to some point, we prove several volume comparison theorems, which can be seen as extensions of already existing results. In fact, under this radial curvature assumption, the model space is the spherically symmetric manifold, which is also called the generalized space form, determined by the bound of the radial curvature, and moreover, volume comparisons are made between annulus or geodesic balls on the original manifold and those on the model space.

Keywords: spherically symmetric manifolds; radial Ricci curvature; radial sectional curvature; volume comparison

MSC 2010: 53C20, 53C21, 52A38

1. Preliminaries

Volume comparison is an important topic in Comparison Geometry, and also has extensive applications in other branches of Differential Geometry. One can easily realize the importance of the theory on volume comparison from the fact that many classical results in geometry could not be obtained without volume comparison results, like Cheng's eigenvalue comparison theorems [5], [6], Cheeger-Yau's heat kernel comparison theorem [4], etc. By improving the classical Bishop's volume comparison theorems to more general forms, we have extended Cheng's eigenvalue comparison results and Cheeger-Yau's heat kernel comparison result to more general forms in [8] and [15], respectively. Here, as in [8], [15], by still using spherically symmetric

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manifolds as model spaces, we generalize the volume comparison theorem (in [8]) to manifolds with *radial Ricci sectional curvature bounded from below or above*, respectively, *with respect to some point*—see Theorem 3.2, (i) of Corollary 3.4 or Theorem 4.3, (i) of Corollary 4.4 for details.

However, in order to state and prove our main results expediently, we need to use some notions from [8], [15], [16]. Let a complete n-dimensional ($n \geqslant 2$) Riemannian manifold M with the metric $\langle \cdot, \cdot \rangle_M$ and the Levi-Civita connection ∇ be given. For any fixed point $p \in M$, let $\mathcal{D}_p$, a star shaped subset of the tangent space T_pM, and d_ξ be defined by

$$\mathcal{D}_p = \{t\xi;\ 0 \leqslant t < d_\xi,\ \xi \in S_p^{n-1}\}$$

and $d_\xi = \sup\{t > 0;\ \gamma_\xi(s) := \exp_p(s\xi)$ is the unique minimal geodesic joining p and $\gamma_\xi(t)\}$, where S_p^{n-1} is the unit sphere with center p in T_pM. Then the exponential map $\exp_p\colon \mathcal{D}_p \to M \setminus \mathrm{Cut}(p)$ is a diffeomorphism from $\mathcal{D}_p$ onto the open set $M \setminus \mathrm{Cut}(p)$, with $\mathrm{Cut}(p)$ the cut locus of p, which is a closed set of zero n-Hausdorff measure. Clearly, this map provides a maximal normal geodesic coordinate chart at p. Then we can introduce an important map. For a fixed vector $\xi \in T_pM$, $|\xi| = 1$, let $\xi^\perp$ be the orthogonal complement of $\{\mathbb{R}\xi\}$ in T_pM, and let $\tau_t\colon T_pM \to T_{\exp_p(t\xi)}M$ be the parallel translation along $\gamma_\xi(t)$. The path of linear transformations $\mathbb{A}(t,\xi)\colon \xi^\perp \to \xi^\perp$ is defined by

$$\mathbb{A}(t,\xi)\eta = (\tau_t)^{-1} Y_\eta(t),$$

where $Y_\eta(t) = d(\exp_p)_{(t\xi)}(t\eta)$ is the Jacobi field along $\gamma_\xi(t)$ satisfying $Y_\eta(0) = 0$, and $(\nabla_t Y_\eta)(0) = \eta$. Clearly, the map $\mathbb{A}(t,\xi)$ satisfies the Jacobi equation $\mathbb{A}'' + \mathcal{R}\mathbb{A} = 0$ with initial conditions $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$. By Gauss's lemma, the Riemannian metric of $M \setminus \mathrm{Cut}(p)$ in the geodesic spherical coordinate chart can be expressed by

$$\mathrm{d}s^2(\exp_p(t\xi)) = \mathrm{d}t^2 + |\mathbb{A}(t,\xi)\,\mathrm{d}\xi|^2, \quad t\xi \in \mathcal{D}_p, \tag{1.1}$$

and so

$$\sqrt{|g|} = \det \mathbb{A}(t,\xi).$$

So, by applying (1.1), the volume $\mathrm{vol}(B(p,r))$ of a geodesic ball $B(p,r)$, with radius r and center p, on M is given by

$$\begin{aligned} \mathrm{vol}(B(p,r)) &= \int_{S_p^{n-1}} \int_0^{\min\{r,d_\xi\}} \sqrt{|g|}\,\mathrm{d}t\,\mathrm{d}\sigma \\ &= \int_{S_p^{n-1}} \int_0^{\min\{r,d_\xi\}} \det(\mathbb{A}(t,\xi))\,\mathrm{d}t\,\mathrm{d}\sigma, \end{aligned} \tag{1.2}$$

where $\mathrm{d}\sigma$ denotes the $(n-1)$-dimensional volume element on $\mathbb{S}^{n-1} \equiv S_p^{n-1} \subseteq T_pM$. Let $\mathrm{inj}(p) := d(p, \mathrm{Cut}(p)) = \min_{\xi} d_\xi$ be the injectivity radius at p. In general, we have $B(p, \mathrm{inj}(p)) \subseteq M \setminus \mathrm{Cut}(p)$. Besides, for $r < \mathrm{inj}(p)$, by (1.2) we can obtain

$$\mathrm{vol}(B(p,r)) = \int_0^r \int_{S_p^{n-1}} \det(\mathbb{A}(t,\xi))\,\mathrm{d}\sigma\,\mathrm{d}t.$$

