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FINITISTIC DIMENSION AND RESTRICTED INJECTIVE DIMENSION

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Abstract. We study the relations between finitistic dimensions and restricted injective dimensions. Let R be a ring and T a left R-module with $A = \operatorname{End}_R T$. If ${}_RT$ is selforthogonal, then we show that $\operatorname{rid}(T_A) \leqslant \operatorname{findim}(A_A) \leqslant \operatorname{findim}({}_RT) + \operatorname{rid}(T_A)$. Moreover, if R is a left noetherian ring and T is a finitely generated left R-module with finite injective dimension, then $\operatorname{rid}(T_A) \leqslant \operatorname{findim}(A_A) \leqslant \operatorname{fin.inj.dim}({}_RR) + \operatorname{rid}(T_A)$. Also we show by an example that the restricted injective dimensions of a module may be strictly smaller than the Gorenstein injective dimension.

Keywords: finitistic dimension; restricted injective dimension; tilting module

MSC 2010: 18G10, 18G20

1. Introduction

Throughout this paper R is a nontrivial associative ring with identity. We denote by R-Mod (respectively, Mod-R) the category of all left (respectively, right) R-modules and by R-mod (respectively, mod-R) the category of all left (respectively, right) R-modules possessing finitely generated projective resolutions. The left little (respectively, big) finitistic (projective) dimension of R, denoted by $\operatorname{findim}({}_RR)$ (respectively, $\operatorname{Findim}({}_RR)$), is defined as the supremum of the projective dimensions of all modules in R-mod (respectively, R-Mod) of finite projective dimension. Clearly, $\operatorname{findim}({}_RR) \leqslant \operatorname{Findim}({}_RR)$. Similarly, one may define the right finitistic dimension of R by using the projective dimensions of right R-modules.

It is well known that $\operatorname{Findim}({}_RR)$ coincides with the Krull dimension of R in case R is commutative and noetherian and that $\operatorname{findim}({}_RR) = \operatorname{depth} R$ in case R is

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commutative local and noetherian. Therefore, in the latter case both the dimensions are finite and they coincide if and only if R is a Cohen-Macaulay ring.

In the case when R is a finite-dimensional algebra over a field, the little and the big finitistic dimensions may also differ; that is, the First Finitistic Dimension Conjecture fails (cf. [7]). However, it is still an open question, known as the Second Finitistic Dimension Conjecture, whether the little finitistic dimension of a finite-dimensional algebra R is always finite. This conjecture is closely related to Nakayama conjecture, Gorenstein symmetry conjecture, Wakamatsu tilting conjecture and other homological conjectures, and attracts many algebraists (cf. [1], [3], [9]).

One can use the injective dimensions of right R-modules to define the right finitistic injective dimension of R, denoted by $\mathrm{fin.inj.dim}(R_R)$, by

$$\mathrm{fin.inj.dim}(R_R) = \sup\{\mathrm{id}(M_R);\ \mathrm{id}(M_R) < \infty,\ M_R \in \mathrm{mod}\text{-}R\}.$$

Note that $\mathrm{findim}({}_RR) = \mathrm{fin.inj.dim}(R_R)$ provided that R is artinian.

As a refinement of Gorenstein flat dimension in some sense, Christensen, Foxby and Frankild in [5] defined the large restricted flat dimension of a right homologically bounded complex X as $\mathrm{Rfd}_R\, X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X);\ T \in \mathcal{F}_0(R)\}$, where $\mathcal{F}_0(R)$ denotes the category of R-modules of finite flat dimension.

The small restricted flat dimension of a right homologically bounded complex X is $\mathrm{rfd}_R\, X = \sup\{\sup(T \otimes^{\mathbf{L}}_R X);\ T \in \mathcal{P}^f_0(R)\}$, where $\mathcal{P}^f_0(R)$ denotes the category of finitely generated R-modules of finite projective dimension.

Dually, the large restricted injective dimension of a left homologically bounded complex Y is defined by $\mathrm{Rid}_R\, Y = \sup\{-\inf(\mathbf{R}\operatorname{Hom}_R(T,Y));\ T \in \mathcal{P}_0(R)\}$, where $\mathcal{P}_0(R)$ denotes the category of R-modules of finite projective dimension.

