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ON THE NÖRLUND MEANS OF VILENKIN-FOURIER SERIES

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Abstract. We prove and discuss some new (H_p, L_p)-type inequalities of weighted maximal operators of Vilenkin-Nörlund means with non-increasing coefficients $\{q_k\colon k \geqslant 0\}$. These results are the best possible in a special sense. As applications, some well-known as well as new results are pointed out in the theory of strong convergence of such Vilenkin-Nörlund means. To fulfil our main aims we also prove some new estimates of independent interest for the kernels of these summability results.

In the special cases of general Nörlund means t_n with non-increasing coefficients analogous results can be obtained for Fejér and Cesàro means by choosing the generating sequence $\{q_k\colon k \geqslant 0\}$ in an appropriate way.

Keywords: Vilenkin system; Vilenkin group; Nörlund means; martingale Hardy space; maximal operator; Vilenkin-Fourier series; strong convergence; inequality

MSC 2010: 42C10, 42B25

1. Introduction

The definitions and notation used in this introduction can be found in our next section. In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means σ^* can be found in Schipp [25] for Walsh series and in Pál, Simon [24] for bounded Vilenkin series. Fujji [6] and Simon [28] verified that σ^* is bounded from H_1 to L_1. Weisz [41] generalized this result and proved boundedness of σ^* from the martingale space H_p to the Lebesgue space L_p for $p > 1/2$. Simon [26] gave a counterexample which shows that boundedness does not hold for $0 < p < 1/2$.

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A counterexample for $p=1/2$ was given by Goginava [14]. Weisz [39] proved that the maximal operator of the Fejér means σ^* is bounded from the Hardy space $H_{1/2}$ to the space $weak$-$L_{1/2}$. Goginava [13] (see also [34]) proved that the weighted maximal operator $\widetilde{\sigma}^*$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2}$. Moreover, the rate of the weights $\{\log^2(n+1)\}_{n=1}^{\infty}$ in the n-th Fejér mean is given exactly. Analogous results for $0<p<1/2$ were proved in [33].

Riesz's logarithmic means with respect to Walsh and Vilenkin systems were studied by several authors. We mention, for instance, the papers by Simon [26], Gát, Nagy [11]. In [30] it was proved that the maximal operator of Riesz's means R^* is bounded from the Hardy space $H_{1/2}$ to the space $weak$-$L_{1/2}$, but is not bounded from the Hardy space H_p to the space L_p, when $0<p\leqslant 1/2$. Moreover, some theorems on boundedness of weighted maximal operators of Riesz's logarithmic means with respect to the Vilenkin-Fourier series were proved.

Móricz and Siddiqi [19] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p functions in norm. The case when $q_k=1/k$ was excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [10] Gát and Goginava investigated some properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. In [37] it was proved that there exists a martingale $f\in H_p$ $(0<p\leqslant 1)$ such that the maximal operator of Nörlund logarithmic means L^* is not bounded in the space L_p. For more information on Nörlund logarithmic means, see the papers of Blahota and Gát [3] and Nagy (see [23], [20] and [21]).

In [17] Goginava investigated the behaviour of Cesàro means of Walsh-Fourier series in detail. In the two-dimensional case approximation properties of Nörlund and Cesàro means were considered by Nagy [22]. Weisz [40] proved that the maximal operator $\sigma^{\alpha,*}$ is bounded from the martingale space H_p to the space L_p for $p>1/(1+\alpha)$. Goginava [15] gave a counterexample which shows that boundedness does not hold for $0<p\leqslant 1/(1+\alpha)$. Simon and Weisz [29] showed that the maximal operator $\sigma^{\alpha,*}$ $(0<\alpha<1)$ of the (C,α) means is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $weak$-$L_{1/(1+\alpha)}$. In [4] it was also proved that the maximal operator $\widetilde{\sigma}^{\alpha,*}$ is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha)}$. Moreover, this result cannot be improved in the following sense:

Theorem BT (Blahota, Tephnadze [4]). *Let $0<\alpha\leqslant 1$ and $\varphi\colon \mathbb{N}_+\to[1,\infty)$ be a non-decreasing function satisfying the condition*

$$\overline{\lim_{n\to\infty}}\frac{\log^{1+\alpha} n}{\varphi(n)}=\infty.$$

Then there exists a martingale $f \in H_{1/(1+\alpha)}(G)$, such that

$$\sup_{n\in\mathbb{N}} \left\| \frac{\sigma_n^\alpha f}{\varphi(n)} \right\|_{1/(1+\alpha)} = \infty.$$

It is well-known that Vilenkin systems do not form bases in the space $L_1(G_m)$. Moreover, there is a function in the Hardy space $H_1(G_m)$ such that the partial sums of f are not bounded in the L_1-norm. However, in Gát [8] (see also [1]) the following strong convergence result was obtained for all $f \in H_1$:

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k f - f\|_1}{k} = 0.$$

Simon [27] (see also [31]) proved that there exists an absolute constant c_p, depending only on p, such that

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^{n} \frac{\|S_k f\|_p^p}{k^{2-p}} \leqslant c_p \|f\|_{H_p}^p, \quad 0 < p \leqslant 1 \tag{1.1}$$

for all $f \in H_p$ and $n \in \mathbb{N}_+$, where $[p]$ denotes the integer part of p. In [35] it was proved that the sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1.1) cannot be improved.

