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POROUS MEDIUM EQUATION AND FAST DIFFUSION EQUATION AS GRADIENT SYSTEMS

Samuel Littig, Jürgen Voigt, Dresden

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Abstract. We show that the Porous Medium Equation and the Fast Diffusion Equation, $\dot{u} - \Delta u^m = f$, with $m \in (0,\infty)$, can be modeled as a gradient system in the Hilbert space $H^{-1}(\Omega)$, and we obtain existence and uniqueness of solutions in this framework. We deal with bounded and certain unbounded open sets $\Omega \subseteq \mathbb{R}^n$ and do not require any boundary regularity. Moreover, the approach is used to discuss the asymptotic behaviour and order preservation of solutions.

Keywords: porous medium equation; gradient system; fast diffusion; asymptotic behaviour; order preservation

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Introduction

The main objective of this paper is to present a treatment of the porous medium equation and the fast diffusion equation (abbreviated PME/FDE)

$$\dot{u} - \Delta u^m = f \tag{0.1}$$

as a gradient system in a functional analytic framework. For $m = 1$, equation (0.1) is the inhomogeneous heat equation. For $m > 1$, equation (0.1) is called the *porous medium equation*. It models the flow of an ideal gas in a homogeneous porous medium, the nonlinear heat transfer and the filtration of incompressible fluids through a porous stratum. For $0 < m < 1$, equation (0.1) is called the *fast diffusion equation*, occurring in plasma physics (cf. [11], Chapter 2).

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A *gradient system* is an evolution equation

$$\dot{u} + \nabla_H \mathscr{E}(u) = f \tag{0.2}$$

in a Hilbert space H, with a Gelfand triple $V \hookrightarrow H \hookrightarrow V'$, where V is a reflexive Banach space. The "energy functional" $\mathscr{E}\colon V \to \mathbb{R}$ and the "forcing term" f (a function with values in H) are given, and u (with values in V) is the solution. We refer to Section 1 for details, and we refer to [5] for the theory of gradient systems.

Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and let $H := H^{-1}(\Omega)$ be the dual of the Sobolev space $H_0^1(\Omega)$. Then we have $V := L_{m+1}(\Omega) \cap H^{-1}(\Omega) \subseteq H^{-1}(\Omega)$ densely, and with $\mathscr{E}\colon V \to \mathbb{R}$ defined by

$$\mathscr{E}(u) := \frac{1}{m+1} \int_\Omega |u(x)|^{m+1}\,\mathrm{d}x, \tag{0.3}$$

we show that the setting of gradient systems can be used for the PME/FDE, in order to obtain solutions $u\colon [0,\infty) \to V$ of Cauchy problems for (0.1), with initial values $u_0 \in V$ and forcing terms $f\colon [0,\infty) \to H$. It is implicit in the setup that the solution has the property that $u(t)^m \in H_0^1(\Omega)$ a.e., and in this sense satisfies (generalized) Dirichlet boundary values zero.

We note that this setting yields a unified treatment for the PME and the FDE (including the heat equation). The existence, uniqueness and asymptotics of solutions of the PME/FDE will be obtained by purely functional analytic arguments; no arguments using elliptic regularity theory, comparison principles or smoothness of the boundary of Ω are needed.

We treat the PME/FDE without any restriction on the parameter m and allow all bounded, but also unbounded Ω, provided a Poincaré inequality holds. Our results on the asymtotic decay rely on the requirement that $L_{m+1}(\Omega)$ should be continuously embedded into $H^{-1}(\Omega)$. This holds for all Ω with finite measure, provided $m \geqslant (n-2)/(n+2)$, and for unbounded Ω in certain cases of fast diffusion ($(n-2)/(n+2) \leqslant m \leqslant 1$).

The importance of the space $H^{-1}(\Omega)$ for the treatment of evolution equations of porous medium type

$$u'(t) - \Delta\beta(u(t)) \ni f(t)$$

(with a suitable "maximal monotone graph" β) was recognized in [3]. Solutions of initial value problems in this context are obtained using the theory of evolution equations involving accretive operators; see for example [3], Theorem 21, and more recently [1], Theorem 53.

The perception that the porous medium equation (0.2) can be treated in the context of gradient systems gives an alternative functional analytic access to the solution theory. Moreover, the additional structure obtained by the energy functional $\mathscr{E}$ yields possibilities for further investigations of the system, e.g. concerning the asymptotic behaviour of solutions.