The small restricted injective dimension of a left homologically bounded complex Y is $\mathrm{rid}_R\, Y = \sup\{-\inf(\mathbf{R}\operatorname{Hom}_R(T,Y));\ T \in \mathcal{P}^f_0(R)\}$.

For right R-modules one has $\mathrm{Rid}(M_R) = \sup\{m \in \mathbb{N};\ \mathrm{Ext}^m_R(T,M) \neq 0$ for some $T \in \mathcal{P}_0(R)\}$; $\mathrm{rid}(N_R) = \sup\{m \in \mathbb{N};\ \mathrm{Ext}^m_R(T,N) \neq 0$ for some $T \in \mathcal{P}^f_0(R)\}$.

In [8], Wei investigated the finitistic dimension in terms of the restricted flat dimension. Inspired by this, we find that the restricted injective dimension is also a useful tool to describe the finitistic dimension.

2. Preliminaries

In this paper, we fix R to be a ring and $T \in R$-Mod with the endomorphism ring A. We denote by $\mathrm{Add}_R\, T$ or $\mathrm{add}_R\, T$ the class of modules isomorphic, respectively, to direct summands of direct or finite direct sums of copies of ${}_RT$. Further, $\mathrm{Prod}_R\, T$

will denote the class of modules isomorphic to direct summands of direct products of copies of ${}_RT$.

Let $\mathcal{C} \subseteq R$-Mod be a category and $M \in R$-Mod. We denote by $\mathcal{C}$-dim$({}_RM)$ the minimal integer m such that there is an exact sequence $0 \to M \to T_0 \to \ldots \to T_m \to 0$ with each $T_i \in \mathcal{C}$ and call it the $\mathcal{C}$-dimension of ${}_RM$. Note that for some ${}_RM$ the $\mathcal{C}$-dimension of ${}_RM$ may not exist. In the latter case, we denote $\mathcal{C}$-dim$({}_RM) = \infty$. The category of all modules $M \in R$-Mod such that $\mathcal{C}$-dim$({}_RM) < \infty$ is denoted by $\widehat{\mathcal{C}}$.

We define Findim$({}_RT)$ to be the supremum of the $\mathrm{Add}_R T$-dimensions of all modules in R-Mod of finite $\mathrm{Add}_R T$-dimension. Similarly, findim$({}_RT)$ is denoted to be the supremum of the $\mathrm{add}_R T$-dimensions of all modules in R-Mod of finite $\mathrm{add}_R T$-dimension.

Recall that $T \in R$-Mod is selforthogonal if $T \in \mathrm{Ker}\,\mathrm{Ext}_R^{i\geqslant 1}(T, -)$, i.e., T belongs to the category of all modules M such that $\mathrm{Ext}_R^i(T, M) = 0$ for all $i \geqslant 1$.

Let A be a ring and $T \in$ Mod-A. Then T_A is said to be Gorenstein injective provided there is an exact sequence of injective modules $\ldots \to I_1 \to I_0 \to I_{-1} \to \ldots$ such that $T \cong \mathrm{Im}(I_1 \twoheadrightarrow I_0)$ and such that $\mathrm{Hom}_A(J, -)$ leaves the sequence exact whenever J_A is an injective module. The Gorenstein injective dimension of T_A is denoted by Gid(T_A). We denote by pd$({}_RT)$ and id$({}_RT)$, respectively, the projective and injective dimension of the module ${}_RT$.

Finally, we recall the definitions of tilting and cotilting modules.

Let R be a ring and $T \in R$-Mod. We say ${}_RT$ is tilting if (1) pd$({}_RT) < \infty$, (2) $\mathrm{Ext}_R^{i\geqslant 1}(T, T^{(X)}) = 0$ for all sets X and (3) there is an exact sequence $0 \to R \to T_0 \to \ldots \to T_n \to 0$ for some n with each $T_i \in \mathrm{Add}_R T$. And ${}_RT$ is classical tilting if (1) pd$({}_RT) < \infty$ and $T \in R$-mod, (2) $\mathrm{Ext}_R^{i\geqslant 1}(T, T) = 0$ and (3) there is an exact sequence $0 \to R \to T_0 \to \ldots \to T_n \to 0$ for some n with each $T_i \in \mathrm{add}_R T$.