Weisz considered the norm convergence of Fejér means of Vilenkin-Fourier series and proved the following:

Theorem W1 (Weisz [42]). *Let $p > 1/2$ and $f \in H_p$. Then there exists an absolute constant c_p, depending only on p, such that*

$$\|\sigma_k f\|_p \leqslant c_p \|f\|_{H_p} \quad \text{for all } f \in H_p \text{ and } k = 1, 2, \dots$$

Theorem W1 implies that

$$\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leqslant c_p \|f\|_{H_p}^p, \quad 1/2 < p < \infty,\ n = 1, 2, \dots$$

If Theorem W1 held for $0 < p \leqslant 1/2$, then we would have

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leqslant c_p \|f\|_{H_p}^p, \quad 0 < p \leqslant 1/2,\ n = 2, 3, \dots \tag{1.2}$$

However, in [36] it was proved that the assumption $p > 1/2$ in Theorem W1 is essential. In particular, we showed that there exists a martingale $f \in H_{1/2}$ such that

$$\sup_n \|\sigma_n f\|_{1/2} = \infty.$$

In [5] it was proved that (1.2) holds, though the Fejér means is not of type (H_p, L_p) for $0 < p \leqslant 1/2$. This result for the (C, α) means $(0 < \alpha < 1)$ when $p = 1/(1+\alpha)$ was generalized in [4].

In this paper we prove and discuss some new (H_p, L_p)-type inequalities for weighted maximal operators of Vilenkin-Nörlund means with non-increasing coefficients. These results are the best possible in a special sense. As applications, some well-known as well as new results are pointed out in the theory of strong convergence of Vilenkin-Nörlund means.

The paper is organized as follows: in order not to disturb our discussions later on some definitions and notation are presented in Section 2. The main results and some of their consequences can be found in Section 3. For the proofs of the main results we need some auxiliary results of independent interest. Also these results are presented in Section 3. The detailed proofs are given in Section 4.

2. Definitions and notation

Denote by $\mathbb{N}_+$ the set of positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ be a sequence of positive integers not less than 2. Denote by

$$Z_{m_n} := \{0, 1, \ldots, m_n - 1\}$$

the additive group of integers modulo m_n.

Define the group G_m as the complete direct product of the groups Z_{m_n} with the product of the discrete topologies of Z_{m_n}'s. In this paper we discuss bounded Vilenkin groups, i.e., the case when $\sup_{n \in \mathbb{N}} m_n < \infty$.

The direct product μ of the measures

$$\mu_n(\{j\}) := 1/m_n, \quad j \in Z_{m_n}$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_n, \ldots), \quad x_n \in Z_{m_n}.$$

It is easy to give a base for the neighbourhood of G_m:

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m\colon\ y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, \quad x \in G_m,\ n \in \mathbb{N}.$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}_+$ and

$$e_n := (0, \dots, x_n = 1, 0, \dots) \in G_m, \quad n \in \mathbb{N}.$$

It is evident that

$$\overline{I_N} = \left(\bigcup_{k=0}^{N-2} \bigcup_{x_k=1}^{m_k-1} \bigcup_{l=k+1}^{N-1} \bigcup_{x_l=1}^{m_l-1} I_{l+1}(x_k e_k + x_l e_l) \right) \cup \left(\bigcup_{k=0}^{N-1} \bigcup_{x_k=1}^{m_k-1} I_N(x_k e_k) \right). \tag{2.1}$$

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1, \quad M_{n+1} := m_n M_n, \quad n \in \mathbb{N},$$

then every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \qquad \text{where } n_k \in Z_{m_k},\ k \in \mathbb{N}_+$$

and only a finite number of n_k's differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. First we define the complex-valued functions $r_k(x)\colon\ G_m \to \mathbb{C}$, the generalized Rademacher functions, by

$$r_k(x) := \exp \frac{2\pi \mathrm{i} x_k}{m_k}, \quad \mathrm{i}^2 = -1,\ x \in G_m,\ k \in \mathbb{N}.$$

Now, define the Vilenkin system $\psi := (\psi_n\colon\ n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad n \in \mathbb{N}.$$

In particular, we call this system the Walsh-Paley system when $m \equiv 2$.

The norm (or quasi-norm) of the space $L_p(G_m)$ $(0 < p < \infty)$ is defined by

$$\|f\|_p^p := \int_{G_m} \|f\|^p \,\mathrm{d}\mu.$$

The space $weak$-$L_p(G_m)$ consists of all measurable functions f for which

$$\|f\|_{\mathrm{weak}-L_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < \infty.$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (see [38]).

Now we introduce analogues of the usual definitions in Fourier analysis. If $f \in L_1(G_m)$ we can define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f\overline{\psi}_n \,\mathrm{d}\mu, \quad S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad n \in \mathbb{N}_+,$$

respectively.