Denote by $r(x) = d(x,p)$ the intrinsic distance to the point $p \in M$. Then, by the definition of a non-zero tangent vector *radial* to a prescribed point on a manifold given on the first page of [13], we know that for $x \in M \setminus (\mathrm{Cut}(p) \cup p)$ the unit vector field

$$v_x := \nabla r(x)$$

is the radial unit tangent vector field. This is because for any $\xi \in S_p^{n-1}$ and $t_0 > 0$ we have $\nabla r(\gamma_\xi(t_0)) = \gamma'_\xi(t_0)$ when the point $\gamma_\xi(t_0) = \exp_p(t_0\xi)$ is away from the cut locus of p. Set

$$l(p) := \sup_{x \in M} r(x). \tag{1.3}$$

Then we have $l(p) = \max_{\xi} d_\xi$ (cf. Section 2 of [8]). Clearly, $l(p) \geqslant \mathrm{inj}(p)$. Define a function $J > 0$ on $\mathcal{D}_p \setminus \{p\}$ by

$$J^{n-1} = \sqrt{|g|} := \sqrt{\det[g_{ij}]}. \tag{1.4}$$

We also need the following fact about $r(x)$ (cf. [17], Proposition 39 on page 266):

$$\partial_r \Delta r + \frac{(\Delta r)^2}{n-1} \leqslant \partial_r \Delta r + |\mathrm{Hess}\, r|^2 = -\,\mathrm{Ric}(\partial_r, \partial_r), \qquad \text{with } \Delta r = \partial_r \ln(\sqrt{|g|}),$$

with $\partial_r = \nabla r$ as a differentiable vector (cf. [17], Proposition 7 on page 47 for the differentiation of ∂_r). Then, together with (1.4), we have

$$J'' + \frac{1}{(n-1)} \mathrm{Ric}(\gamma'_\xi(t), \gamma'_\xi(t)) J \leqslant 0, \tag{1.5}$$

$$J(t,\xi) = t + O(t^2), \tag{1.6}$$

$$J'(t,\xi) = 1 + O(t). \tag{1.7}$$

The facts (1.5), (1.6), and (1.7) play a fundamental role in the derivation of Lemma 2.1, which is the *key* to proving Theorem 4.3.

We will compare our manifolds with model manifolds which are spherically symmetric with respect to a base point and whose radial curvatures bound those of the original manifolds. First, we need the following definition, which allows us to make the concept clear and can be found in [8], [15], [16] and maybe other literatures.

Definition 1.1. A domain $\Omega = \exp_p([0,l)\times S_p^{n-1}) \subset M\setminus \mathrm{Cut}(p)$, with $l < \mathrm{inj}(p)$, is said to be spherically symmetric with respect to a point p if the matrix $\mathbb{A}(t,\xi)$ satisfies $\mathbb{A}(t,\xi) = f(t)I$ for a function $f \in C^2([0,l))$, $l \in (0,\infty]$ with $f(0) = 0$, $f'(0) = 1$, and $f|(0,l) > 0$.

So, by (1.1), on the set Ω given in Definition 1.1 the Riemannian metric of M can be expressed by

$$\mathrm{d}s^2(\exp_p(t\xi)) = \mathrm{d}t^2 + f^2(t)|\,\mathrm{d}\xi|^2, \quad \xi \in S_p^{n-1}, \quad 0 \leqslant t < l, \tag{1.8}$$

with $|\mathrm{d}\xi|^2$ the round metric on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. Spherically symmetric manifolds were named generalized space forms by Katz and Kondo [13], and a standard model for such manifolds is given by the quotient manifold of the warped product $[0,l)\times_f \mathbb{S}^{n-1}$ equipped with the metric (1.8), where f satisfies the conditions of Definition 1.1, and all pairs $(0,\xi)$ are identified with a single point p (see [2]). That is to say, $M^* = [0,l) \times_{f(t)} \mathbb{S}^{n-1}$ with $f(t)$ satisfying the conditions in Definition 1.1 is a spherically symmetric manifold with p the base point and (1.8) as its metric. If $l = \infty$, then M^* has a pole at $p = \{0\} \times_f \mathbb{S}^{n-1}$, and vice versa. If l is finite and $f(l) = 0$, then M^* "closes". For M^* and $r < l$, by (1.2) we have

$$\mathrm{vol}(B(p,r)) = w_n \int_0^r f^{n-1}(t)\,\mathrm{d}t,$$

and moreover, by applying the co-area formula, the volume of the boundary $\partial B(p,r)$ is given by

$$\mathrm{vol}(\partial B(p,r)) = w_n f^{n-1}(r),$$

where w_n denotes the $(n-1)$-volume of the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$. A space form with constant curvature k is also a spherically symmetric manifold, and in this special case we have

$$f(t) = \begin{cases} \dfrac{\sin\sqrt{k}t}{\sqrt{k}}, & l = \dfrac{\pi}{\sqrt{k}}, \quad k > 0, \\ t, & l = \infty, \quad k = 0, \\ \dfrac{\sinh\sqrt{-k}t}{\sqrt{-k}}, & l = \infty, \quad k < 0. \end{cases}$$

Readers can learn more about the model manifolds, like the regularity of the metric, the existence of the model manifold for a given open manifold, etc., from [8], [16].