Dually, we say ${}_RT$ is cotilting if (1) id$({}_RT) < \infty$, (2) $\mathrm{Ext}_R^{i\geqslant 1}(T^X, T) = 0$ for all sets X and there exists (3) an injective cogenerator E and a long exact sequence $0 \to T_n \to \ldots \to T_0 \to E \to 0$ for some n with each $T_i \in \mathrm{Prod}_R T$.

3. Finitistic dimension of endomorphism rings

First note that the following relations between finitistic dimensions and restricted injective dimensions.

Lemma 3.1. *Let A be a ring and $T_A \in$ Mod-A.*

(1) Rid$(T_A) \leqslant$ Findim(A_A).
(2) rid$(T_A) \leqslant$ findim(A_A).

P r o o f. We show that if $\operatorname{Findim}(A_A) = n < \infty$ or $\operatorname{findim}(A_A) = n < \infty$, then $\operatorname{Rid}(T_A) \leqslant n$ or $\operatorname{rid}(T_A) \leqslant n$, respectively. (1) It is sufficient to show that $\operatorname{Ext}_A^{n+1}(M,T) = 0$ for any M_A with finite projective dimension. Since $\operatorname{Findim}(A_A) = n$, we have $\operatorname{pd}(M_A) \leqslant n$. Hence $\operatorname{Ext}_A^{n+1}(M,T) = 0$. (2) Similarly. □

Lemma 3.2. *If T is a selforthogonal left R-module with $A = \operatorname{End}_R T$, then*

(1) $\operatorname{add}_R T\text{-}\dim({}_RM) = \operatorname{pd}(\operatorname{Hom}_R(M,T)_A)$ *for any* $M \in \widehat{\operatorname{add}_R T}$;

(2) $\operatorname{findim}({}_RT) \leqslant \operatorname{pd}({}_RT)$.

P r o o f. (1) Suppose that $\operatorname{add}_R T\text{-}\dim({}_RM) = m$. There is an exact sequence of minimal length

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \longrightarrow \cdots \xrightarrow{f^m} T^m \longrightarrow 0$$

with $T^i \in \operatorname{add}_R T$ for each $0 \leqslant i \leqslant m$. By applying the functor $\operatorname{Hom}_R(-,T)$ to the above sequence, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R(T^m,T) &\xrightarrow{\operatorname{Hom}_R(f^m,T)} \operatorname{Hom}_R(T^{m-1},T) \longrightarrow \cdots \\ \longrightarrow \operatorname{Hom}_R(T^0,T) &\xrightarrow{\operatorname{Hom}_R(f^0,T)} \operatorname{Hom}_R(M,T) \longrightarrow 0 \end{aligned}$$

as T is selforthogonal. Note that the above sequence is a projective resolution of the A-module $\operatorname{Hom}_R(M,T)$ and so $\operatorname{pd}(\operatorname{Hom}_R(M,T)_A) \leqslant m$. If $\operatorname{pd}(\operatorname{Hom}_R(M,T)_A) < m$, then it is easy to see that $\operatorname{Coker}\operatorname{Hom}_R(f^m,T)$ is a projective A-module. Now by applying the functor $\operatorname{Hom}_A(-,T)$ to the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(T^m,T) \xrightarrow{\operatorname{Hom}_R(f^m,T)} \operatorname{Hom}_R(T^{m-1},T) \longrightarrow \operatorname{Coker}\operatorname{Hom}_R(f^m,T) \longrightarrow 0$$

we obtain that $\operatorname{Ker} f^m \cong \operatorname{Hom}_A(\operatorname{Coker}\operatorname{Hom}(f^m,T),T) \in \operatorname{add}_R T$. This shows that $\operatorname{add}_R T\text{-}\dim({}_RM) < m$, a contradiction. Therefore,

$$\operatorname{add}_R T\text{-}\dim({}_RM) = \operatorname{pd}(\operatorname{Hom}_R(M,T)_A)$$

for any $M \in \widehat{\operatorname{add}_R T}$.