Recall that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n, \\ 0 & \text{if } x \notin I_n. \end{cases} \tag{2.2}$$

It is also known that (see [2], [9] and [16])

$$D_{sM_n} = D_{M_n} \sum_{k=0}^{s-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s-1} r_n^k, \tag{2.3}$$

and

$$D_{sM_n - j} = D_{sM_n} - \psi_{sM_n - 1}\overline{D_j}, \quad j = 1, \dots, M_n - 1. \tag{2.4}$$

The σ-algebra generated by the intervals $\{I_n(x)\colon\ x \in G_m\}$ will be denoted by $\mathcal{F}_n$ $(n \in \mathbb{N})$. Denote by $f = (f^{(n)}, n \in \mathbb{N})$ a martingale with respect to $\mathcal{F}_n$ $(n \in \mathbb{N})$. (For details see e.g. [42].)

The maximal function of a martingale f is defined by

$$f^* := \sup_{n \in \mathbb{N}} \|f^{(n)}\|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f = (f^{(n)}, n \in \mathbb{N})$ is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \to \infty} \int_{G_m} f^{(k)} \overline{\psi}_i \,\mathrm{d}\mu.$$

Let $\{q_n\colon\ n \geqslant 0\}$ be a sequence of non-negative numbers. The n-th Nörlund mean is defined by

$$t_n f := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=0}^{n-1} q_k.$$

It is well known that

$$t_n f(x) = \int_{G_m} f(t) F_n(x-t)\,\mathrm{d}t,$$

where F_n are the so called Nörlund kernels

$$F_n := \frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} D_k.$$

We always assume that $q_0 > 0$ and $\lim_{n\to\infty} Q_n = \infty$. In this case (see [18]) the summability method generated by $\{q_n\colon\ n \geqslant 0\}$ is regular if and only if

$$\lim_{n\to\infty} \frac{q_{n-1}}{Q_n} = \infty.$$

If $q_n \equiv 1$, then we get the usual n-th Fejér mean and Fejér kernel

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k,$$

respectively.

Let $t, n \in \mathbb{N}$. It is known that (see [7])

$$K_{M_n}(x) = \begin{cases} 0 & \text{if } x - x_t e_t \notin I_n,\ x \in I_t \setminus I_{t+1}, \\ \dfrac{M_t}{1 - r_t(x)} & \text{if } x - x_t e_t \in I_n,\ x \in I_t \setminus I_{t+1}, \\ (M_n + 1)/2 & \text{if } x \in I_n. \end{cases} \tag{2.5}$$

The (C,α)-means (Cesàro means) of the Vilenkin-Fourier series are defined by

$$\sigma_n^\alpha f := \frac{1}{A_n^\alpha} \sum_{k=1}^{n} A_{n-k}^{\alpha-1} S_k f,$$

where

$$A_0^\alpha := 0, \quad A_n^\alpha := \frac{(\alpha+1)\dots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \dots$$

The n-th Riesz logarithmic mean R_n and the Nörlund logarithmic mean L_n are defined by

$$R_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{k}, \quad L_n f := \frac{1}{l_n} \sum_{k=1}^{n-1} \frac{S_k f}{n-k},$$

respectively, where

$$l_n := \sum_{k=1}^{n-1} \frac{1}{k}.$$

For the martingale f we consider the maximal operators

$$t^* f := \sup_{n\in\mathbb{N}} |t_n f|, \quad \sigma^* f := \sup_{n\in\mathbb{N}} |\sigma_n f|, \quad \sigma^{\alpha,*} f := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha f|,$$
$$R^* f := \sup_{n\in\mathbb{N}} |R_n f|, \quad L^* f := \sup_{n\in\mathbb{N}} |L_n f|.$$

We also define the weighted maximal operators

$$\widetilde{t}^* f := \sup_{n\in\mathbb{N}} \frac{|t_n f|}{\log^{1+\alpha}(n+1)},$$
$$\widetilde{\sigma}^{\alpha,*} f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n^\alpha f|}{\log^{1+\alpha}(n+1)},$$
$$\widetilde{\sigma}^* f := \sup_{n\in\mathbb{N}} \frac{|\sigma_n f|}{\log^2(n+1)}.$$

A bounded measurable function a is called a p-atom, if there exists an interval I such that

$$\int_I a \,\mathrm{d}\mu = 0, \quad \|a\|_\infty \leqslant \mu(I)^{-1/p}, \quad \operatorname{supp}(a) \subset I.$$

3. Results

Main results and some of their consequences

Theorem 3.1. *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha \leqslant 1$, and let $\{q_n \colon n \geqslant 0\}$ be a non-increasing sequence of numbers such that*

$$\frac{n^\alpha}{Q_n} = O(1) \quad \text{as } n \to \infty, \tag{3.1}$$

and

$$\frac{q_n - q_{n+1}}{n^{\alpha-2}} = O(1) \quad \text{as } n \to \infty. \tag{3.2}$$

Then there exists an absolute constant c_α, depending only on α, such that

$$\|\widetilde{t}^* f\|_{1/(1+\alpha)} \leqslant c_\alpha \|f\|_{H_{1/(1+\alpha)}}.$$