We will use the following concepts.

Definition 1.2. Given a continuous function $k\colon [0,l) \to \mathbb{R}$, we say that M has a radial sectional curvature upper bound k along any unit-speed minimizing geodesic starting from a point $p \in M$ if

$$K(v_x, V) \leqslant k(r(x)), \quad x \in M \setminus (\mathrm{Cut}(p) \cup \{p\}), \tag{1.9}$$

where $V \perp v_x$, $V \in S_x^{n-1} \subseteq T_x M$, and $K(v_x, V)$ is the sectional curvature of the plane spanned by v_x and V.

Definition 1.3. Given a continuous function $k\colon [0,l) \to \mathbb{R}$, we say that M has a radial Ricci curvature lower bound $(n-1)k$ along any unit-speed minimizing geodesic starting from a point $p \in M$ if

$$\mathrm{Ric}(v_x, v_x) \geqslant (n-1)k(r(x)), \quad x \in M \setminus (\mathrm{Cut}(p) \cup \{p\}), \tag{1.10}$$

where Ric denotes the Ricci curvature of M.

Remark 1.4. Since the radial distance is given by $r(x) = d(p,x) =: t(x)$ for $x = \gamma_\xi(t)$, the parameter t may be seen as the argument of the continuous function $k\colon [0,l) \to \mathbb{R}$ in Definitions 1.2 and 1.3. Additionally, $\mathrm{d}/\mathrm{d}t\big|_x = \nabla r(x) = v_x$, which implies that our conditions (1.9) and (1.10) become $K(\mathrm{d}/\mathrm{d}t, \xi) \leqslant k(t)$ and $\mathrm{Ric}(\mathrm{d}/\mathrm{d}t, \mathrm{d}/\mathrm{d}t) \geqslant (n-1)k(t)$, respectively. Besides, for convenience, if a manifold satisfies (1.9) or (1.10) then we say that M has a *radial sectional curvature upper bound with respect to a point p* or a *radial Ricci curvature lower bound with respect to a point p*, respectively, that is to say, its *radial sectional curvature is bounded from above with respect to p* or *radial Ricci curvature is bounded from below with respect to p*. At the end, we would like to recall the history of radial curvature briefly and also mention some comparison theorems for radial curvature partially.

It was for the first time that Klingenberg introduced the notion of radial curvature in [14] to study compact Riemannian manifolds with radial curvatures pinched between 1/4 and 1. After that, mathematicians have been paying attention to the radial curvatures. In general, the reference manifolds for comparison theorems are space forms. However, Elerath [7] employed a Von Mangoldt surface of revolution (i.e., a complete surface of revolution homeomorphic to Euclidean plane whose Gaussian curvature is non-increasing along each meridian) $\widetilde{Z} \subset \mathbb{R}^3$ with nonnegative Gaussian curvature as the reference surface to prove the generalized Topologov comparison theorem (we write GTCT for short) successfully for complete open Riemannian manifolds with radial curvatures bounded from below by that of $\widetilde{Z}$.

For complete open Riemannian manifolds whose radial Ricci curvatures are bounded from below by a nonnegative smooth function $\zeta(t)$ of the distance parameter with respect to some point (as described in Definition 1.3), together with other constraints for $\zeta(t)$, Abresch proved the GTCT in [1] (these special manifolds were called "*asymptotically nonnegatively curved*" manifolds therein). Of course, there are other types of GTCT, which we do not need to mention here. From these facts, we know that mathematicians have investigated manifolds with radial curvatures bounded by some continuous function of the distance parameter (of the original manifolds), and generalized some classical comparison theorems. By the way, we may find definitions similar to Definition 1.3 in [10], [11], [18].

The paper is organized as follows. In Section 1, we give some useful definitions and notions, and also recall some fundamental knowledge about the model manifolds. An important lemma will be given in the next section, which is the footstone to prove the volume comparison theorems in Section 3. Several volume comparison results for manifolds with *radial Ricci curvature bounded from below with respect to some point* will be given in Section 3. For manifolds with *radial sectional curvature bounded from above with respect to some point*, we give several volume comparison theorems in the last section.

2. A key lemma

In this section, we give a conclusion which will play an important role in the derivation of volume comparison theorems in the next section. In fact, this conclusion with its proof has been shown in [8], [16] and might be covered in some other literature as a special case, for instance [9], but, in order to emphasize the importance of this result and let readers know it well without checking its proof somewhere else, we also want to give here its proof in detail.

We define a quantity on $M \setminus \mathrm{Cut}(p)$ by

$$\theta(t,\xi) = \Big(\frac{J(t,\xi)}{f(t)}\Big)^{n-1}.$$

Theorem 2.1. *Given* $\xi \in S_p^{n-1} \subseteq T_pM$ *and a model space* $M^- = [0,l) \times_f \mathbb{S}^{n-1}$ *with respect to* p^-*, under the curvature assumption on the radial Ricci tensor,* $\mathrm{Ric}(v_x, v_x) \geqslant -(n-1)f''(t)/f(t)$ *on* M*, for* $x = \gamma_\xi(t) = \exp_p(t\xi)$ *with* $t < \min\{d_\xi, l\}$*, the function* θ *is nonincreasing in* t*. In particular, for all* $t < \min\{d_\xi, l\}$ *we have* $J(t,\xi) \leqslant f(t)$*. Furthermore, this inequality is strict for all* $t \in (t_0, t_1]$ *with* $0 \leqslant t_0 < t_1 < \min\{d_\xi, l\}$*, if the above curvature assumption holds with a strict inequality for* t *in the same interval.*

