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ORDER COMPLEX OF IDEALS IN A COMMUTATIVE RING WITH IDENTITY

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Abstract. Order complex is an important object associated to a partially ordered set. Following a suggestion from V. A. Vassiliev (1994), we investigate an order complex associated to the partially ordered set of nontrivial ideals in a commutative ring with identity. We determine the homotopy type of the geometric realization for the order complex associated to a general commutative ring with identity. We show that this complex is contractible except for semilocal rings with trivial Jacobson radical when it is homotopy equivalent to a sphere.

Keywords: ideal; commutative ring; order complex; homotopy type

MSC 2010: 55P15, 06A07, 05E40, 13A99

1. Introduction

Order complex is an important object associated to a poset. Since posets occur in many areas, so do order complexes. For some recent and interesting occurrences of order complexes in algebraic contexts, the reader may consult [1], [3], [5], [6], [8]–[10].

In [11], the author suggested as interesting to investigate the order complex associated to nontrivial ideals in a commutative ring. It is the purpose of this paper to determine the homotopy type for order complexes associated to general commutative rings with identity.

In the next section we collect some definitions mainly concerning simplicial complexes which are needed for the discussion in Section 3. Section 3 is devoted to the exposition of our main results.

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2. Background notions and results

For further information concerning simplicial complexes, geometric realization, simplicial maps, etc. the reader is referred to [4], [7]. For information concerning the necessary notions and results from homotopy theory the reader is referred to [2]. We present here only the basic notions, mainly to fix notation and terminology.

Let $a_0, a_1, \ldots, a_n$ be elements in some $\mathbb{R}^N$. They are said to be *geometrically* (or *affinely*) independent if from $\sum_{i=0}^{n} \lambda_i a_i = 0$, where $\sum_{i=0}^{n} \lambda_i = 0$, it follows that $\lambda_0 = \lambda_1 = \ldots = \lambda_n = 0$. If these points are geometrically independent their convex hull σ forms a *geometric* n-simplex. Convex hulls of subsets of $\{a_0, a_1, \ldots, a_n\}$ form *faces* σ and a_i are their *vertices*. The *standard geometric n-simplex* Δ^n is given by:

$$\Delta^n := \{(x_0, x_1, \ldots, x_n) \in \mathbb{R}_+^{n+1} \colon x_0 + x_1 + \ldots + x_n = 1\},$$

where $\mathbb{R}_+$ is the set of all nonnegative real numbers. There is an affine bijection between any geometric n-simplex and the standard geometric n-simplex. In what follows, we will say simply simplex instead of geometric simplex.

Let us denote by $\mathbb{R}^{\oplus J}$ the direct sum of $|J|$ (where J may be infinite; $|J|$ stands here for the cardinality of J) copies of $\mathbb{R}$ (so, it is a subset of $\mathbb{R}^J$ consisting of those $x = (x_j)_{j \in J} \in \mathbb{R}^J$ such that $x_j = 0$ for all but finitely many $j \in J$). A (geometric) simplicial complex K in $\mathbb{R}^{\oplus J}$ is a collection of simplices in $\mathbb{R}^{\oplus J}$ which satisfy two conditions:

(1) Every face of a simplex in K is a simplex in K.
(2) The intersection of two simplices in K is a face of both of them.

By $|K|$ we denote the union of all simplices from K. This set is given topology as follows. A set $F \subset |K|$ is closed if and only if $F \cap \sigma$ is closed in σ for every $\sigma \in K$ (σ itself has the subspace topology induced by the n-dimensional plane determined by its vertices).

A map $f \colon |K| \to |L|$, where K and L are simplicial complexes, is a simplicial map if it satisfies the following two conditions.

(1) If $a_0, \ldots, a_n$ are vertices of a simplex in K, then $f(a_0), \ldots, f(a_n)$ are vertices of a simplex in L.
(2) If $x \in |K|$ is such that $x = \sum_{i=0}^{n} \lambda_i a_i$, for some $a_0, a_1, \ldots, a_n$ which are vertices of a simplex in K, then $f(x) = \sum_{i=0}^{n} \lambda_i f(a_i)$.

