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A NOTE ON THE MULTIPLIER IDEALS OF MONOMIAL IDEALS

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Abstract. Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal and $\mathcal{J}(\mathfrak{a}^c)$ the multiplier ideal of $\mathfrak{a}$ with coefficient c. Then $\mathcal{J}(\mathfrak{a}^c)$ is also a monomial ideal of $\mathbb{C}[x_1, \ldots, x_n]$, and the equality $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}$ implies that $0 < c < n + 1$. We mainly discuss the problem when $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$ or $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ for all $0 < \varepsilon < 1$. It is proved that if $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$ then $\mathfrak{a}$ is principal, and if $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ holds for all $0 < \varepsilon < 1$ then $\mathfrak{a} = (x_1, \ldots, x_n)$. One global result is also obtained. Let $\tilde{\mathfrak{a}}$ be the ideal sheaf on $\mathbb{P}^{n-1}$ associated with $\mathfrak{a}$. Then it is proved that the equality $\mathcal{J}(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$ implies that $\tilde{\mathfrak{a}}$ is principal.

Keywords: multiplier ideal; monomial ideal; convex set

MSC 2010: 14F18

1. Introduction

Multiplier ideal sheaves are fundamental topics in higher dimensional algebraic geometry. In commutative algebra, there are objects similar to adjoint ideals and test ideals (cf. [2], [7], [11], [12], [13], [14]).

Let X be a smooth quasiprojective complex algebraic variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ an ideal sheaf on X. Let $f\colon Y \to X$ be a log resolution of $\mathfrak{a}$ with $f^{-1}(\mathfrak{a}) = \mathcal{O}_Y(-E)$. For any rational number $c > 0$, the multiplier ideal of $\mathfrak{a}$ with coefficient c is defined to be

$$\mathcal{J}(\mathfrak{a}^c) = f_*\mathcal{O}_Y(K_{Y/X} - \llcorner cE \lrcorner),$$

where $K_{Y/X} = K_Y - f^*K_X$ is the relative canonical divisor and $\llcorner - \lrcorner$ is the rounddown for $\mathbb{Q}$-divisors. Then $\mathcal{J}(\mathfrak{a}^c)$ is an ideal sheaf on X, and $\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a}) \subseteq \mathcal{O}_X$.

If $\mathfrak{a}$ is invertible, then $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$. In general, $\mathcal{J}(\mathfrak{a}) \neq \mathfrak{a}$ because of the singularity of $\mathfrak{a}$. We are interested in the problem of when $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}$.

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When X is an affine variety of dimension n and $\mathfrak{a}$ is monomial, $\mathcal{J}(\mathfrak{a}^c)$ can be described explicitly by a remarkable theorem of Howald [9]. In this case, $\mathcal{J}(\mathfrak{a}^c)$ is also a monomial ideal. Note that, by Skoda's Theorem, one has

$$\mathcal{J}(\mathfrak{a}^{n+1+\varepsilon}) = \mathfrak{a}^2 \mathcal{J}(\mathfrak{a}^{n-1+\varepsilon}) \subseteq \mathfrak{a}^2 \neq \mathfrak{a}$$

for any $\varepsilon \geqslant 0$. It follows that the equality $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}$ implies that $0 < c < n+1$. For any $0 < c < 1$, as $\mathcal{J}(\mathfrak{a}) \subseteq \mathcal{J}(\mathfrak{a}^c)$, we see that $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}$ implies that $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$. In the rest of the paper we analyse the case when $\mathcal{J}(\mathfrak{a}^c) = \mathfrak{a}$ for $c \in [1, n+1)$. We will mainly discuss the problem when c is at or near to the endpoints of the interval $[1, n+1)$, i.e., $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$ or $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ for all $0 < \varepsilon < 1$.

Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a monomial ideal. It is well-known that $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$ when $\mathfrak{a}$ is principal. The main result in Section 2 claims that the converse is also true. Similar results hold also for multiplier ideals of monomial ideals on affine toric varieties, adjoint ideals and test ideals. Then we prove that if $\mathfrak{a}$ is not principal then $\mathcal{J}(\mathfrak{a}^m) \neq \mathcal{J}(\mathfrak{a})^m$ for $m \gg 0$.

