Communications in Mathematics

Vishnuvardhana S.V. Venkatesha

Almost pseudo symmetric Sasakian manifold admitting a type of quarter symmetric metric connection

Communications in Mathematics, Vol. 23 (2015), No. 2, 163–169

Persistent URL: `http://dml.cz/dmlcz/144804`

Communications in Mathematics 23 (2015) 163–169

Almost pseudo symmetric Sasakian manifold admitting a type of quarter symmetric metric connection

Vishnuvardhana S.V. and Venkatesha

Abstract. In the present paper we have obtained the necessary condition for the existence of almost pseudo symmetric and almost pseudo Ricci symmetric Sasakian manifold admitting a type of quarter symmetric metric connection.

1 Introduction

Cartan [2] initiated the study of Riemannian symmetric spaces and obtained a classification of these spaces. The class of Riemannian symmetric manifolds is a very natural generalization of the class of manifolds of constant curvature. Many authors have been studied the notion of locally symmetric manifolds by extending into several manifolds such as recurrent manifolds [20], pseudo-Riemannian manifold with recurrent concircular curvature tensor [14], semi-symmetric manifolds [17], pseudo symmetric manifolds [3], weakly symmetric manifolds [18], almost pseudo symmetric manifolds [8], etc.

A non-flat Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be almost pseudo symmetric $(APS)_n$ [8], if the curvature tensor R satisfies the condition

$$\begin{aligned}(\nabla_X R)(Y,Z)W = {} & [A(X)+B(X)]R(Y,Z)W + A(Y)R(X,Z)W \\ & + A(Z)R(Y,X)W + A(W)R(Y,Z)X + g(R(Y,Z)W,X)P\,,\end{aligned} \tag{1}$$

where A, B are two nonzero 1-forms defined by

$$A(X) = g(X,P), \qquad B(X) = g(X,Q). \tag{2}$$

If in particular $A = B$ in (1) then the manifold reduces to a pseudo symmetric manifold introduced by M. C. Chaki [3].

2010 MSC: 53C05, 53C15, 53D10

Key words: Almost pseudo symmetric manifold, pseudo Ricci symmetric manifold, almost pseudo Ricci symmetric manifold, quarter-symmetric metric connection

Recently Gazi, Pal and Mallick with U.C. De studied almost pseudo conformally symmetric manifolds [9], almost pseudo-Z-symmetric manifolds [11] and almost pseudo concircularly symmetric manifolds [10]. Also Yilmaz in [21] studied decomposable almost pseudo conharmonically symmetric manifolds.

In 2007, Chaki and Kawaguchi [5] introduced the notion of almost pseudo Ricci symmetric manifolds as an extended class of pseudo symmetric manifold. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci symmetric manifold $(APRS)_n$, if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = [A(X) + B(X)]S(Y,Z) + A(Y)S(X,Z) + A(Z)S(X,Y), \quad (3)$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and A, B are two nonzero 1-forms defined as in (2). If, in particular, $B = A$ then almost pseudo Ricci symmetric manifold reduces to pseudo Ricci symmetric manifold [4]. It may be mentioned that almost pseudo Ricci symmetric manifold is not a particular case of weakly Ricci symmetric manifold, introduced by Tamassy and Binh [19]. Since then, several papers [7], [13], [16] have appeared concerning different aspects of almost pseudo Ricci symmetric manifold.

Motivated by the above study, in the present paper we have studied the existence of almost pseudo symmetric and almost pseudo Ricci-symmetric Sasakian manifolds admitting a quarter-symmetric metric connection. The paper is organized as follows: In Section 2, we have given a brief introduction about Sasakian manifolds and some formulae for quarter-symmetric metric connection. In the next section, it is shown that almost pseudo symmetric Sasakian manifold satisfies cyclic Ricci tensor only when $3A(X) + B(X) = 0$. Section 4 is devoted to study of almost pseudo symmetric Sasakian manifold with respect to quarter symmetric metric connection, here we proved that there is no almost pseudo symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless $3A+B$ vanishes everywhere. In the last section we studied almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection.

2 Preliminaries

It is known that in a Sasakian manifold M^n, the following relations hold [1], [15]:

$$\phi^2 = -I + \eta \, o \, \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad g(X,\xi) = \eta(X), \quad (4)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (5)$$

$$(\nabla_X \phi)Y = R(\xi, X)Y, \quad \nabla_X \xi = -\phi X, \quad (6)$$

$$d\eta(\phi X, \xi) = 0, \quad d\eta(X,\xi) = 0, \quad (7)$$

$$(a) \quad g(R(\xi,X)Y,\xi) = g(X,Y) - \eta(X)\eta(Y), \quad (b) \quad R(\xi,X)\xi = -X + \eta(X)\xi, \quad (8)$$

$$S(X,\xi) = (n-1)\eta(X), \quad (9)$$

$$g(R(X,Y)\xi, Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \quad (10)$$

for any vector fields X, Y, Z on M^n.