It is clear than any simplicial map is continuous.

An *abstract simplicial complex* $\mathcal{K}$ is a collection of finite nonempty sets such that if $A \in \mathcal{K}$, and $\emptyset \neq B \subseteq A$, then $B \in \mathcal{K}$. The union $\cup\mathcal{K}$ is the set of all vertices of $\mathcal{K}$. If $A \in \mathcal{K}$, and A has $n+1$ elements, we refer to A as *n-simplex of $\mathcal{K}$*. To any abstract simplicial complex $\mathcal{K}$ one can associate the appropriate geometric simplicial complex K and we will refer to $|K|$ as to the *geometric realization* of $\mathcal{K}$. In what follows we will denote abstract simplicial complexes in the same manner as geometric ones by K, L, etc.

3. Order complex

For any poset P, one can define an abstract simplicial complex, called the order complex of P, by taking as n-simplices chains of $n+1$ elements from the poset P from which we exclude the greatest element and the least element (if they exist). For a general discussion concerning order complexes we refer the reader to [4], [12].

Let us make this notion more explicit for the case of ideals of a commutative ring.

Definition 3.1. Let R be a commutative ring with identity, and let $I^*(R)$, the set of all proper nonzero ideals of R, be the vertex set. We define order complex $\Delta(R)$ as follows:

$$\{I_0, I_1, \dots, I_n\} \in \Delta(R) \quad \text{if and only if } I_0 \subset I_1 \subset \dots \subset I_n.$$

For example, Figure 1 illustrates both the poset $I^*(\mathbb{Z}_{60})$ and $|\Delta(\mathbb{Z}_{60})|$.

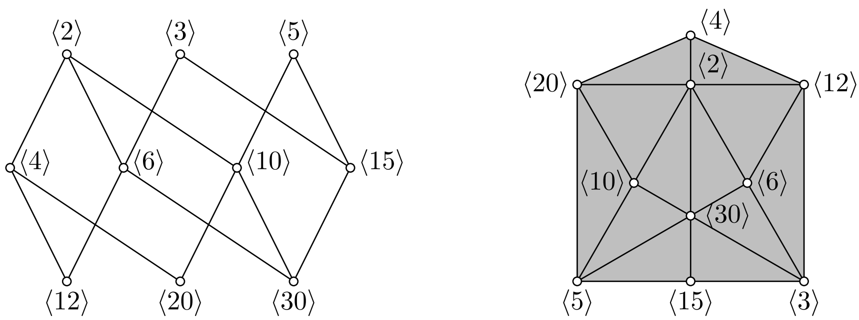

Figure 1.

Let us first observe that in the case the ring R is local, the resulting complex $\Delta(R)$ is a cone, therefore $|\Delta(R)|$ is contractible. Namely, every ideal is contained in the maximal ideal M, so every chain of ideals $I_0 \subset I_1 \subset \dots \subset I_m$ (if $I_m \neq M$) may be extended into the chain $I_0 \subset I_1 \subset \dots \subset I_m \subset M$, so the order complex is a cone with vertex M.

Now let us deal with the case of semilocal rings. First, we look at semilocal rings in which Jacobson radical is trivial.