Section 3 is devoted to the discussion of the problem of when $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ for all $0 < \varepsilon < 1$. It is also well-known that $\mathcal{J}((x_1, \dots, x_n)^{n+1-\varepsilon}) = (x_1, \dots, x_n)$ for all $0 < \varepsilon < 1$. The main theorem in this section states that the converse is also true.

Let $\mathfrak{a} \subseteq \mathbb{C}[x_0, x_1, \dots, x_n]$ be a monomial ideal and $\tilde{\mathfrak{a}}$ the ideal sheaf on $\mathbb{P}^n$ associated with $\mathfrak{a}$. Gluing local results, we get a global result in the last section, which says that if $\mathcal{J}(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$ then $\tilde{\mathfrak{a}}$ is principal.

2. When $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$

Let $\mathfrak{a}$ be a monomial ideal. Howald [9] gave a description of multiplier ideals of $\mathfrak{a}$ by convex sets. There are similar descriptions for multiplier ideals of monomial ideals on affine toric varieties, adjoint ideals and test ideals. In order to discuss uniformly these ideals together, we prove first a theorem on convex sets.

Before stating the theorem, it is not difficult to see that the following result holds because of the discreteness of integral numbers.

Lemma 2.1. *Suppose that a domain $D \subseteq \mathbb{R}^n$ is contained in the zero set of the equation $p_1x_1 + \ldots + p_nx_n = p$ where at least two of $p_1, \dots, p_n$ are nonzero. Then there exists a point $(a_1, \dots, a_n) \in D$ such that $a_1, \dots, a_n$ are all non-integral numbers.*

Let $\alpha = (a_1, \dots, a_n)$, $\beta = (b_1, \dots, b_n) \in \mathbb{R}^n$. Denote that $\alpha \leqslant \beta$ if $a_i \leqslant b_i$, $i = 1, \dots, n$. Let $\alpha_1, \dots, \alpha_s \in \mathbb{N}^n$ where $\mathbb{N}$ contains 0. We say that $\{\alpha_1, \dots, \alpha_s\}$ is a reduced set if $\alpha_i \nleqslant \alpha_j$ for any $i \neq j$. Set $P(\alpha_1, \dots, \alpha_s)$ to be the convex hull in $\mathbb{R}^n$ of the set $\{\beta \in \mathbb{N}^n \colon \beta \geqslant \alpha_i \text{ for some } i\}$. Let $w \in \mathbb{R}^n$. Set

$$\mathcal{J}(\alpha_1, \dots, \alpha_s; w) = \{\beta \in \mathbb{N}^n \colon \beta + w \in \operatorname{Int}(P(\alpha_1, \dots, \alpha_s))\},$$

where $\operatorname{Int}(A)$ denotes the interior of a set A. Then it is clear that $P(\alpha_1, \dots, \alpha_s) \cap \mathbb{N}^n \subseteq \mathcal{J}(\alpha_1, \dots, \alpha_s; w)$ provided that $w > (0, \dots, 0)$.

Theorem 2.2. *Let $\{\alpha_1, \dots, \alpha_s\}$ be a reduced set in $\mathbb{N}^n$ and $w \geqslant (1, \dots, 1) \in \mathbb{R}^n$. If $s \geqslant 2$, then*

$$P(\alpha_1, \dots, \alpha_s) \cap \mathbb{N}^n \neq \mathcal{J}(\alpha_1, \dots, \alpha_s; w).$$

P r o o f. Let S be the set of all non-coordinate hyperplanes which bound the convex hull $P(\alpha_1, \dots, \alpha_s)$. Let $H \in S$. Then the equation of H has the following form:

$$p_1 x_1 + \dots + p_n x_n = p, \quad p_i \geqslant 0, \ p > 0.$$

If all the equations have the form $x_i = q_i$, then $S = \{H_{i_j} \colon j = 1, \dots, r\}$, where $H_{i_j} \colon x_{i_j} = q_{i_j}$. It follows that

$$P(\alpha_1, \dots, \alpha_s) = \bigcap_{j=1}^{r} \{\beta \in \mathbb{R}^n_{\geqslant 0} \colon \beta \geqslant (0, \dots, 0, q_{i_j}, 0, \dots, 0)\} = P(\alpha),$$