Proof. From the assumption on the radial Ricci curvature tensor and (1.5), with initial conditions (1.6) and (1.7), the function $J(t,\xi)$ satisfies the differential inequality

$$\begin{cases} J'' + k(t)J \leqslant 0, \quad 0 \leqslant t \leqslant l, \\ J(0) = 0, \quad J'(0) = 1, \end{cases} \tag{2.1}$$

where $k(t) = -f''(t)/f(t)$. On the other hand, $y(t) = f(t)$ is the unique solution of the equation

$$\begin{cases} y'' + k(t)y = 0, \\ y(0) = 0, \quad y'(0) = 1, \\ y > 0 \quad \text{on } (0, l). \end{cases} \tag{2.2}$$

Consequently, on an interval $(0, l)$ on which $y(t) = f(t) > 0$, we have $J''f - f''J \leqslant 0$, that is, $(J'f - f'J)' \leqslant 0$. The initial conditions for $J(t)$ and $f(t)$ then yield $J'f - f'J \leqslant 0$. Hence, $(J/y)' = (J/f)' \leqslant 0$ whenever $y(t) = f(t) > 0$ on $(0, l)$. Thus J/f is a nonincreasing function. Furthermore, by applying L'Hôpital's rule, we have

$$\lim_{t \to 0} \frac{J(t, \xi)}{f(t)} = \lim_{t \to 0} \frac{J'(t, \xi)}{f'(t)} = 1.$$

Consequently, for $t < d(\xi)$, $J(t, \xi) \leqslant f(t)$ holds. If the radial Ricci curvature is strictly greater than $-f''(t)/f(t)$ for $0 \leqslant t_0 < t \leqslant t_1$, then $(J/f)' < 0$, i.e. J/f is strictly decreasing on the interval $(t_0, t_1]$, which implies the last assertion. $\square$

Remark 2.2. The proof of the first part of the above theorem may be found in [10] but with an opposite sign for $k(t)$. (I.e. the authors required $y''(t) - k(t)y(t) = 0$. Clearly, they are wrong.) We do not find any explanation for this different sign, since even if the curvature tensor is defined with the opposite sign to the one chosen by us, the Ricci tensor always agrees.

3. Volume comparison theorems for manifolds with radial Ricci curvature bounded from below

For an n-dimensional complete Riemannian manifold M and a point $p \in M$, as before, $r(x) = d(p, x)$ and S_p^{n-1} denote the intrinsic distance to the point p and the unit sphere with center p in the tangent space T_pM, respectively. For any measurable subset Γ of S_p^{n-1}, we define an annulus $A^{\Gamma}_{r,R}(p)$ as follows:

$$A^{\Gamma}_{r,R}(p) = \{x \in M;\ r \leqslant r(x) \leqslant R, \text{ and any minimal geodesic } \gamma \text{ from } p \text{ to } x \text{ satisfies } \gamma'(0) \in \Gamma\}. \tag{3.1}$$

In what follows, we will show that if *M has a radial Ricci curvature lower bound $(n-1)k(t) = -(n-1)f''(t)/f(t)$ with respect to p*, then we may give an estimate for the volume $\mathrm{vol}(A^{\Gamma}_{r,R}(p))$ of $A^{\Gamma}_{r,R}(p)$ by using the corresponding quantity of its model manifold $M^- := [0,l) \times_f \mathbb{S}^{n-1}$, with the base point p^-, determined by solving the initial value problem

$$\begin{cases} -f''(t) = k(t)f(t), \quad 0 < t < l, \quad f|(0,l) > 0, \\ f(0) = 0, \\ f'(0) = 1. \end{cases}$$

In order to prove this estimate, we need the following conclusion (cf. Lemma 3.2 in [19]).

Lemma 3.1 ([19]). *Let g, h be two positive functions defined over $[0,\infty)$. If g/h is non-increasing, then for any $R > r > 0$, $S > s > 0$, $s > r$, $S > R$, we have*

$$\frac{\int_s^S g(t)\,\mathrm{d}t}{\int_r^R g(t)\,\mathrm{d}t} \leqslant \frac{\int_s^S h(t)\,\mathrm{d}t}{\int_r^R h(t)\,\mathrm{d}t}.$$

By applying Lemmas 2.1 and 3.1, we can prove the following result.

Theorem 3.2. *Let M be an n-dimensional complete Riemannian manifold. For a point $p \in M$, let $r \leqslant R$, $s \leqslant S$, $r \leqslant s$, $R \leqslant S < \min\{l(p), l\}$, with $l(p)$ defined by (1.3). Then, under the curvature assumption in Theorem 2.1, we have*

$$\frac{\mathrm{vol}(A^{\Gamma}_{r,R}(p))}{\mathrm{vol}(A^{\Gamma}_{s,S}(p))} \geqslant \frac{\mathrm{vol}^{M^-}(A^{\Gamma}_{r,R}(p^-))}{\mathrm{vol}^{M^-}(A^{\Gamma}_{s,S}(p^-))}, \tag{3.2}$$