Proposition 3.2. *Let R be a semilocal ring. If $|\max(R)| = n > 1$ and $J(R) = \{0\}$, then $|\Delta(R)| \simeq \dot{\Delta}^{n-1}$.*

P r o o f. Since $J(R) = 0$, Chinese remainder theorem shows that R is isomorphic to a direct product of finitely many fields, $R \cong F_1 \times \ldots \times F_n$. So, every nonzero proper ideal is of the form $I_1 \times \ldots \times I_n$ where I_j is either $\{0\}$ or F_j and not all of them are $\{0\}$ and not all of them are F_j. So, ideals are in one-to-one correspondence with proper nonempty subsets of $\{1, \ldots, n\}$. Simplices in the order complex are, therefore, chains of proper subsets of $\{1, \ldots, n\}$ and it is clear that this is exactly the barycentric subdivision of $\dot{\Delta}^{n-1}$. $\square$

Note that, in this case, $|\Delta(R)|$ is connected unless $n = 2$. Figure 2 illustrates the poset and the order complex in case of $n = 4$ (to simplify notation, 1101 stands for the ideal $F_1 \times F_2 \times \{0\} \times F_4$, etc.). Note the identifications which show that $|\Delta(R)|$ in this case is the boundary of a tetrahedron.

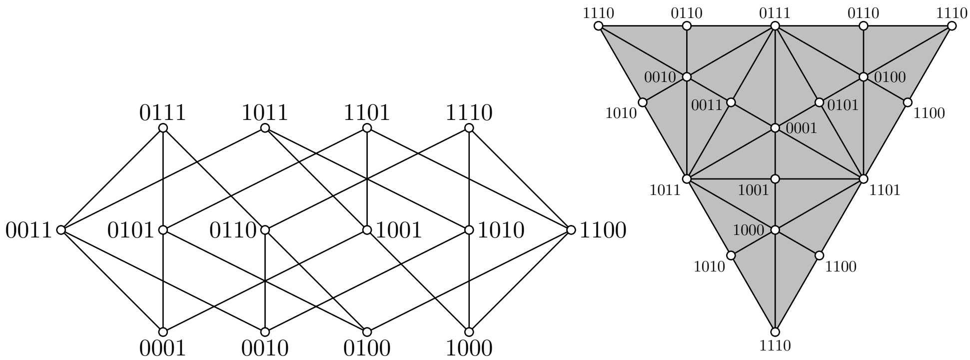

Figure 2.

The next proposition deals with semilocal rings in which $J(R) \neq \{0\}$.

Proposition 3.3. *Let R be a semilocal ring. If $J(R) \neq \{0\}$, then $|\Delta(R)|$ is contractible.*

P r o o f. Let us first show that there is a maximal ideal M in R such that $M \cap I \neq \{0\}$ for all $I \neq \{0\}$. In particular, this will show that $|\Delta(R)|$ is connected. Let x be any nonzero element in $J(R)$. The ideal $\langle x \rangle + \mathrm{Ann}(x)$ is a proper nonzero ideal. Namely, if $\langle x \rangle + \mathrm{Ann}(x) = R$, then $1 = rx + a$, for some $r \in R$ and $a \in \mathrm{Ann}(x)$. Since $x \in J(R)$, by the well-known property of $J(R)$, $a = 1 - rx \in U(R)$. Since $ax = 0$, it would follow that $x = 0$, which is not true.

So, let M be a maximal ideal containing $\langle x \rangle + \mathrm{Ann}(x)$. Let I be a nonzero ideal. If t is any nonzero element in I, let us look at tx.

1) If $tx = 0$, then $t \in \mathrm{Ann}(x)$, so $t \in I \cap M$, and $I \cap M \neq \{0\}$.

2) If $tx \neq 0$, then tx is a nonzero element in $I \cap M$.

Let us denote by $K(M)$ the subcomplex of $\Delta(R)$ whose vertices are nonzero ideals contained in M. This subcomplex is obviously a cone over M, so $|K(M)|$ is contractible.