where $\alpha = (0, \dots, 0, q_{i_1}, 0, \dots, 0, q_{i_r}, 0, \dots, 0)$. This contradicts the assumption that $\{\alpha_1, \dots, \alpha_s\}$ is a reduced set and $s \geqslant 2$. Therefore there exists $H \in S$ whose equation has the form $p_1 x_1 + \dots + p_n x_n = p$, $p_i \geqslant 0$, $p > 0$, where at least two of $p_1, \dots, p_n$ are nonzero. Then, by Lemma 2.1, there exists a point $Q = (q_1, \dots, q_n)$ on the boundary of $P(\alpha_1, \dots, \alpha_s)$ with all the q_i non-integral.

Denote the least integer not less than q_i by $\ulcorner q_i \urcorner$ and the maximal integer not bigger than q_i by $\llcorner q_i \lrcorner$. Set $\ulcorner Q \urcorner = (\ulcorner q_1 \urcorner, \dots, \ulcorner q_n \urcorner)$ and $\llcorner Q \lrcorner = (\llcorner q_1 \lrcorner, \dots, \llcorner q_n \lrcorner)$. Then $q_i' = \ulcorner q_i \urcorner - q_i > 0$, $q_i'' = q_i - \llcorner q_i \lrcorner > 0$, $i = 1, \dots, n$. Thus, assuming that Q is on the boundary, $\ulcorner Q \urcorner = Q + (q_1', \dots, q_n')$ and $\llcorner Q \lrcorner = Q - (q_1'', \dots, q_n'')$, we see that $\ulcorner Q \urcorner \in \operatorname{Int}(P(\alpha_1, \dots, \alpha_s))$ and $\llcorner Q \lrcorner \notin P(\alpha_1, \dots, \alpha_s)$. Note that $\llcorner Q \lrcorner + (1, \dots, 1) = \ulcorner Q \urcorner$ and $w \geqslant (1, \dots, 1)$. Necessarily $\llcorner Q \lrcorner + w \in \operatorname{Int}(P(\alpha_1, \dots, \alpha_s))$, so that $\llcorner Q \lrcorner \in \mathcal{J}(\alpha_1, \dots, \alpha_s; w)$. Hence $P(\alpha_1, \dots, \alpha_s) \cap \mathbb{N}^n \neq \mathcal{J}(\alpha_1, \dots, \alpha_s; w)$. $\square$

Let K be a field and $K[x_1, \dots, x_n]$ a polynomial ring over K. Let I be an ideal of $K[x_1, \dots, x_n]$. When I is generated by monomials, we say that I is a monomial ideal and its minimal generating set is denoted by $G(I)$.

Every monomial $x_1^{a_1} \dots x_n^{a_n} = \underline{x}^{\alpha} \in K[x_1, \dots, x_n]$ corresponds to its exponent vector $\alpha = (a_1, \dots, a_n) \in \mathbb{N}^n$. Let $I \subseteq K[x_1, \dots, x_n]$ be a monomial ideal. The convex hull in $\mathbb{R}^n$ of the set of all exponent vectors of monomials of I is called the Newton polygon of I, denoted by $P(I)$. Then the set of monomials in the integral closure $\overline{I}$ of I is just the set of all monomials $\underline{x}^\alpha$ with $\alpha \in P(I)$ (cf. [15], Proposition 1.4.6). For any rational number $c > 0$, set $cP(I) = \{c\alpha\colon \alpha \in P(I)\}$.

In the case $X = \mathbb{A}^n$, Howald [9] gave an explicit description of $\mathcal{J}(\mathfrak{a}^c)$.

Howald's theorem. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a monomial ideal and $c > 0$ a rational number. Then $\mathcal{J}(\mathfrak{a}^c)$ is a monomial ideal and*

$$\mathcal{J}(\mathfrak{a}^c) = (\underline{x}^\alpha\colon\ \alpha + (1, \dots, 1) \in \operatorname{Int}(cP(\mathfrak{a})) \cap \mathbb{N}^n).$$

Notice that, as $P(\mathfrak{a}) = P(\overline{\mathfrak{a}})$, one has that $\mathfrak{a} \subseteq \overline{\mathfrak{a}} \subseteq \mathcal{J}(\overline{\mathfrak{a}}) = \mathcal{J}(\mathfrak{a})$.