with equality if and only if the equality in the curvature assumption holds, i.e. $\mathrm{Ric}(v_x, v_x) \geqslant -(n-1)f''(t)/f(t)$ on M, for $x = \gamma_\xi(t) = \exp_p(t\xi)$ with $t < \min\{d_\xi, l\}$ and $\xi \in \Gamma$, where $\mathrm{vol}(A^{\Gamma}_{r,R}(p))$ is defined by (3.1) and $\mathrm{vol}(A^{\Gamma}_{s,S}(p))$ has the same meaning as $\mathrm{vol}(A^{\Gamma}_{r,R}(p))$ with just r and R being replaced by s and S, respectively; moreover, $\mathrm{vol}^{M^-}(A^{\Gamma}_{\cdot,\cdot}(p^-))$ denotes the corresponding annulus on the model manifold $M^- = [0,l) \times_f \mathbb{S}^{n-1}$ with the base point p^-.

Proof. Here we will use a method similar to that of the proof of Theorem 3.1 in [19]. In fact, by Lemma 3.1 we know that if we want to prove (3.2), it suffices to show that

$$\frac{\mathrm{vol}(A^{\Gamma}_{x,y}(p))}{\mathrm{vol}^{M^-}(A^{\Gamma}_{x,y}(p^-))}$$

is nonincreasing for $0 \leqslant x \leqslant y < \min\{l(p), l\}$, where

$$\mathrm{vol}(A^{\Gamma}_{x,y}(p)) = \int_\Gamma \mathrm{d}\sigma \int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t,$$

with, as before, $\mathrm{d}\sigma$ being the $(n-1)$-volume element on $\mathbb{S}^{n-1}$, and $\mathbb{A}(t,\xi)$ and d_ξ defined in Section 1 as the path of linear transformations and the distance to the cut locus of p in the direction $\xi \in S_p^{n-1}$, respectively.

By Lemma 2.1, for any $\xi \in S_p^{n-1}$ and $t < d_\xi$ we have

$$\Big(\frac{\mathbb{A}(t,\xi)}{\mathbb{A}^{M^-}(t,p^-)}\Big)' = \Big(\frac{J^{n-1}(t,\xi)}{f^{n-1}(t)}\Big)' = (n-1)\Big(\frac{J(t,\xi)}{f(t)}\Big)^{n-2}\Big(\frac{J}{f}\Big)' \leqslant 0,$$

with $\mathbb{A}^{M^-}(t,p^-)$ being the path of linear transformations (defined on a subspace of $T_{p^-}M^-$) for the model space M^- and the function $J(t,\xi)$ defined by (1.4). This implies that $\mathbb{A}(t,\xi)/\mathbb{A}^{M^-}(t,p^-)$ is nonincreasing (with respect to the variable t) for any $\xi \in S_p^{n-1}$ and $t < d_\xi$. Then, together with Lemma 3.1, we obtain

$$\frac{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t} \geqslant \frac{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t}$$

for $y \leqslant z$, which yields that

$$\begin{aligned}(3.3)\qquad \int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t &\geqslant \frac{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t \\ &\geqslant \frac{\int_{x}^{\min\{y,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t}{\int_{x}^{\min\{z,d_\xi\}} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t \\ &\geqslant \frac{\int_{x}^{y} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t}{\int_{x}^{z} \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t,\end{aligned}$$

where the second inequality follows from the fact that the division $\int_x^a \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t/\int_x^b \mathbb{A}^{M^-}(t,p^-)\,\mathrm{d}t$ (i.e. $\int_x^a f^{n-1}(t)\,\mathrm{d}t/\int_x^b f^{n-1}(t)\,\mathrm{d}t$) is nonincreasing when $a < b$, and the last inequality always holds in three cases $d_\xi \leqslant y \leqslant z$, $y \leqslant d_\xi \leqslant z$, and $y \leqslant z \leqslant d_\xi$. Integrating both sides of (3.3) results in

$$\mathrm{vol}(A_{x,y}^\Gamma(p)) \geqslant \frac{\mathrm{vol}^{M^-}(A_{x,y}^\Gamma(p^-))}{\mathrm{vol}^{M^-}(A_{x,z}^\Gamma(p^-))}\,\mathrm{vol}(A_{x,z}^\Gamma(p)),$$

which implies the conclusion of Theorem 3.2. □

Remark 3.3. Since $\mathbb{A}^{M^-}(t,p^-) = f^{n-1}(t)$, where $f(t)$ satisfies $f''(t)+k(t)f(t)=0$ with $f(0)=0$, $f'(0)=1$ and $f(t)>0$ on $(0,l)$, inequality (3.2) becomes

$$\frac{\mathrm{vol}(A_{s,S}^\Gamma(p))}{\mathrm{vol}(A_{r,R}^\Gamma(p))} \leqslant \frac{\int_s^S f^{n-1}(t)\,\mathrm{d}t}{\int_r^R f^{n-1}(t)\,\mathrm{d}t}.$$

It may seem that the above inequality looks the *same* as the inequality (1.3) in Theorem 1.3 of [10]. However, the main volume comparison result (1.3) in [10] is wrong, since, as pointed out in Remark 2.2, the authors made a mistake on the sign of $k(t)$, which lead to the result that they did not get the essence that the radial curvature lower bound of the original manifold M determines the warping function $f(t)$ of M^-, and naturally, they did not pursue to find the model space M^- therein. By the way, we would like to point out that the volume comparison inequality in Theorem 3.1 of [19] has a wrong direction, which we believe is a negligible clerical error, but its proof is correct. Besides, it is clear that Theorem 3.2 is an extension of the corresponding volume comparison results in [19], where the space form with constant curvature is used as the model space.