Corollary 3.2 (Blahota, Tephnadze [4]). *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Then there exists an absolute constant c_α, depending only on α, such that*

$$\|\widetilde{\sigma}^{\alpha,*} f\|_{1/(1+\alpha)} \leqslant c_\alpha \|f\|_{H_{1/(1+\alpha)}}.$$

Corollary 3.3 (Goginava [13], Tephnadze [34]). *Let $f \in H_{1/2}$. Then there exists an absolute constant c such that*

$$\|\widetilde{\sigma}^* f\|_{1/2} \leqslant c\|f\|_{H_{1/2}}.$$

Theorem 3.4. *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$, and let $\{q_n \colon n \geqslant 0\}$ be a non-increasing sequence of numbers satisfying conditions* (3.1) *and* (3.2). *Then there exists an absolute constant c_α, depending only on α, such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|t_k f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{k} \leqslant c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.$$

Corollary 3.5 (Blahota, Tephnadze [4]). *Let $f \in H_{1/(1+\alpha)}$, where $0 < \alpha < 1$. Then there exists an absolute constant c_α, depending only on α, such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k^\alpha f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}}{k} \leqslant c_\alpha \|f\|_{H_{1/(1+\alpha)}}^{1/(1+\alpha)}.$$

Corollary 3.6 (Blahota, Tephnadze [5], Tephnadze [32]). *Let $f \in H_{1/2}$. Then there exists an absolute constant c such that*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|\sigma_k f\|_{1/2}^{1/2}}{k} \leqslant c\|f\|_{H_{1/2}}^{1/2}.$$

Remark 3.7. For some non-increasing sequence $\{q_n\colon\ n \geqslant 0\}$ of numbers, conditions (3.1) and (3.2) can be true or false independently.

Remark 3.8. Since Cesàro means satisfy conditions (3.1) and (3.2), we immediately obtain from Theorem BT that the rate of the weights $\{\log^{1+\alpha}(n+1)\}_{n=1}^{\infty}$ in the n-th Nörlund mean cannot be improved.

Some auxiliary results

Weisz proved that the following is true:

Lemma 3.9 (Weisz [42]). *Suppose that an operator T is σ-linear and for some $0 < p \leqslant 1$*

$$\int_{\overline{I}} |Ta|^p \,\mathrm{d}\mu \leqslant c_p < \infty$$

for every p-atom a, where I denotes the support of the atom. If T is bounded from L_∞ to L_∞, then

$$\|Tf\|_p \leqslant c_p \|f\|_{H_p}.$$

We also state three new lemmas we need for the proofs of our main results but which are also of independent interest:

Lemma 3.10. *Let $sM_n < r \leqslant (s+1)M_n$, where $1 \leqslant s < m_n$. Then*

$$\begin{aligned}(3.3)\qquad Q_r F_r = Q_r D_{sM_n} - w_{sM_n-1} &\sum_{l=1}^{sM_n-2} (q_{r-sM_n+l} - q_{r-sM_n+l+1}) l \overline{K_l} \\ &- \psi_{sM_n-1}(sM_n-1) q_{r-1} \overline{K_{sM_n-1}} + \psi_{sM_n} Q_{r-sM_n} F_{r-sM_n}.\end{aligned}$$

The next lemma is a generalization of an analogous estimate of the Cesàro means (see [12]).

Lemma 3.11. *Let $0 < \alpha \leqslant 1$ and let $\{q_n\colon\ n \geqslant 0\}$ be a non-increasing sequence of numbers satisfying conditions (3.1) and (3.2). Then*

$$|F_n| \leqslant \frac{c_\alpha}{n^\alpha} \left\{ \sum_{j=0}^{|n|} M_j^\alpha |K_{M_j}| \right\}.$$

Lemma 3.12. *Let $0 < \alpha \leqslant 1$ and let $\{q_n\colon n \geqslant 0\}$ be a non-increasing sequence of numbers satisfying conditions* (3.1) *and* (3.2)*. If $r \geqslant M_N$, then*

$$\int_{I_N} |F_r(x-t)|\,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_l^\alpha M_k}{r^\alpha M_N}, \quad x \in I_{l+1}(s_k e_k + s_l e_l),$$

where

$$1 \leqslant s_k \leqslant m_k - 1, \quad 1 \leqslant s_l \leqslant m_l - 1, \quad k = 0, \dots, N-2,\ l = k+2, \dots, N-1,$$

and

$$\int_{I_N} |F_r(x-t)|\,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_k}{M_N}, \quad x \in I_N(s_k e_k),$$

where

$$1 \leqslant s_k \leqslant m_k - 1, \quad k = 0, \dots, N-1.$$

4. Proofs

P r o o f of Lemma 3.10. In [16] Goginava proved a similar equality for the kernel of Nörlund logarithmic mean L_n. We will use his method.