Let $\underline{x}^\alpha$, $\underline{x}^\beta$ be two monomials in $K[x_1, \dots, x_n]$. Then $\underline{x}^\alpha \mid \underline{x}^\beta$ if and only if $\alpha \leqslant \beta$. Thus $\{\underline{x}^{\alpha_1}, \dots, \underline{x}^{\alpha_s}\}$ forms a minimal generating set for some monomial ideal in $K[x_1, \dots, x_n]$ if and only if $\{\alpha_1, \dots, \alpha_s\}$ forms a reduced set in $\mathbb{N}^n$. Let $I \subseteq K[x_1, \dots, x_n]$ be a monomial ideal and $G(I) = \{\underline{x}^{\alpha_1}, \dots, \underline{x}^{\alpha_s}\}$. Then one has that $P(I) = P(\alpha_1, \dots, \alpha_s)$.

For any monomial ideal $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ with $G(\mathfrak{a}) = \{\underline{x}^{\alpha_1}, \dots, \underline{x}^{\alpha_s}\}$, it follows from Howald's theorem that

$$\mathcal{J}(\mathfrak{a}) = (\underline{x}^\alpha\colon\ \alpha \in \mathcal{J}(\alpha_1, \dots, \alpha_s; (1, \dots, 1))).$$

By Theorem 2.2, we get the necessary part of the following theorem, while the sufficient part is well-known, which can also be seen directly by Howald's theorem.

Theorem 2.3. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be a monomial ideal. Then $\mathcal{J}(\mathfrak{a}) = \mathfrak{a}$ if and only if $\mathfrak{a}$ is principal.*

Remark 2.4. We can get results similar to Theorem 2.3 from Theorem 2.2 for multiplier ideals of monomial ideals on affine toric varieties, adjoint ideals and test ideals since there are similar descriptions for these ideals in [1], [10], [7]. For toric varieties and the other unexplained notions, we refer to [4] and [6].

Let $\mathfrak{a}, \mathfrak{b} \subseteq \mathbb{C}[x_1, \dots, x_n]$ be two monomial ideals. It is proved in [3] that the following subadditivity property holds:

$$\mathcal{J}(\mathfrak{a}\mathfrak{b}) \subseteq \mathcal{J}(\mathfrak{a})\mathcal{J}(\mathfrak{b}),$$

which will be used in the sequel.

Corollary 2.5. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be monomial ideals in $\mathbb{C}[x_1, \ldots, x_n]$. Suppose that*

$$\mathfrak{a}_1 \ldots \mathfrak{a}_m = \mathcal{J}(\mathfrak{a}_1) \ldots \mathcal{J}(\mathfrak{a}_m).$$

Then $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are all principal.

P r o o f. By the subadditivity property, we have

$$\mathfrak{a}_1 \ldots \mathfrak{a}_m \subseteq \mathcal{J}(\mathfrak{a}_1 \ldots \mathfrak{a}_m) \subseteq \mathcal{J}(\mathfrak{a}_1) \ldots \mathcal{J}(\mathfrak{a}_m).$$

It follows from the hypotheses that $\mathfrak{a}_1 \ldots \mathfrak{a}_m = \mathcal{J}(\mathfrak{a}_1 \ldots \mathfrak{a}_m)$. Then, by Theorem 2.3, $\mathfrak{a}_1 \ldots \mathfrak{a}_m$ is principal. Hence $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are all principal. $\square$

Thus, for any monomial ideal $\mathfrak{a}$ in $\mathbb{C}[x_1, \ldots, x_n]$ and any integer $m > 0$, $\mathfrak{a}^m = \mathcal{J}(\mathfrak{a})^m$ holds when and only when $\mathfrak{a}$ is principal.

By the subadditivity property, we have that $\mathcal{J}(\mathfrak{a}^m) \subseteq \mathcal{J}(\mathfrak{a})^m$ holds for any integer $m > 0$. It is clear that the equality holds when $\mathfrak{a}$ is principal. Notice that $\mathfrak{a}$ is principal if and only if $\overline{\mathfrak{a}}$ is principal.