By applying Theorem 3.2, we are able to get the following volume comparison results without any big difficulty.

Corollary 3.4. *Let M be an n-dimensional complete Riemannian manifold. Then, under the curvature assumption in Theorem* 2.1, *we have:*

(i) (*The Gromov-type relative volume comparison theorem* I) *the inequality*

$$\frac{\operatorname{vol}(B(p,r))}{\operatorname{vol}(B(p,R))} \geqslant \frac{\operatorname{vol}(V_n(p^-,r))}{\operatorname{vol}(V_n(p^-,R))} \tag{3.4}$$

holds for $r \leqslant R < \min\{l, l(p)\}$, with $l(p)$ defined by (1.3), *where, as before,* $\operatorname{vol}(B(p,r))$ *and* $\operatorname{vol}(V_n(p^-,r))$ *denote the volume of the geodesic ball $B(p,r)$, with center p and radius r, on M and the volume of the geodesic ball $V_n(p^-,r)$, with center p^- and radius r, on $M^- = [0,l) \times_{f(t)} \mathbb{S}^{n-1}$ with the base point p^-, respectively. The equality in* (3.4) *holds if and only if $B(p,r)$ and $B(p,R)$ are isometric to $V_n(p^-,r)$ and $V_n(p^-,R)$, respectively.*

(ii) (*The Bishop-type volume comparison theorem* I) *for $r_0 < \min\{l(p), l\}$ the inequality*

$$\operatorname{vol}(B(p,r_0)) \leqslant \operatorname{vol}(V_n(p^-,r_0)) \tag{3.5}$$

holds, with equality if and only if $B(p,r_0)$ is isometric to $V_n(p^-,r_0)$.

Proof. Choosing $s = r = 0$ and $\Gamma = S_p^{n-1}$, the assertion (3.4) of (i) can be derived directly by applying Theorem 3.2. When the equality in (3.4) holds, then by Lemma 2.1 we have $J(t,\xi) = f(t)$, for all $t < \min\{d_\xi, l\}$ and $\xi \in S_p^{n-1}$. As in the proof of Bishop's comparison theorem II on pages 72–73 of [3], this implies that $\operatorname{tr} U^2 = (\operatorname{tr} U)^2/(n-1)$, where $U = \mathbb{A}'\mathbb{A}^{-1}$, and thus U is a scalar matrix and so is

$\mathbb{A}$ with $\mathbb{A}(\xi, t) = f(t)I$. Hence, the metric of $B(p, r_0)$ is of the form (1.8), that is $B(p, r_0)$ is isometric to $V_n(p^-, r_0)$. For $r < r_0 < \min\{l, l(p)\}$, by (3.4) we have

$$\frac{\operatorname{vol}(B(p, r_0))}{\operatorname{vol}(V_n(p^-, r_0))} \leqslant \frac{\operatorname{vol}(B(p, r))}{\operatorname{vol}(V_n(p^-, r))}.$$

Letting $r \to 0$, together with the facts that $\mathbb{A}(0, \xi) = 0$, $\mathbb{A}'(0, \xi) = I$, $f(0) = 0$ and $f'(0) = 1$, we obtain

$$\frac{\operatorname{vol}(B(p, r_0))}{\operatorname{vol}(V_n(p^-, r_0))} \leqslant \lim_{r \to 0} \frac{\operatorname{vol}(B(p, r))}{\operatorname{vol}(V_n(p^-, r))} = 1$$

by using L'Hôpital's rule. This implies the assertion (3.5) of (ii). The last assertion of (ii) can be obtained by applying (i) directly. □

Remark 3.5. (1) One may find that the volume comparison result (3.4) was claimed to be proved in [11], [18] (in fact, as explained by Shiohama, [18] is the first draft of [11]. This leads to the result that one may find some clerical errors in [18]). However, we find that the way in the proof for the volume comparison result in [11], [18] is different from the one we have used here. Besides, we have shown much more interesting conclusions for the model manifold in [8], [15], [16], like the regularity of the metric on the model manifold, and the existence of the model space, etc. So, we still think that it is meaningful to give (3.4) here even if it might be proved in [11], [18] using a different method. By the way, by using Lemma 2.1 directly, we can also get the *Bishop-type volume comparison theorem* I, which was pointed out in [8], [16].

(2) Although the last assertion of (i) of Corollary 3.4 can be derived by using a method similar to that of the proof of Bishop's comparison theorem II on pages 72–73 of [3], the volume inequality (i.e. the first assertion) cannot be obtained in this way. This is because the fact that the function $\psi(t) = (n-1)C_k(t)/S_k(t)$ defined on page 73 of [3], with

$$S_k(t) = \begin{cases} \dfrac{\sin\sqrt{k}t}{\sqrt{k}}, & k > 0, \\ t, & k = 0, \\ \dfrac{\sinh\sqrt{-k}t}{\sqrt{-k}}, & k < 0, \end{cases} \quad \text{and} \quad C_k(t) = S_k'(t),$$

is nonincreasing on its domain of definition (that is, $[0, \pi/\sqrt{k})$ if $k > 0$, and $[0, \infty)$ if $k \leqslant 0$) is necessary for completing the proof. However, in our case, $\psi(t)$ becomes $\psi(t) = (n-1)f'(t)/f(t)$, from which we cannot get any information about the monotonicity of $\psi(t)$. This leads to the invalidity of the way in the proof of Bishop's comparison theorem II on pages 72–73 of [3] for proving the first assertion in (i) of Corollary 3.4.