(2) Let $M \in \widehat{\operatorname{add}_R T}$. There is an exact sequence

$$0 \longrightarrow M \xrightarrow{f^0} T^0 \xrightarrow{f^1} T^1 \longrightarrow \cdots \xrightarrow{f^m} T^m \longrightarrow 0$$

with $T^i \in \operatorname{add}_R T$ for each $0 \leqslant i \leqslant m$. Assume that $\operatorname{pd}({}_RT) = s$. If $m \leqslant s$, then there is nothing to prove. So we suppose that $m > s$. Write $K_i = \operatorname{Ker} f^{i+1}$

for each $0 \leqslant i \leqslant m$. Note that $K_0 = M$ and $K_m = T^m$ in this case. Since T is selforthogonal, we have that $\mathrm{Ext}^j_R(K_m, T^i) = 0$ for all $j \geqslant 1$ and $0 \leqslant i \leqslant m$. Note that $\mathrm{Ext}^1_R(K_m, K_{m-1}) \cong \mathrm{Ext}^m_R(K_m, K_0)$ by dimension shifting. Since $\mathrm{pd}({}_RT) = s$ and $m > s$, we have $\mathrm{Ext}^1_R(K_m, K_{m-1}) = 0$. Hence the sequence

$$0 \longrightarrow K_{m-1} \longrightarrow T^{m-1} \longrightarrow T^m \longrightarrow 0$$

splits. It follows that $\mathrm{findim}({}_RT) \leqslant \mathrm{pd}({}_RT)$. □

Lemma 3.3. *Let R be a ring and ${}_RT \in R$-Mod with $A = \mathrm{End}_R T$. If ${}_RT$ is selforthogonal, then for any $Y \in$ mod-A with $\mathrm{Ext}^{i\geqslant 1}_A(Y, T) = 0$ and $\mathrm{pd}(Y_A) < \infty$, one has $Y \cong \mathrm{Hom}_R(\mathrm{Hom}_A(Y, T), T)$ canonically and $\mathrm{add}_R T$-$\mathrm{dim}(\mathrm{Hom}_A(Y, T)) < \infty$.*

Proof. Since $Y \in$ mod-A and $\mathrm{pd}(Y_A) < \infty$, we can take a finitely generated projective resolution of Y_A,

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Y \longrightarrow 0,$$

with each P_i finitely generated and projective. Note that $Y_A \in \mathrm{Ker}\,\mathrm{Ext}^{i\geqslant 1}_A(-, T)$ and $\mathrm{Ker}\,\mathrm{Ext}^{i\geqslant 1}_A(-, T)$ is closed under kernels of epimorphisms. Therefore, we have the following exact sequence by applying the functor $\mathrm{Hom}_A(-, T)$:

$$0 \longrightarrow \mathrm{Hom}_A(Y, T) \longrightarrow \mathrm{Hom}_A(P_0, T) \longrightarrow \cdots \longrightarrow \mathrm{Hom}_A(P_n, T) \longrightarrow 0.$$

Note that $\mathrm{Hom}_A(P_i, T) \in \mathrm{add}_R T$ for $0 \leqslant i \leqslant n$. As $\mathrm{add}_R T$-$\mathrm{dim}(\mathrm{Hom}_A(Y, T)) < \infty$. Moreover, by applying the functor $\mathrm{Hom}_R(-, T)$ to the above sequence, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_A(P_n, T), T) \longrightarrow \cdots$$
$$\longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_A(P_0, T), T) \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_A(Y, T), T) \longrightarrow 0$$

as T is selforthogonal and $\mathrm{Ker}\,\mathrm{Ext}^{i\geqslant 1}_R(-, T)$ is closed under kernels of epimorphisms. Since P_i is finitely generated and projective, we have that

$$\mathrm{Hom}_R(\mathrm{Hom}_A(P_i, T), T) \cong P_i \otimes_A \mathrm{Hom}_R(T, T) \cong P_i$$

for each i. It follows that $Y_A \cong \mathrm{Hom}_R(\mathrm{Hom}_A(Y, T), T)$ canonically. □

Now we can prove one of our main results.