Here we consider a quarter symmetric metric connection $\tilde{\nabla}$ on a Sasakian manifold given by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{11}$$

The relation between curvature tensor $\tilde{R}(X,Y)Z$ of M^n with respect to quarter symmetric metric connection $\tilde{\nabla}$ and the Riemannian curvature tensor $R(X,Y)Z$ with respect to the connection ∇ is given by [12]

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi Z + \eta(X)g(Y,Z)\xi \\ - \eta(Y)g(X,Z)\xi + \{\eta(Y)X - \eta(X)Y\}\eta(Z)\,,\end{aligned} \tag{12}$$

where $R(X,Y)Z$ is the Riemannian curvature of the manifold. Also from (12) we obtain

$$\tilde{S}(X,Y) = S(Y,Z) - 2d\eta(\phi Z,Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{13}$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ respectively. From (13) it is clear that in a Sasakian manifold the Ricci tensor with respect to the quarter-symmetric metric connection is symmetric.

Now contracting (13) we have

$$\tilde{r} = r + 2(n-1), \tag{14}$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ respectively.

3 Almost Pseudo Symmetric Sasakian manifold Satisfying Cyclic Ricci tensor

On taking the cyclic sum of (3), we get

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = [3A(X) + B(X)]S(Y,Z) \\ + [3A(Y) + B(Y)]S(X,Z) + [3A(Z) + B(Z)]S(X,Y).\end{aligned} \tag{15}$$

Let M^n admits a cyclic Ricci tensor. Then (15) reduces to

$$[3A(X) + B(X)]S(Y,Z) + [3A(Y) + B(Y)]S(X,Z) + [3A(Z) + B(Z)]S(X,Y) = 0. \tag{16}$$

Taking $Z = \xi$ in (16) and using (9), we have

$$\begin{aligned}[3A(X) + B(X)](n-1)\eta(Y) + [3A(Y) + B(Y)](n-1)\eta(X) \\ + [3A(\xi) + B(\xi)]S(X,Y) = 0.\end{aligned} \tag{17}$$

Now putting $Y = \xi$ in the above equation and by making use of (2), (4) and (9), we obtain

$$(n-1)[3A(X) + B(X)] + [3\eta(P) + \eta(Q)](n-1)\eta(X) + [3\eta(P) + \eta(Q)]S(X,\xi) = 0. \tag{18}$$

Again taking $X = \xi$ in (18) and using (2), (4) and (9), we get

$$3\eta(P) + \eta(Q) = 0. \tag{19}$$

From equations (18) and (19), it follows that

$$3A(X) + B(X) = 0. \tag{20}$$

Thus we can state:

Theorem 1. *An almost pseudo symmetric Sasakian manifold satisfies cyclic Ricci tensor if and only if* $3A(X) + B(X) = 0$ *for any vector fields* X, Y, Z *on* M^n.

4 Almost pseudo symmetric Sasakian manifold with respect to quarter symmetric metric connection

Definition 1. A Sasakian manifold (M^n, g) $(n \geq 2)$ is said to be almost pseudo symmetric $(APS)_n$ with respect to quarter symmetric metric connection, if there exist 1-forms A and B and a vector field P such that

$$\begin{aligned}(\bar{\nabla}_X \bar{R})(Y, Z)W = {} & [A(X) + B(X)]\bar{R}(Y, Z)W \\ & + A(Y)\bar{R}(X, Z)W + A(Z)\bar{R}(Y, X)W \\ & + A(W)\bar{R}(Y, Z)X + g(\bar{R}(Y, Z)W, X)P,\end{aligned} \tag{21}$$

Theorem 2. *There is no almost pseudo symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless* $3A + B$ *vanishes everywhere.*

Proof. Contracting (21), we get

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, W) = {} & [A(X) + B(X)]\tilde{S}(Z, W) + A(\tilde{R}(X, Z)W) \\ & + A(Z)\tilde{S}(X, W) + A(W)\tilde{S}(Z, X) + A(\tilde{R}(X, W)Z).\end{aligned} \tag{22}$$