4. Volume comparison theorems for manifolds with radial sectional curvature bounded from above

Now, we will recall a Laplacian comparison result which will play an important role in the derivation of the volume comparison theorems in this section. In [12], Kasue proved that for smooth complete connected Riemannian n-manifolds ($n \geqslant 2$) with *radial sectional curvature bounded from above* by $k(t)$, $0 < t < l$ (a continuous function) with respect to a fixed point p, the inequality

$$\Delta r(x) \geqslant (n-1)\frac{f'(t)}{f(t)}, \tag{4.1}$$

with $r(x) = d(x,p)$ and $x = \gamma_\xi(t)$ defined as before, holds, where $f(t)$ is a solution of

$$\begin{cases} f''(t) + k(t)f(t) = 0, \quad 0 < t < l, \; f\big|(0,l) > 0, \\ f(0) = 0, \quad f'(0) = 1. \end{cases} \tag{4.2}$$

In order to prove this estimate, we need the following conclusions (cf. Lemmas 2.3 and 3.2 in [19]).

Lemma 4.1 ([19]). *Given a complete Riemannian manifold M and a point $p \in M$, we have $\Delta r(x) = \mathbb{A}'(t,\xi)/\mathbb{A}(t,\xi)$, where, as before, $r(x) = d(x,p)$, with $x = \gamma_\xi(t)$, and $\mathbb{A}(t,\xi)$ denote the distance to the point p and the path of linear transformations, respectively.*

Remark 4.2. The above lemma shows us the connection between the Laplacian of the Riemannian distance function and the volume element of the original manifold. Besides, the equality in (4.1) holds if and only if $\mathbb{A}(t,\xi) = f(t)I$ for some $\xi \in S_p^{n-1}$, which implies that the given manifold has a *radial sectional curvature* $k(t) = -f''(t)/f(t)$ along the direction $\xi \in S_p^{n-1}$, i.e., $K(v_x,\xi) = -f''(t)/f(t)$ with $x = \gamma_\xi(t)$ and $\xi \in S_p^{n-1}$.

By applying Lemmas 4.1 and 3.1, together with (4.1), we can prove the following result.

Theorem 4.3. *Let M be an n-dimensional complete Riemannian manifold. For a point $p \in M$, let $r \leqslant R$, $s \leqslant S$, $r \leqslant s$, $R \leqslant S < \min\{\mathrm{inj}(p), l\}$. If M has a radial sectional curvature upper bound $k(t) = -f''(t)/f(t)$, then we have*

$$\frac{\mathrm{vol}(A^{\Gamma}_{r,R}(p))}{\mathrm{vol}(A^{\Gamma}_{s,S}(p))} \leqslant \frac{\mathrm{vol}^{M^+}(A^{\Gamma}_{r,R}(p^+))}{\mathrm{vol}^{M^+}(A^{\Gamma}_{s,S}(p^+))}, \tag{4.3}$$

with equality if and only if the equality in the curvature assumption holds, i.e., $K(V, v_x) \leqslant -(n-1)f''(t)/f(t)$ *on* M *for* $V \perp v_x$, $V \in S_x^{n-1} \subseteq T_x M$, $x = \gamma_\xi(t) = \exp_p(t\xi)$ *with* $t < \min\{\mathrm{inj}(p), l\}$ *and* $\xi \in \Gamma$, *where* $\mathrm{vol}(A^\Gamma_{r,R}(p))$ *is defined by* (3.1) *and* $\mathrm{vol}(A^\Gamma_{s,S}(p))$ *has the same meaning as* $\mathrm{vol}(A^\Gamma_{r,R}(p))$ *with just* r *and* R *being replaced by* s *and* S, *respectively; moreover,* $\mathrm{vol}^{M^+}(A^\Gamma_{\cdot,\cdot}(p^+))$ *denotes the corresponding annulus on the model manifold* $M^+ = [0,l) \times_f \mathbb{S}^{n-1}$ *with the base point* p^+.

Proof. By Lemma 3.1, we know that if we want to prove (4.3), it suffices to show that

$$\frac{\mathrm{vol}(A^\Gamma_{x,y}(p))}{\mathrm{vol}^{M^+}(A^\Gamma_{x,y}(p^+))}$$

is nondecreasing for $0 \leqslant x \leqslant y < \min\{\mathrm{inj}(p), l\}$, where

$$\mathrm{vol}(A^\Gamma_{x,y}(p)) = \int_\Gamma \mathrm{d}\sigma \int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t$$

with, as before, $\mathrm{d}\sigma$ being the $(n-1)$-volume element on $\mathbb{S}^{n-1}$, and $\mathbb{A}(t,\xi)$ and d_ξ the path of linear transformations and the distance to the cut locus of p in the direction $\xi \in S_p^{n-1}$, respectively.