Let $sM_n < r \leqslant (s+1)M_n$, where $1 \leqslant s < m_n$. It is easy to show that

$$\sum_{k=1}^{r} q_{r-k} D_k = \sum_{l=1}^{sM_n} q_{r-l} D_l + \sum_{l=sM_n+1}^{r} q_{r-l} D_l := I + II. \tag{4.1}$$

By combining (2.4) and Abel transformation we get that

$$\begin{aligned} I &= \sum_{l=0}^{sM_n-1} q_{r-sM_n+l} D_{sM_n-l} = \sum_{l=1}^{sM_n-1} q_{r-sM_n+l} D_{sM_n-l} + q_{r-sM_n} D_{sM_n} \\ &= D_{sM_n} \sum_{l=0}^{sM_n-1} q_{r-sM_n+l} - \psi_{sM_n-1} \sum_{l=1}^{sM_n-1} q_{r-sM_n+l} \overline{D_l} \\ &= (Q_r - Q_{r-sM_n}) D_{sM_n} - \psi_{sM_n-1} \sum_{l=1}^{sM_n-2} (q_{r-sM_n+l} - q_{r-sM_n+l+1}) l \overline{K_l} \\ &\quad - \psi_{sM_n-1} q_{r-1} (sM_n - 1) \overline{K_{sM_n-1}}. \end{aligned} \tag{4.2}$$

Since

$$D_{j+sM_n} = D_{sM_n} + \psi_{sM_n} D_j, \quad j = 1, 2, \dots, sM_n - 1,$$

for II we have that

$$II = \sum_{l=1}^{r-sM_n} q_{r-sM_n-l} D_{l+sM_n} = Q_{r-sM_n} D_{sM_n} + \psi_{sM_n} Q_{r-sM_n} F_{r-sM_n}. \tag{4.3}$$

By combining (4.1)–(4.3) we obtain (3.3) and the proof is complete.

P r o o f of Lemma 3.11. Let $sM_n < k \leqslant (s+1)M_n$, where $1 \leqslant s < m_n$, and let the sequence $\{q_k\colon\ k \geqslant 0\}$ be non-increasing and satisfy the condition

$$\frac{q_0 n}{Q_n} = O(1) \quad \text{as } n \to \infty. \tag{4.4}$$

By using Abel transformation we get that

$$Q_n = \sum_{j=1}^{n} q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n \tag{4.5}$$

and

$$F_n = \frac{1}{Q_n} \left(\sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \right). \tag{4.6}$$

Since

$$n|K_n| \leqslant c \sum_{A=0}^{|n|} M_A |K_{M_A}|, \tag{4.7}$$

by combining (4.5) and (4.6) we immediately get that

$$\begin{aligned} |F_n| &\leqslant \frac{c}{Q_n} \left(\sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \sum_{A=0}^{|n|} M_A |K_{M_A}| \\ &= \frac{c}{Q_n} \left(\sum_{j=1}^{n-1} -(q_{n-j} - q_{n-j-1}) + q_0 \right) \sum_{A=0}^{|n|} M_A |K_{M_A}| \\ &\leqslant c \frac{2q_0 - q_{n-1}}{Q_n} \sum_{A=0}^{|n|} M_A |K_{M_A}| \leqslant \frac{c}{Q_n} \sum_{A=0}^{\|n\|} M_A |K_{M_A}| \leqslant \frac{c}{n} \sum_{A=0}^{|n|} M_A |K_{M_A}|. \end{aligned}$$

Since the case $q_0 n / Q_n = \overline{O}(1)$, as $n \to \infty$, have already been considered, we can exclude it.

Let $0 < \alpha < 1$. We may assume that $\{q_k\colon\ k \geqslant 0\}$ satisfies conditions (3.1) and (3.2) and in addition, the condition

$$\frac{Q_n}{q_0 n} = o(1) \quad \text{as } n \to \infty.$$

It follows that

$$q_n = q_0 \frac{q_n n}{q_0 n} \leqslant q_0 \frac{Q_n}{q_0 n} = o(1) \quad \text{as } n \to \infty. \tag{4.8}$$

By using (4.8) we immediately get that

$$q_n = \sum_{l=n}^{\infty} (q_l - q_{l+1}) \leqslant \sum_{l=n}^{\infty} \frac{1}{l^{2-\alpha}} \leqslant \frac{c}{n^{1-\alpha}} \tag{4.9}$$

and

$$Q_n = \sum_{l=0}^{n-1} q_l \leqslant \sum_{l=1}^{n} \frac{c}{l^{1-\alpha}} \leqslant cn^{\alpha}. \tag{4.10}$$

Let $sM_n \leqslant k \leqslant (s+1)M_n$. It is easy to show that

$$Q_k |D_{sM_n}| \leqslant cM_n^{\alpha} |D_{sM_n}| \tag{4.11}$$

and

$$(sM_n - 1) q_{k-1} |K_{sM_n - 1}| \leqslant ck^{\alpha-1} M_n |K_{sM_n - 1}| \leqslant cM_n^{\alpha} |K_{sM_n - 1}|. \tag{4.12}$$

Let

$$n = s_{n_1} M_{n_1} + s_{n_2} M_{n_2} + \ldots + s_{n_r} M_{n_r}, \quad n_1 > n_2 > \ldots > n_r,$$

and

$$n^{(k)} = s_{n_{k+1}} M_{n_{k+1}} + \ldots + s_{n_r} M_{n_r}, \quad 1 \leqslant s_{n_l} \leqslant m_l - 1,\ l = 1, \ldots, r.$$