Theorem 3.4. *Let R be a ring and ${}_RT \in R$-Mod with $A = \operatorname{End}_R T$. If ${}_RT$ is selforthogonal, then one has*

$$\operatorname{rid}(T_A) \leqslant \operatorname{findim}(A_A) \leqslant \operatorname{findim}({}_RT) + \operatorname{rid}(T_A).$$

Proof. By Lemma 3.1, we have $\operatorname{rid}(T_A) \leqslant \operatorname{findim}(A_A)$. If $\operatorname{findim}({}_RT)$ or $\operatorname{rid}(T_A)$ is infinite, then we have nothing to prove. Now assume that $\operatorname{findim}({}_RT) = m < \infty$ and $\operatorname{rid}(T_A) = n < \infty$. We will show that $\operatorname{findim}(A_A) \leqslant m + n$.

Let $Y_A \in$ mod-A with $\operatorname{pd}(Y_A) < \infty$. By taking a finitely generated projective resolution of Y_A, we have the exact sequence

$$0 \longrightarrow P_r \longrightarrow \cdots \longrightarrow P_0 \longrightarrow Y \longrightarrow 0,$$

where P_i is finitely generated and projective for each $0 \leqslant i \leqslant r$. Now we claim that $\operatorname{pd}(\Omega^n(Y_A)) \leqslant m$ and so $\operatorname{pd}(Y_A) \leqslant n + m$, where $\Omega^n(Y_A)$ denotes the nth syzygy of the A-module Y. Thus the conclusion will follow from the arbitrarity of the choice of Y_A.

In fact, since $\operatorname{rid}(T_A) = n$, we have that $\Omega^n(Y_A) \in \operatorname{Ker}\operatorname{Ext}_A^{i\geqslant 1}(-,T)$. It is easy to see that $\operatorname{pd}(\Omega^n(Y_A)_A) < \infty$. By Lemma 3.3, we have that

$$\Omega^n(Y_A)_A \cong \operatorname{Hom}_R(\operatorname{Hom}_A(\Omega^n(Y_A),T),T)$$

canonically and $\operatorname{add}_R T$-$\dim(\operatorname{Hom}_A(\Omega^n(Y_A),T)) < \infty$. It follows that

$$\operatorname{add}_R T\text{-}\dim(\operatorname{Hom}_A(\Omega^n(Y_A),T)) \leqslant \operatorname{findim}({}_RT) = m.$$

Now by Lemma 3.2, we obtain that

$$\operatorname{pd}(\Omega^n(Y_A)_A) = \operatorname{pd}(\operatorname{Hom}_R(\operatorname{Hom}_A(\Omega^n(Y_A),T),T)_A) \leqslant m,$$

as desired. □

Corollary 3.5. *Let R be a ring and ${}_RT \in R$-Mod with $A = \operatorname{End}_R T$. If ${}_RT$ is selforthogonal, then* $\operatorname{rid}(T_A) \leqslant \operatorname{findim}(A_A) \leqslant \operatorname{pd}({}_RT) + \operatorname{rid}(T_A)$.

Proof. The result follows from Lemma 3.2 and Theorem 3.4. □

Corollary 3.6. *Let R be a ring and ${}_RT \in R$-Mod with $A = \operatorname{End}_R T$. If ${}_RT$ is selforthogonal with* $\operatorname{add}_R T$ *closed under kernels of epimorphisms, then* $\operatorname{findim}(A_A) = \operatorname{rid}(T_A)$.

Proof. It is easy to see that $\operatorname{findim}({}_RT) = 0$. Now the result follows from Theorem 3.4. □

Corollary 3.7. *If A is an Artin algebra, then* $\operatorname{findim}(A_A)=\operatorname{rid}(A_A)$.