There is an example when $\mathfrak{a}$ is not principal and $\mathcal{J}(\mathfrak{a}^m) = \mathcal{J}(\mathfrak{a})^m$ holds for some $m > 0$. Taking $\mathfrak{a} = (x_1, \ldots, x_n)$ as the threshold $\operatorname{lct}(\mathfrak{a}) = n$ (see the following section), one has that $\mathcal{J}(\mathfrak{a}^m) = \mathcal{J}(\mathfrak{a})^m = \mathcal{O}_{\mathbb{A}^n}$ for $m = 1, \ldots, n-1$. However, there is an upper bound for such m.

Theorem 2.6. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal. Suppose that $\mathfrak{a}$ is not principal. Then there exists m_0 such that for all $m \geqslant m_0$,*

$$\mathcal{J}(\mathfrak{a}^m) \neq \mathcal{J}(\mathfrak{a})^m.$$

P r o o f. Since $\mathfrak{a}$ is not principal, so neither is $\overline{\mathfrak{a}}$, it follows from Theorem 2.3 that there exists a monomial $x_1^{a_1} \ldots x_n^{a_n} \in \mathcal{J}(\mathfrak{a}) \setminus \overline{\mathfrak{a}}$. Let $P = (a_1, \ldots, a_n)$ and $P' = (a_1 + 1, \ldots, a_n + 1)$ be two points. Then $P \notin P(\mathfrak{a})$ and $P' \in \operatorname{Int}(P(\mathfrak{a}))$. Thus the line segment $\overline{PP'}$ must pass the boundary of $P(\mathfrak{a})$. Let $Q \in \overline{PP'}$ be on the boundary. Then $Q = (a_1 + \lambda, \ldots, a_n + \lambda)$ for some $0 < \lambda < 1$. Take an integer m_0 such that $m_0 \lambda > 1$. Suppose that $m \geqslant m_0$. Since $\operatorname{Int}(P(\mathfrak{a}^m)) \subseteq m \operatorname{Int}(P(\mathfrak{a}))$, it follows that the point $(m(a_1 + \lambda), \ldots, m(a_n + \lambda))$ is on the boundary of $P(\mathfrak{a}^m)$. Note that

$$(m(a_1 + \lambda), \ldots, m(a_n + \lambda)) = (ma_1 + 1, \ldots, ma_n + 1) + (\delta, \ldots, \delta)$$

with $\delta > 0$. Then it holds that $(ma_1 + 1, \ldots, ma_n + 1) \notin \operatorname{Int}(P(\mathfrak{a}^m))$. Therefore $x_1^{ma_1} \ldots x_n^{ma_n} \notin \mathcal{J}(\mathfrak{a}^m)$. However $x_1^{ma_1} \ldots x_n^{ma_n} \in \mathcal{J}(\mathfrak{a})^m$, which proves that $\mathcal{J}(\mathfrak{a}^m) \neq \mathcal{J}(\mathfrak{a})^m$. $\square$

Corollary 2.7. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal. Then $\mathcal{J}(\mathfrak{a}^m) = \mathcal{J}(\mathfrak{a})^m$ for all $m > 0$ if and only if $\mathfrak{a}$ is principal.*

Remark 2.8. In some cases, the bound m_0 in Theorem 2.6 depends on the threshold of $\mathfrak{a}$. Let us return to the proof of Theorem 2.6 where the bound m_0 comes from the line segment between two points with integral components. Now suppose that the reciprocal $m = 1/\mathrm{lct}(\mathfrak{a})$ of the threshold of $\mathfrak{a}$ is not an integer. Then the point $(m, \ldots, m)$ is on the boundary of $P(\mathfrak{a})$ (see the next section), and the points $(\ulcorner m \urcorner, \ldots, \ulcorner m \urcorner) \in \mathrm{Int}(P(\mathfrak{a}))$, while $(\llcorner m \lrcorner, \ldots, \llcorner m \lrcorner) \notin P(\mathfrak{a})$. Then any $m_0 > 1/(m - \llcorner m \lrcorner) = 1/(1/\mathrm{lct}(\mathfrak{a}) - [1/\mathrm{lct}(\mathfrak{a})])$ is a required bound.