Suppose that $I_0 \subset I_1 \subset \ldots \subset I_n$, so $\{I_0, I_1, \ldots, I_n\}$ forms a simplex in $\Delta(R)$. Then $0 \neq I_0 \cap M \subseteq I_1 \cap M \subseteq \ldots \subseteq I_n \cap M$, so $\{I_0 \cap M, I_1 \cap M, \ldots, I_n \cap M\}$ is a simplex of $K(M)$ whose dimension certainly may be smaller that the dimension of the original simplex. This shows that the map $I \mapsto I \cap M$ defines a simplicial map $f\colon |\Delta(R)| \to |K(M)|$. As an example, the reader may wish to check Figure 1, where $M = \langle 2 \rangle$. We claim that this map is a strong deformation retraction. Namely, every simplex $\{I_0, \ldots, I_n\}$ in $\Delta(R)$ is a face of a simplex $\{I_0 \cap M, \ldots, I_n \cap M, I_0, \ldots, I_n\}$ and our simplicial map is nothing but the projection of that larger simplex onto its face $\{I_0 \cap M, \ldots, I_n \cap M\}$. This is clearly a (strong) deformation retraction since every point in $|\Delta(R)|$ is mapped onto its image along a line inside the appropriate simplex.

So, $|\Delta(R)| \simeq |K(M)|$ and the last space is contractible since it is a cone. $\square$

Let us now proceed to the case of a ring R with infinitely many maximal ideals. First we prove the following lemma.

Lemma 3.4. *Suppose that R is such that* $\max(R)$ *is infinite. If K_0 is a finite subcomplex of $\Delta(R)$, then there is a subcomplex K_1 such that K_0 is a subcomplex of K_1 and $|K_1|$ is contractible.*

P r o o f. Suppose that $\{I_1, \ldots, I_m\}$ is the set of all vertices in K_0. Let us first prove that there exists a maximal ideal M in R such that $I_k \cap M \neq \{0\}$ for all $k = \overline{1, m}$. This will, as before, show that $|\Delta(R)|$ is connected.

Let M_k be a maximal ideal containing $\mathrm{Ann}(I_k)$ and let M be a maximal ideal such that $M \neq M_k$ for all k. We claim that $M \cap I_k \neq \{0\}$.

Since $M \not\subseteq M_1 \cup \ldots \cup M_m$, by the Prime Avoidance Theorem we can choose an element $x \in M \setminus (M_1 \cup \ldots \cup M_m)$. Let $k \in \{1, \ldots, m\}$. Since $x \notin \mathrm{Ann}(I_k)$, there is an element $t_k \in I_k$ such that $xt_k \neq 0$. So, the element xt_k is a nonzero element in $M \cap I_k$ and $M \cap I_k \neq \{0\}$.

We now proceed as in the proof of the previous proposition. Let K_1 be a subcomplex $K_0 \cup K(M)$, where $K(M)$ is a subcomplex of $\Delta(R)$ formed by the ideals of R contained in M. We construct a simplicial map $f\colon |K_1| \to |K(M)|$ given on vertices by $I \mapsto I \cap M$. As before, this map shows that $|K(M)|$ is a strong deformation retract of $|K_1|$. Since $|K(M)|$ is contractible, this shows that $|K_0|$ is contractible as well. $\square$

Let us now characterize the homotopy type of $|\Delta(R)|$ for a ring R with infinitely many maximal ideals.

Proposition 3.5. *If* $\max(R)$ *is infinite, then* $|\Delta(R)|$ *is contractible.*

Proof. Since $|\Delta(R)|$ has the homotopy type of a CW complex, we may use the Whitehead theorem. We only need to show that all homotopy groups of $|\Delta(R)|$ are trivial. Suppose that $n \geqslant 1$ and that $g\colon S^n \to |\Delta(R)|$ is a continuous map. Since the image $g[S^n]$ is compact, by Lemma 2.5 in [7], there is a finite subcomplex K_0 such that $g[S^n] \subseteq |K_0|$. By Lemma 3.4, there is a subcomplex K_1 such that $K_0 \subset K_1$ and $|K_1|$ is contractible. So, the map g may be factored through the contractible space $|K_1|$ and it is homotopically trivial. We conclude that $\pi_n(|\Delta(R)|, *)$ is trivial. Since this holds for all n, by Whitehead's theorem we get that $|\Delta(R)|$ is contractible. $\square$

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