Substituting $W = \xi$ in (22) and then using the relations (12) and (13), we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{S})(Z, \xi) = {} & 2n\eta(Z)A(X) + 2(n-1)\eta(Z)B(X) + 2(n-2)\eta(X)A(Z) \\ & + \eta(P)\big\{S(Z, X) - 2d\eta(\phi X, Z) + g(X, Z) + (n-2)\eta(X)\eta(Z)\big\} \\ & + 2\eta(Z)A(X) - 2g(X, Z)A(\xi).\end{aligned} \tag{23}$$

Now, we know that

$$(\tilde{\nabla}_X \tilde{S})(Z, Y) = \tilde{\nabla}_X(\tilde{S}(Z, Y)) - \tilde{S}(\tilde{\nabla}_X Z, Y) - \tilde{S}(Z, \tilde{\nabla}_X Y). \tag{24}$$

Replacing Y with ξ in the above equation and using (4), (11) and (13), we get

$$(\tilde{\nabla}_X \tilde{S})(Z, \xi) = S(Z, \phi X) + (1 - 2n)g(Z, \phi X). \tag{25}$$

By virtue of (23) and (25), we obtain

$$\begin{aligned}S(Z, \phi X) + (1 - 2n)g(Z, \phi X) = {} & 2n\eta(Z)A(X) \\ & + 2(n-1)\eta(Z)B(X) + 2(n-2)\eta(X)A(Z) \\ & + \eta(P)\big\{S(Z, X) - 2d\eta(\phi X, Z) + g(X, Z) + (n-2)\eta(X)\eta(Z)\big\} \\ & + 2\eta(Z)A(X) - 2g(X, Z)A(\xi).\end{aligned} \tag{26}$$

Taking $X = Z = \xi$ in (26) and using (4) and (9), we obtain

$$3A(\xi) + B(\xi) = 0. \tag{27}$$

Putting $Z = \xi$ in (22) and by making use of equations (4), (7), (12) and (13), we get

$$\begin{aligned} S(\phi X, W) + (1-2n)g(\phi X, W) &= 2(n+1)A(X)\eta(W) \\ &+ 2(n-1)B(X)\eta(W) - 2g(X, W)A(\xi) \\ &+ A(\xi)\{S(X, W) - 2d\eta(\phi W, X) + g(X, W) + (n-2)\eta(X)\eta(W)\} \\ &+ 2(n-2)\eta(X)A(W). \end{aligned} \tag{28}$$

By taking $X = \xi$ in (28) and then using (4) and (9), it follows that

$$0 = 2(2n-1)A(\xi)\eta(W) + 2(n-1)B(\xi)\eta(W) + 2(n-2)A(W). \tag{29}$$

Again putting $W = \xi$ in (28) and using (4) and (9), we have

$$0 = 2(n+1)A(X) + 2(n-1)B(X) + 4(n-2)A(\xi)\eta(X). \tag{30}$$

Adding (29) with (30) by replacing W by X and in view of (27), we get

$$4nA(X) - 2A(X) + 2(n-1)B(X) + 2(n-2)A(\xi)\eta(X) = 0. \tag{31}$$

Further replacing W by X in (29) and then adding with (31), in view of (27) we arrive at

$$3A(X) + B(X) = 0$$

□

5 Almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection

A non-flat n-dimensional Riemannian manifold M^n $(n \geq 2)$ is said to be almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection if there exist 1-forms A and B such that

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = [A(X) + B(X)]\tilde{S}(Y, Z) + A(Y)\tilde{S}(X, Z) + A(Z)\tilde{S}(X, Y). \tag{32}$$

Theorem 3. *There is no almost pseudo Ricci symmetric Sasakian manifold admitting a quarter symmetric metric connection, unless $3A + B = 0$ everywhere.*

Proof. Assume that M^n is an almost pseudo Ricci symmetric Sasakian manifold with respect to quarter symmetric metric connection. Replacing Z with ξ in (32) and then using (25), we get

$$\begin{aligned} &S(\phi X, Y) + (1-2n)g(\phi X, Y) \\ &\qquad = [A(X) + B(X)]\tilde{S}(Y, \xi) + A(Y)\tilde{S}(X, \xi) + A(\xi)\tilde{S}(X, Y). \end{aligned} \tag{33}$$

By substituting $X = Y = \xi$ in (33) and then using (13), one can get

$$3A(\xi) + B(\xi) = 0. \tag{34}$$

Taking $X = \xi$ in (33) and in view of (13), we have

$$0 = 2(n-1)\{2A(\xi)\eta(Y) + B(\xi)\eta(Y) + A(Y)\}. \tag{35}$$

Putting $Y = \xi$ in (33) and using (13), we get

$$0 = 2(n-1)\{A(X) + B(X) + 2A(\xi)\eta(X)\}. \tag{36}$$

Adding (35) with (36) by replacing Y by X and in view of (34), we obtain

$$2(n-1)\{2A(X) + B(X) + A(\xi)\eta(X)\} = 0. \tag{37}$$

Replacing Y by X in (35) and then adding with (37), in view of (34) we get

$$3A(X) + B(X) = 0. \qquad \square$$

Acknowledgement

Vishnuvardhana S.V. was supported by the Department of Science and Technology, INDIA though the JRF [IF140186] DST/INSPIRE FELLOWSHIP/2014/181.