P r o o f. It is clear. □

Lemma 3.8. *Let A be an Artin algebra. If* $\operatorname{fin.inj.dim}({}_AA)<\infty$, *then there exists a cotilting module T such that* ${}^{\perp}T={}^{\perp}A\text{-mod}$ *and* $\operatorname{id}({}_AT)=\operatorname{fin.inj.dim}({}_AA)$.

P r o o f. See [4], Proposition 2.1. □

Proposition 3.9. *Let R be an Artin algebra with* $\operatorname{fin.inj.dim}({}_RR)<\infty$. *If ${}_RT$ is a classical cotilting module with $A=\operatorname{End}_R T$ such that* ${}^{\perp}T={}^{\perp}R\text{-mod}$ *and* $\operatorname{id}({}_RT)=\operatorname{fin.inj.dim}({}_RR)$, *then* $\operatorname{add}_R T$ *is closed under kernels of epimorphisms. In particular,*

$$\operatorname{findim}(A_A)=\operatorname{rid}(T_A)\leqslant\operatorname{id}({}_RT)=\operatorname{fin.inj.dim}({}_RR).$$

P r o o f. Let $0\to M\to T_0\to T_1\to 0$ be an exact sequence with $T_0,T_1\in\operatorname{add}_R T$. Clearly, ${}_RM\in R\text{-mod}$. By hypothesis, we have $\operatorname{Ext}^1_R(T,M)=0$ and so the above exact sequence splits. Hence ${}_RM\in\operatorname{add}_R T$, i.e., $\operatorname{add}_R T$ is closed under kernels of epimorphisms. The remaining part follows from a dual argument of [8], Lemma 1.2 (2), Corollary 3.6 and Lemma 3.8. □

Proposition 3.10. *Let R be a ring and* ${}_RT\in R\text{-Mod}$ *with $A=\operatorname{End}_R T$. If ${}_RT$ is a selforthogonal module with finite injective dimension, then*

$$\operatorname{rid}(T_A)\leqslant\operatorname{findim}(A_A)\leqslant\operatorname{Fin.inj.dim}({}_RR)+\operatorname{rid}(T_A).$$

Moreover, if R is left noetherian and ${}_RT\in R\text{-mod}$, *then*

$$\operatorname{rid}(T_A)\leqslant\operatorname{findim}(A_A)\leqslant\operatorname{fin.inj.dim}({}_RR)+\operatorname{rid}(T_A).$$

P r o o f. It is sufficient to show that $\operatorname{Fin.inj.dim}({}_RR)\geqslant\operatorname{findim}({}_RT)$ by Theorem 3.4. Assume that $\operatorname{Fin.inj.dim}({}_RR)=t<\infty$. Let $M\in\widehat{\operatorname{add}_R T}$. We have an exact sequence

$$0\longrightarrow M\xrightarrow{f^0}T^0\xrightarrow{f^1}T^1\longrightarrow\cdots\xrightarrow{f^m}T^m\longrightarrow 0$$

with each $T^i\in\operatorname{add}_R T$. Since $\operatorname{id}({}_RT)<\infty$, we have $\operatorname{id}({}_RM)<\infty$ and so $\operatorname{id}({}_RM)\leqslant t$. It is easy to see that $\operatorname{Ext}^{i\geqslant 1}_R(\operatorname{Ker} f^i,T)=0$ as T is selforthogonal. If $m>t$, then by the dimension shifting we have that

$$\operatorname{Ext}^1_R(\operatorname{Ker} f^{t+2},\operatorname{Ker} f^{t+1})\cong\operatorname{Ext}^{t+1}_R(\operatorname{Ker} f^{t+2},M)=0.$$

It follows that $\operatorname{Ker} f^{t+1}\in\operatorname{add}_R T$. Consequently, $\operatorname{add}_R T\text{-}\dim({}_RM)\leqslant t$. Therefore, $\operatorname{findim}({}_RT)\leqslant\operatorname{Fin.inj.dim}({}_RR)$. The last statement is clear. □

Corollary 3.11. *Let R be a ring and ${}_RT \in R$-Mod with $A = \operatorname{End}_R T$. If ${}_RT$ is injective, then* $\operatorname{findim}(A_A) \leqslant \operatorname{Fin.inj.dim}({}_RR) + \operatorname{rid}(T_A)$. *Moreover, if R is left noetherian and ${}_RT$ is finitely generated, then* $\operatorname{findim}(A_A) \leqslant \operatorname{fin.inj.dim}({}_RR) + \operatorname{rid}(T_A)$.