By (4.1), (4.2) and Lemma 4.1, for any $\xi \in S_p^{n-1}$ and $t < d_\xi$ we have

$$\Big(\frac{\mathbb{A}(t,\xi)}{\mathbb{A}^{M^+}(t,p^+)}\Big)' = \Big(\frac{\mathbb{A}(t,\xi)}{f^{n-1}(t)}\Big)' = \frac{1}{f^{n-1}(t)}\Big(\mathbb{A}'(t,\xi) - (n-1)\frac{f'(t)}{f(t)}\mathbb{A}(t,\xi)\Big) \geqslant 0,$$

with $\mathbb{A}^{M^+}(t,p^+)$ being the path of linear transformations (defined on a subspace of $T_{p^+}M^+$) for the model space M^+. This implies that $\mathbb{A}(t,\xi)/\mathbb{A}^{M^+}(t,p^+)$ is nonincreasing (with respect to the variable t) for any $\xi \in S_p^{n-1}$ and $t < d_\xi$. This, together with Lemma 3.1, yields

$$\frac{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t} \leqslant \frac{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}$$

for $y \leqslant z$, which indicates that

$$\begin{aligned}(4.4)\qquad \int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t &\leqslant \frac{\int_{\min\{x,d_\xi\}}^{\min\{y,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}{\int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t \\ &\leqslant \frac{\int_{x}^{\min\{y,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}{\int_{x}^{\min\{z,d_\xi\}} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t \\ &\leqslant \frac{\int_{x}^{y} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}{\int_{x}^{z} \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t} \int_{\min\{x,d_\xi\}}^{\min\{z,d_\xi\}} \mathbb{A}(t,\xi)\,\mathrm{d}t,\end{aligned}$$

where the second inequality follows from the fact that

$$\frac{\int_x^a \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}{\int_x^b \mathbb{A}^{M^+}(t,p^+)\,\mathrm{d}t}$$

(i.e. $\int_x^a f^{n-1}(t)\,\mathrm{d}t/\int_x^b f^{n-1}(t)\,\mathrm{d}t$) is nonincreasing when $a<b$, and the last inequality always holds in three cases $d_\xi \leqslant y \leqslant z$, $y \leqslant d_\xi \leqslant z$ and $y \leqslant z \leqslant d_\xi$. Integrating both sides of (4.4) results in

$$\operatorname{vol}(A^\Gamma_{x,y}(p)) \leqslant \frac{\operatorname{vol}^{M^+}(A^\Gamma_{x,y}(p^+))}{\operatorname{vol}^{M^+}(A^\Gamma_{x,z}(p^+))}\operatorname{vol}(A^\Gamma_{x,z}(p)),$$

which implies the first conclusion of Theorem 4.3, i.e., (4.3). The second assertion of Theorem 4.3 follows from the last part of Remark 4.2. □

By using Theorem 4.3, we obtain the following volume comparison results.

Corollary 4.4. *Let M be an n-dimensional complete Riemannian manifold. Then, under the curvature assumption in Theorem 4.3, we have:*
(i) (*The Gromov-type relative volume comparison theorem* II) *the inequality*

$$\frac{\operatorname{vol}(B(p,r))}{\operatorname{vol}(B(p,R))} \leqslant \frac{\operatorname{vol}(V_n(p^+,r))}{\operatorname{vol}(V_n(p^+,R))} \tag{4.5}$$

holds for $r \leqslant R < \min\{l, \operatorname{inj}(p)\}$, where, as before, $\operatorname{vol}(B(p,r))$ and $\operatorname{vol}(V_n(p^+,r))$ denote the volume of the geodesic ball $B(p,r)$, with center p and radius r, on M and the volume of the geodesic ball $V_n(p^+,r)$, with center p^+ and radius r, on $M^+ = [0,l) \times_{f(t)} \mathbb{S}^{n-1}$ with the base point p^+, respectively. The equality in (4.5) holds if and only if $B(p,r)$ and $B(p,R)$ are isometric to $V_n(p^+,r)$ and $V_n(p^+,R)$, respectively.

(ii) (*The Bishop-type volume comparison theorem* II) *for $r_0 < \min\{\operatorname{inj}(p), l\}$, the inequality*

$$\operatorname{vol}(B(p,r_0)) \geqslant \operatorname{vol}(V_n(p^+,r_0)) \tag{4.6}$$

holds, with equality if and only if $B(p,r_0)$ is isometric to $V_n(p^-,r_0)$.

Proof. Choosing $s=r=0$ and $\Gamma = S^{n-1}_p$, the assertion (4.5) of (i) can be derived directly by applying Theorem 4.3. The characterization for the equality in (4.5) can be obtained by using Theorem 4.3 directly.

For $r < r_0 < \min\{l, \mathrm{inj}(p)\}$, by (4.5) we have

$$\frac{\mathrm{vol}(B(p,r_0))}{\mathrm{vol}(V_n(p^+,r_0))} \geqslant \frac{\mathrm{vol}(B(p,r))}{\mathrm{vol}(V_n(p^+,r))}.$$

Letting $r \to 0$, together with the facts that $\mathbb{A}(0,\xi) = 0$, $\mathbb{A}'(0,\xi) = I$, $f(0) = 0$ and $f'(0) = 1$, we obtain

$$\frac{\mathrm{vol}(B(p,r_0))}{\mathrm{vol}(V_n(p^+,r_0))} \leqslant \lim_{r\to 0} \frac{\mathrm{vol}(B(p,r))}{\mathrm{vol}(V_n(p^+,r))} = 1$$

by using L'Hôpital's rule. This implies the assertion (4.6) of (ii). □

Remark 4.5. The *Bishop-type volume comparison theorem* II above was proved in [8], [16] by a different method.

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