3. When $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ for all $0 < \varepsilon < 1$

Let $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be a monomial ideal of zero-dimension. The log canonical threshold $\mathrm{lct}(\mathfrak{a})$ of $\mathfrak{a}$ is defined as

$$\mathrm{lct}(\mathfrak{a}) = \inf\{c > 0\colon\ \mathcal{J}(\mathfrak{a}^c) \neq \mathcal{O}_{\mathbb{A}^n}\}.$$

Then

$$\begin{aligned}\mathrm{lct}(\mathfrak{a}) &= \sup\{c > 0\colon\ \mathcal{J}(\mathfrak{a}^c) = \mathcal{O}_{\mathbb{A}^n}\}\\ &= \sup\{c > 0\colon\ (1, \ldots, 1) \in \mathrm{Int}(cP(\mathfrak{a}))\}\\ &= \sup\Big\{c > 0\colon\ \Big(\frac{1}{c}, \ldots, \frac{1}{c}\Big) \in \mathrm{Int}(P(\mathfrak{a}))\Big\}.\end{aligned}$$

Hence $\mathrm{lct}(\mathfrak{a})$ is just the number t such that the point $(1/t, \ldots, 1/t)$ is on the boundary of $P(\mathfrak{a})$. Furthermore, note that $\mathrm{lct}(\mathfrak{a}) \leqslant n$ because $\mathcal{J}(\mathfrak{a}^n) \subseteq \mathfrak{a}$ by Skoda's theorem (cf. [11], Theorem 11.1.1).

When $\mathfrak{a} = (x_1^{a_1}, \ldots, x_n^{a_n})$ with $a_i > 0$, $i = 1, \ldots, n$, then $\mathrm{lct}(\mathfrak{a}) = 1/a_1 + \ldots + 1/a_n$ (cf. [2], Example 4.5, or [11], Example 9.3.15). This implies the sufficient part of the following theorem.

Theorem 3.1. $\mathrm{lct}(\mathfrak{a}) = n$ *if and only if* $\mathfrak{a} = (x_1, \ldots, x_n)$.

Proof. Suppose that $\mathrm{lct}(\mathfrak{a}) = n$. Let H be a non-coordinate hyperplane bounding $P(\mathfrak{a})$. Then, by the lemma below, the equation of H has the form:

$$a_1x_1 + \ldots + a_nx_n = 1, \quad 0 \leqslant a_i \leqslant 1,\ i = 1, \ldots, n.$$

Consider the intersection point of H with the diagonal line $x_1 = x_2 = \ldots = x_n$. It is clear that the point is $\Big(1/\sum\limits_{i=1}^{n} a_i, \ldots, 1/\sum\limits_{i=1}^{n} a_i\Big)$, which is on the boundary of $P(\mathfrak{a})$.

Then it is necessary to have $\operatorname{lct}(\mathfrak{a}) = \sum_{i=1}^{n} a_i$. This implies that $a_1 = a_2 = \ldots = a_n = 1$. Then the equation of H is $x_1 + \ldots + x_n = 1$. Thus H is the unique non-coordinate hyperplane of $P(\mathfrak{a})$. This proves that $\mathfrak{a} = (x_1, \ldots, x_n)$. □

Lemma 3.2. *Let $I \subseteq K[x_1, \ldots, x_n]$ be a monomial ideal. Suppose that*

$$a_1x_1 + \ldots + a_nx_n = 1, \quad a_i \geqslant 0$$

is the equation of some hyperplane H bounding $P(I)$. Then $a_i \leqslant 1$, $i = 1, \ldots, n$.

Proof. Suppose that, for example, $a_1 \neq 0$, let us show that $a_1 \leqslant 1$.

We claim that there exists one point $(m_1, \ldots, m_n) \in \mathbb{N}^n$ on H with $m_1 \neq 0$. Otherwise, all the exponent vectors of I which determine H are on the coordinate hyperplane $P_1\colon\ x_1 = 0$. Then $H = H \cap P_1$. Note that the equation of the hyperplane $H \cap P_1$ in $\mathbb{R}^{n-1}$ is $a_2x_2 + \ldots + a_nx_n = 1$. It follows that the equation of H in $\mathbb{R}^n$ should also be $a_2x_2 + \ldots + a_nx_n = 1$, a contradiction.