Note that the restricted injective dimensions may be strictly smaller than the Gorenstein injective dimension as the following example shows.

Example 3.12. There exists a finite dimensional algebra A satisfying the following statement: There is a right A-module T_A such that

$$\operatorname{rid}(T_A) = \operatorname{Rid}(T_A) < \operatorname{Gid}(T_A) = \operatorname{id}(T_A) < \infty.$$

Proof. By [6], for any arbitrary finite numbers m and n, there is a finite dimensional algebra A with $\operatorname{findim}({}_AA) = \operatorname{Findim}({}_AA) = m$ and $\operatorname{findim}(A_A) = \operatorname{Findim}(A_A) = n$. It is well known that $\operatorname{findim}({}_AA) = \operatorname{fin.inj.dim}(A_A)$ and $\operatorname{Findim}({}_AA) = \operatorname{Fin.inj.dim}(A_A)$. Let us take $m > 0$ and $n = 0$. We have $\operatorname{rid}(T_A) = \operatorname{Rid}(T_A) = 0$ by Lemma 3.1. Now we take $T_A \in \text{mod-}A$ with $\operatorname{id}(T_A) = m$. By the remark after [2], Theorem 2.3, we have $0 = \operatorname{rid}(T_A) = \operatorname{Rid}(T_A) < \operatorname{Gid}(T_A) = \operatorname{id}(T_A) < \infty$. □

Recall that for an Artin algebra R, ${}_RT \in R$-mod is classical cotilting if

(1) $\operatorname{id}({}_RT) < \infty$,
(2) $\operatorname{Ext}_R^{i \geqslant 1}(T, T) = 0$ and
(3) there is an exact sequence $0 \to T_n \to \ldots \to T_0 \to_R (DR) \to 0$ for some n with each $T_i \in \operatorname{add}_R T$, where D is the usual duality in Artin algebras.

Proposition 3.13. *Let R, A be Artin algebras and $T \in R$-mod with $A = \operatorname{End}_R T$. If ${}_RT$ is classical tilting and classical cotilting, then*

$$\max\{\operatorname{findim}(R_R) - \operatorname{pd}({}_RT), \operatorname{pd}_R T\} \leqslant \operatorname{findim}(A_A) \leqslant \operatorname{pd}({}_RT) + \operatorname{id}({}_RT).$$

Proof. Since $\operatorname{rid}(T_A) \leqslant \operatorname{id}(T_A) = \operatorname{id}({}_RT)$, the second inequality above follows from Corollary 3.5. Now consider the classical tilting and cotilting module ${}_A(DT)_R$. By Proposition 3.10, we have the inequalities

$$\begin{aligned}\operatorname{findim}(R_R) &\leqslant \operatorname{fin.inj.dim}({}_AA) + \operatorname{rid}((DT)_R) \leqslant \operatorname{fin.inj.dim}({}_AA) + \operatorname{id}((DT)_R)\\ &= \operatorname{findim}(A_A) + \operatorname{id}((DT)_R).\end{aligned}$$

It follows that $\operatorname{findim}(R_R) - \operatorname{id}((DT)_R) \leqslant \operatorname{findim}(A_A)$. Note that $\operatorname{id}((DT)_R) = \operatorname{pd}({}_RT)$ and so $\operatorname{findim}(R_R) - \operatorname{pd}({}_RT) \leqslant \operatorname{findim}(A_A)$. In addition,

$$\operatorname{findim}(A_A) = \operatorname{fin.inj.dim}({}_AA) \geqslant \operatorname{id}({}_A(DT)) = \operatorname{id}((DT)_R) = \operatorname{pd}({}_RT).$$

Thus the first inequality holds. □

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