References

[1] D.E. Blair: *Contact manifolds in Riemannian geometry*. Springer-Verlag, Berlin (1976). Lecture Notes in Mathematics, 509

[2] E. Cartan: Sur une classe remarquable d'espaces de Riemann. *Soc. Math., France* 54 (1926) 214–264.

[3] M.C. Chaki: On pseudo symmetric manifolds. *Analele Stiint, Univ.Al-I. Cuza* 33 (1987) 53–58.

[4] M.C. Chaki: On pseudo Ricci symmetric manifolds. *Bulg. J. Phys.* 15 (1988) 526–531.

[5] M.C. Chaki, T. Kawaguchi: On almost pseudo Ricci symmetric manifolds. *Tensor N. S.* 68 (2007) 10–14.

[6] A. De: On almost pseudo symmetric manifolds admitting a type of semi-symmetric non-metric connection. *Acta Math. Hungar.* 125 (2009) 183–190.

[7] U.C. De, A. De: On almost pseudo-conformally symmetric Ricci-recurrent manifolds with applications to relativity. *Czechoslovak Mathematical Journal* 62 (2012) 1055–1072.

[8] U.C. De, A.K. Gazi: On almost pseudo symmetric manifolds. *Ann. Univ. Sci. Budap.* 51 (2008) 53–68.

[9] U.C. De, A.K. Gazi: On almost pseudo conformally symmetric manifolds. *Demonstratio Mathematica* 4 (2009) 861–878.

[10] U.C. De, S. Mallick: On almost pseudo concircularly symmetric manifolds. *The Journal of Mathematics and Computer Science* 4 (3) (2012) 317–330.

[11] U.C. De, P. Pal: On Almost Pseudo-Z-symmetric Manifolds. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 53 (1) (2014) 25–43.

[12] U.C. De, J. Sengupta: Quarter-symmetric metric connection on a Sasakian manifold. *Commun. Fac. Sci. Univ. Ank. Series A1.* 49 (2000) 7–13.

[13] S.K. Hui, F.O. Zengin: On almost pseudo concircular Ricci symmetric manifolds. *Boletin de la Asociacion Matematica Venezolana* 1 (2012) 57–69.

[14] K. Olszak, Z. Olszak: On pseudo-riemannian manifolds with recurrent concircular curvature tensor. *Acta Mathematica Hungar.* 137 (1-2) (2012) 64–71.

[15] S. Sasaki: *Lecture notes on almost contact manifolds, part 1.* Tohoku University (1975).

[16] A.A. Shaikh, S.K. Hui, C.S. Bagewadi: On Quasi-Conformally Flat Almost Pseudo Ricci Symmetric Manifolds. *Tamsui Oxford Journal of Mathematical Sciences* 26 (2) (2010) 203–219.

[17] Z.I. Szabo: Structure theorems on riemannian spaces satisfying $R(X,Y)R=0$. *Journal of Differential Geometry* 17 (1982) 531–582.

[18] L. Tamassy, T.Q. Binh: On weakly symmetric and weakly projectively symmetric riemannian manifolds. *Colloquia Mathematica Societatis Janos Bolyai* 56 (1989) 663–670.

[19] L. Tamassy, T.Q. Binh: On weak symmetries of Einstein and Sasakian manifolds. *Tensor* 53 (1993) 140–148. International Conference on Differential Geometry and its Applications (Bucharest, 1992)

[20] A.G. Walker: On ruse's space of recurrent curvature. *Proceedings of the London Mathematical Society* 52 (1951) 36–64.

[21] H.B. Yilmaz: On Decomposable Almost Pseudo Conharmonically Symmetric Manifolds. *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 51 (1) (2012) 111–124.

Authors' address:
Department of Mathematics, Kuvempu University, Shankaraghatta, 577 451, Shimoga, Karnataka, India

E-mail: `vensmath@gmail.com`, `svvishnuvardhana@gmail.com`

Received: 24 August, 2015
Accepted for publication: 30 October, 2015
Communicated by: Olga Rossi