Let $(m_1, m_2, \ldots, m_n) \in \mathbb{N}^n$ be a point on H with $m_1 \neq 0$. Then

$$a_1m_1 + a_2m_2 + \ldots + a_nm_n = 1.$$

This implies that $a_1 \leqslant 1$, as required. □

By the definition of the threshold, Theorem 3.1 is equivalent to asserting that $\mathcal{J}(\mathfrak{a}^{n-\varepsilon}) = \mathcal{O}_{\mathbb{A}^n}$ for all $0 < \varepsilon < 1$ if and only if $\mathfrak{a} = (x_1, \ldots, x_n)$. On the other hand, for any $0 < \varepsilon < 1$, by Skoda's Theorem (cf. [11], Theorem 11.1.1), one has that $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{n-\varepsilon})$. Then, by the Nakayama Lemma, $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ if and only if $\mathcal{J}(\mathfrak{a}^{n-\varepsilon}) = \mathcal{O}_{\mathbb{A}^n}$. It follows from Theorem 3.1 that the following theorem holds.

Theorem 3.3. $\mathcal{J}(\mathfrak{a}^{n+1-\varepsilon}) = \mathfrak{a}$ *holds for all* $0 < \varepsilon < 1$ *when and only when* $\mathfrak{a} = (x_1, \ldots, x_n)$.

4. A global result

Let $\mathfrak{a}$ be a monomial ideal of $\mathbb{C}[x_0, x_1, \ldots, x_n]$ and $\tilde{\mathfrak{a}}$ the sheaf on $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ associated with $\mathfrak{a}$. The ideal sheaf $\tilde{\mathfrak{a}}$ is said to be principal if its stalks are all principal. In this last section, we consider the conclusions of the sheaf equality $\mathcal{J}(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$. We will adopt the notation in [8].

Theorem 4.1. *Let $\mathfrak{a} \subseteq \mathbb{C}[x_0, x_1, \ldots, x_n]$ be a monomial ideal. If $\mathcal{J}(\tilde{\mathfrak{a}}) = \tilde{\mathfrak{a}}$ as sheaves on $\mathbb{P}^n$, then $\tilde{\mathfrak{a}}$ is principal.*

Proof. Consider the restrictions on $D_+(x_i)$, $i = 0, 1, \ldots, n$. By the Restriction theorem on multiplier ideals (cf. [5], Proposition 7.5), we have that

$$\mathcal{J}(\tilde{\mathfrak{a}}|_{D_+(x_i)}) \subseteq \mathcal{J}(\tilde{\mathfrak{a}})|_{D_+(x_i)}.$$

Then $\mathcal{J}(\tilde{\mathfrak{a}}|_{D_+(x_i)}) \subseteq \tilde{\mathfrak{a}}|_{D_+(x_i)}$, i.e., $\mathcal{J}((\mathfrak{a}_{(x_i)})^{\sim}) \subseteq (\mathfrak{a}_{(x_i)})^{\sim}$ on $D_+(x_i)$. It follows that $\mathcal{J}((\mathfrak{a}_{(x_i)})^{\sim}) = (\mathfrak{a}_{(x_i)})^{\sim}$ on $D_+(x_i)$. Then, by Theorem 2.3, as an ideal in $\mathbb{C}[x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i]$, $\mathfrak{a}_{(x_i)} = \tilde{\mathfrak{a}}(D_+(x_i))$ is principal. Therefore, the ideal sheaf $\tilde{\mathfrak{a}}$ is principal. □

Remark 4.2. Notice that $\mathfrak{a}$ may not be principal, while $\tilde{\mathfrak{a}}$ is principal. Set

$$\mathfrak{a} = (x_0^2 x_1 x_2, x_0 x_1^2 x_2, x_0 x_1 x_2^2) \subseteq \mathbb{C}[x_0, x_1, x_2].$$

Then $\mathfrak{a}$ is not principal, while $\tilde{\mathfrak{a}}$ is principal.

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