By combining (4.11), (4.12) and Lemma 3.10 we have that

$$\begin{aligned} |Q_n F_n| \leqslant c_\alpha \Bigg(M_{n_1}^{\alpha} |D_{s_{n_1} M_{n_1}}| + \sum_{l=1}^{s_{n_1} M_{n_1} - 1} &|(n^{(1)} + l)^{\alpha - 2}|\, |l K_l| \\ &+ M_{n_1}^{\alpha} |K_{s_{n_1} M_{n_1} - 1}| + |Q_{n^{(1)}} F_{n^{(1)}}| \Bigg). \end{aligned}$$

By repeating this process r-times we get that

$$|Q_nF_n| \leqslant c_\alpha \sum_{k=1}^{r} \Bigg(M_{n_k}^{\alpha} |D_{s_{n_k}M_{n_k}}| + \sum_{l=1}^{s_{n_k}M_{n_k}-1} (n^{(k)}+l)^{\alpha-2} |lK_l| \\ + M_{n_k}^{\alpha} |K_{s_{n_k}M_{n_k}-1}| \Bigg) := I + II + III.$$

By combining (2.2), (2.3) and (2.5) we obtain that

$$I \leqslant c_\alpha \sum_{k=1}^{n} M_k^{\alpha} |D_{M_k}| \leqslant c_\alpha \sum_{k=1}^{n} M_k^{\alpha} |K_{M_k}|$$

and

$$III \leqslant c_\alpha \sum_{k=1}^{r} M_{n_k}^{\alpha-1} |M_{n_k}K_{s_{n_k}M_{n_k}} - D_{s_{n_k}M_{n_k}}| \leqslant c_\alpha \sum_{k=1}^{r} M_k^{\alpha} |K_{M_k}|.$$

Moreover,

$$\begin{aligned} II &= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_k} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} (n^{(k)}+l)^{\alpha-2} |lK_l| \\ &= c_\alpha \sum_{k=1}^{r} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}^{s_AM_A-1}} (n^{(k)}+l)^{\alpha-2} |lK_l| \\ &\quad + c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} \sum_{s_AM_A-1}^{l=s_{A-1}M_{A-1}} (n^{(k)}+l)^{\alpha-2} |lK_l| \\ &\leqslant c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} |lK_l| \\ &\quad + c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-2} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} |lK_l| := II_1 + II_2. \end{aligned}$$

By combining (2.5) and (4.7) for II_1 we get that

$$\begin{aligned} II_1 &\leqslant c_\alpha \sum_{k=1}^{r} M_{n_{k+1}}^{\alpha-2} \sum_{A=1}^{n_{k+1}} \sum_{l=s_{A-1}M_{A-1}}^{s_AM_A-1} \sum_{j=0}^{A} M_j |K_{M_j}| \\ &\leqslant c_\alpha \sum_{k=1}^{n_1} M_k^{\alpha-2} \sum_{A=1}^{k} M_A \sum_{j=0}^{A} M_j |K_{M_j}| \leqslant c_\alpha \sum_{k=0}^{n_1} M_k^{\alpha-1} \sum_{j=0}^{k} M_j |K_{M_j}| \\ &= c_\alpha \sum_{j=0}^{n_1} M_j |K_{M_j}| \sum_{k=j}^{n_1} M_k^{\alpha-1} \leqslant c_\alpha \sum_{j=0}^{n_1} M_j^{\alpha} |K_{M_j}|. \end{aligned}$$

By using (4.7) for II_2 we have similarly that

$$
\begin{aligned}
II_2 &\leqslant c_\alpha \sum_{k=1}^{r} \sum_{A=n_{k+1}+1}^{n_k} M_A^{\alpha-1} \sum_{j=0}^{A} M_j |K_{M_j}| \\
&\leqslant c_\alpha \sum_{A=1}^{n_1} M_A^{\alpha-1} \sum_{j=0}^{A} M_j |K_{M_j}| \leqslant c_\alpha \sum_{j=0}^{n_1} M_j^{\alpha} |K_{M_j}|.
\end{aligned}
$$

The proof is completed by combining the estimates above. □

P r o o f of Lemma 3.12. Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, $1 \leqslant s_k \leqslant m_k - 1$, $1 \leqslant s_l \leqslant m_l - 1$. Then, by applying (2.5), we have that

$$K_{M_n}(x) = 0, \qquad \text{when } n > l > k.$$

Suppose that $k < n \leqslant l$. By using (2.5) we get that

$$|K_{M_n}(x)| \leqslant c M_k.$$

Let $n \leqslant k < l$. Then

$$|K_{M_n}(x)| = \frac{M_n + 1}{2} \leqslant c M_k.$$

If we now apply Lemma 3.11 we can conclude that

$$Q_r |F_r(x)| \leqslant c_\alpha \sum_{A=0}^{l} M_A^{\alpha} |K_{M_A}(x)| \leqslant c_\alpha \sum_{A=0}^{l} M_A^{\alpha} M_k \leqslant c_\alpha M_l^{\alpha} M_k. \tag{4.13}$$

Let $x \in I_{l+1}(s_k e_k + s_l e_l)$ for some $0 \leqslant k < l \leqslant N-1$. Since $x - t \in I_{l+1}(s_k e_k + s_l e_l)$ for $t \in I_N$ and $r \geqslant M_N$, from (4.13) we obtain that

$$\int_{I_N} |F_r(x-t)| \,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_l^{\alpha} M_k}{r^{\alpha} M_N}. \tag{4.14}$$

Let $x \in I_N(s_k e_k)$, $k = 0, \ldots, N-1$. Then, by applying Lemma 3.11 and (2.5), we have that

$$\int_{I_N} Q_r |F_r(x-t)| \,\mathrm{d}\mu(t) \leqslant \sum_{A=0}^{|r|} M_A^{\alpha} \int_{I_N} |K_{M_A}(x-t)| \,\mathrm{d}\mu(t). \tag{4.15}$$

Let $x \in I_N(s_k e_k)$, $k = 0, \ldots, N-1$, $t \in I_N$ and $x_q \neq t_q$, where $N \leqslant q \leqslant |r| - 1$. By combining (2.5) and (4.15) we get that

$$\int_{I_N} Q_r |F_r(x-t)| \,\mathrm{d}\mu(t) \leqslant c_\alpha \sum_{A=0}^{q-1} M_A^{\alpha} \int_{I_N} M_k \,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_k M_q^{\alpha}}{M_N}.$$

Hence,

$$\int_{I_N} |F_r(x-t)| \,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_k M_q^\alpha}{r^\alpha M_N} \leqslant \frac{c_\alpha M_k}{M_N}. \tag{4.16}$$

Let $x \in I_N(s_k e_k)$, $k = 0, \ldots, N-1$, $t \in I_N$ and $x_N = t_N, \ldots, x_{|r|-1} = t_{|r|-1}$. By applying again (2.5) and (4.15) we have that

$$\int_{I_N} |F_r(x-t)| \,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha}{r^\alpha} \sum_{A=0}^{|r|-1} M_A^\alpha \int_{I_N} M_k \,\mathrm{d}\mu(t) \leqslant \frac{c_\alpha M_k}{M_N}. \tag{4.17}$$

By combining (4.14), (4.16) and (4.17) we complete the proof of Lemma 3.12. $\square$

P r o o f of Theorem 3.1. According to Lemma 3.9 the proof of the first part of Theorem 3.1 will be complete if we show that

$$\int_{\overline{I_N}} |\tilde{t}^* a(x)|^{1/(1+\alpha)} \,\mathrm{d}\mu(x) < \infty$$

for every $1/(1+\alpha)$-atom a. We may assume that a is an arbitrary $1/(1+\alpha)$-atom with support I, $\mu(I) = M_N^{-1}$ and $I = I_N$. It is easy to see that $t_n(a) = 0$ when $n \leqslant M_N$. Therefore, we can suppose that $n > M_N$.

Let $x \in I_N$. Since t_n is bounded from L_∞ to L_∞ (the boundedness follows from Lemma 3.11 and $\|a\|_\infty \leqslant M_N^{1+\alpha}$ we obtain that

$$\begin{aligned} |t_n a(x)| &\leqslant \int_{I_N} |a(t)|\,|F_n(x-t)| \,\mathrm{d}\mu(t) \leqslant \|a\|_\infty \int_{I_N} |F_n(x-t)| \,\mathrm{d}\mu(t) \\ &\leqslant c_\alpha M_N^{1+\alpha} \int_{I_N} |F_n(x-t)| \,\mathrm{d}\mu(t). \end{aligned}$$

Let $x \in I_{l+1}(s_k e_k + s_l e_l)$, $0 \leqslant k < l < N$. From Lemma 3.12 we get that

$$|t_n a(x)| \leqslant \frac{c_\alpha M_l^\alpha M_k M_N^\alpha}{n^\alpha}. \tag{4.18}$$

Let $x \in I_N(s_k e_k)$, $0 \leqslant k < N$. Lemma 3.12 implies that

$$|t_n a(x)| \leqslant c_\alpha M_k M_N^\alpha. \tag{4.19}$$

By combining (2.1) and (4.18)–(4.19) we obtain that

$$\begin{aligned}
&\int_{\overline{I_N}} |\tilde{t^*} a(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\quad= \sum_{k=0}^{N-2}\sum_{s_k=1}^{m_k-1}\sum_{l=k+1}^{N-1}\sum_{s_l=1}^{m_l-1}\int_{I_{l+1}(s_ke_k+s_le_l)} \sup_{n>M_N}\Big|\frac{t_na(x)}{\log^{1+\alpha}(n+1)}\Big|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\qquad+\sum_{k=0}^{N-1}\sum_{s_k=1}^{m_k-1}\int_{I_N(s_ke_k)} \sup_{n>M_N}\Big|\frac{t_na(x)}{\log^{1+\alpha}(n+1)}\Big|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\quad\leqslant \frac{c_\alpha}{N}\sum_{k=0}^{N-2}\sum_{s_k=1}^{m_k-1}\sum_{l=k+1}^{N-1}\sum_{s_l=1}^{m_l-1}\int_{I_{l+1}(s_ke_k+s_le_l)} \sup_{n>M_N}|t_na(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\qquad+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\sum_{s_k=1}^{m_k-1}\int_{I_N(s_ke_k)} \sup_{n>M_N}|t_na(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\quad\leqslant \frac{c_\alpha}{N}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{(m_k-1)(m_l-1)}{M_{l+1}}\frac{(M_l^\alpha M_k)^{1/(1+\alpha)}M_N^{\alpha/(1+\alpha)}}{M^{\alpha/(1+\alpha)}}\\
&\qquad+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\frac{(m_k-1)}{M_N}M_N^{\alpha/(1+\alpha)}M_k^{1/(1+\alpha)}\\
&\quad\leqslant \frac{c_\alpha}{N}\sum_{k=0}^{N-2}\sum_{l=k+1}^{N-1}\frac{(M_l^\alpha M_k)^{1/(1+\alpha)}}{M_{l+1}}+\frac{c_\alpha}{N}\sum_{k=0}^{N-1}\frac{M_k^{1/(1+\alpha)}}{M_N^{1/(1+\alpha)}}\leqslant c_\alpha<\infty.
\end{aligned}$$

The proof is complete. □

P r o o f of Theorem 3.4. By Lemma 3.9 the proof of Theorem 3.4 will be complete, if we show that

$$\frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}\|t_ka\|_{1/(1+\alpha)}^{1/(1+\alpha)}\leqslant c_\alpha<\infty$$

for every $1/(1+\alpha)$-atom a. Analogously to the proof of Theorem 3.1 we may assume that a is an arbitrary $1/(1+\alpha)$-atom with support I, $\mu(I)=M_N^{-1}$ and $I=I_N$ and $n>M_N$.

Let $x\in I_N$. Since t_m is bounded from L_∞ to L_∞ (the boundedness follows from Lemma 3.11) and $\|a\|_\infty\leqslant M_N^{1+\alpha}$, we obtain that

$$\int_{I_N}|t_na(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu\leqslant\|a(x)\|_\infty^{1/(1+\alpha)}M_N^{-1}\leqslant c_\alpha<\infty.$$

Hence

$$\frac{1}{\log n}\sum_{k=M_N}^{n}\frac{1}{k}\int_{I_N}|t_ka(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu\leqslant\frac{c_\alpha}{\log n}\sum_{k=1}^{n}\frac{1}{k}\leqslant c_\alpha<\infty.$$

By combining (2.1) and (4.18)–(4.19) we can conclude that

$$\begin{aligned}
&\frac{1}{\log n}\sum_{k=M_N}^{n}\frac{1}{k}\int_{\overline{I_N}}|t_k a(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\quad=\frac{1}{\log n}\sum_{k=M_N}^{n}\sum_{r=0}^{N-2}\sum_{s_r=1}^{m_r-1}\sum_{l=r+1}^{N-1}\sum_{s_l=1}^{m_l-1}\frac{1}{k}\int_{I_{l+1}(s_re_r+s_le_l)}|t_k a(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\qquad+\frac{1}{\log n}\sum_{k=M_N}^{n}\sum_{r=0}^{N-1}\sum_{s_r=1}^{m_r-1}\frac{1}{k}\int_{I_N(s_re_r)}|t_k a(x)|^{1/(1+\alpha)}\,\mathrm{d}\mu(x)\\
&\quad\leqslant\frac{1}{\log n}\biggl(\sum_{k=M_N}^{n}\frac{c_\alpha M_N^{\alpha/(1+\alpha)}}{k^{\alpha/(1+\alpha)+1}}+\sum_{k=M_N}^{n}\frac{c_\alpha}{k}\biggr)<c_\alpha<\infty.
\end{aligned}$$

The proof is complete. □

Proof of Remark 3.7. Let us see an example. Let

$$q_n:=\begin{cases}\dfrac{1}{\sqrt{n}} & \text{if } n\in\mathbb{N}_+,\\ 0 & \text{if } n=0.\end{cases}$$

Then this sequence is non-increasing and non-negative.

1. Let $0<\alpha<1/2$ be arbitrary. It is easy to see that if $n>2$, then

$$Q_n>\sum_{k=1}^{n-2}\frac{1}{\sqrt{k}}>\int_1^{n-1}\frac{1}{\sqrt{x}}\,\mathrm{d}x=2\sqrt{n-1}-2.$$

Recalling $\alpha<1/2$ we obtain

$$0<\frac{n^\alpha}{Q_n}<\frac{n^\alpha}{2\sqrt{n-1}-2}=O(1).$$

On the other hand,

$$\frac{q_n-q_{n+1}}{n^{\alpha-2}}=\frac{1}{\sqrt{1+1/n}\bigl(\sqrt{1+1/n}+1\bigr)}n^{1/2-\alpha}\neq O(1).$$

2. Analogously we can show that in the case of $\alpha>1/2$ the situation is the opposite. □

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