In Section 1 we first give an outline of gradient systems, mainly referring to and using the notation of [5]. We show that the term Δu^m can be interpreted as the gradient in the space $H^{-1}(\Omega)$ of the "energy functional" $\mathscr{E}$ defined in (0.3) (Proposition 1.7), and then draw the conclusion concerning the existence and uniqueness of solutions of the Cauchy problem for the PME/FDE (Theorem 1.8). The section closes with the proof that the solutions obtained in this way are also "weak solutions" as considered, for instance, in Vazquez' monograph [11] (Proposition 1.10).

Under the additional assumption that $L_{m+1}(\Omega)$ is imbedded into $H^{-1}(\Omega)$ we derive the asymptotic behaviour of solutions of the PME/FDE in Section 2. Using a Grönwall type procedure the PME, the heat equation, and the FDE are treated simultaneously (Theorem 2.5). In particular, for the FDE, the usual extinction of the solution in finite time is shown.

In Section 3 we treat order and positivity preservation of the time development of the PME/FDE. For these results we cannot simply rely on "gradient system methods", but rather use a mixture of subgradient system procedures and classical techniques (Theorem 3.1).

The paper is an outgrowth of a project from phase 2 of the 13th International Internet Seminar on Gradient Systems, 2009/10. Besides the lectures of the ISEM Team, R. Chill and E. Fašangová, the material presented in [7] and [11] and ideas contained in [8] led to our approach. In contrast to the recent treatment in [2], Section 4, Example 3, where the PME is considered as a gradient system with a non-constant metric, we need no a priori assumption that the solution takes values in an interval $[\varepsilon, 1/\varepsilon]$.

Concerning notation, we only mention some rather fundamental issues and comment on additional notions when they occur in the text. We mention that the functions in all function spaces are real-valued, and accordingly all vector spaces are over $\mathbb{R}$. We will denote duality forms on dual pairs and scalar products by $\langle\cdot\,,\cdot\rangle$, with indices indicating the pair of spaces or the underlying Hilbert space, respectively.

1. The PME/FDE as a gradient system

We start by describing the context of gradient systems in more detail. Let V be a reflexive Banach space that embeds continuously and densely into a Hilbert space H. Moreover, let $\mathscr{E}$ be a continuously differentiable functional defined on V. By the Gelfand triple setting $V \hookrightarrow H = H' \hookrightarrow V'$ we can define the gradient $\nabla_H \mathscr{E}$ of $\mathscr{E}$

in H as the restriction of the derivative $\mathscr{E}'\colon V \to V'$ of $\mathscr{E}$ in the image on H, that is, $\nabla_H \mathscr{E}$ is the operator given by

$$D(\nabla_H \mathscr{E}) := \{u \in V;\ \exists v =: \nabla_H \mathscr{E}(u) \in H\ \forall w \in V\colon\ \langle v, w\rangle_H = \langle \mathscr{E}'(u), w\rangle_{V',V}\}.$$

A gradient system is a differential equation of the form

$$\dot{u}(t) + \nabla_H \mathscr{E}(u(t)) = f(t), \tag{1.1}$$

where the spaces V and H and the functional $\mathscr{E}$ are as above, $t \in I$, where I is an interval in $\mathbb{R}$, and $f \in L_{2,\mathrm{loc}}(I, H)$ is given. A solution of (1.1) is a measurable function $u\colon I \to V$ such that

$$\begin{aligned} &u \in W^1_{2,\mathrm{loc}}(I; H) \cap L_\infty(I; V),\\ &u(t) \in D(\nabla_H \mathscr{E}) \text{ for almost all } t \in I, \text{ and}\\ &\text{equation (1.1) holds for almost all } t \in I. \end{aligned}$$

Note that by the Sobolev embedding $W^1_2(I, H) \hookrightarrow C(I, H)$ it makes sense to evaluate a solution u pointwise and an initial value $u(0) = u_0$ has a well-defined meaning.

The central theorem for existence and uniqueness of gradient systems is the following theorem, which we essentially quote from [5], Theorem 6.1.

Theorem 1.1. *Let V be a separable reflexive Banach space that is continuously and densely embedded into a Hilbert space H, and suppose that $\mathscr{E}\colon V \to \mathbb{R}$ is an H-elliptic (i.e., there exists $\omega \geqslant 0$ such that the function $\mathscr{E}_\omega\colon V \to \mathbb{R}$, $u \mapsto \mathscr{E}(u) + \frac{1}{2}\omega\|u\|^2_H$ is coercive and convex), convex, continuously differentiable function and that $\mathscr{E}'$ maps bounded sets of V to bounded sets of V'. Then for all $T > 0$, $f \in L_2(0, T; H)$ and $u_0 \in V$ the gradient system with initial value,*

$$\begin{cases} \dot{u} + \nabla_H \mathscr{E}(u) = f,\\ u(0) = u_0, \end{cases} \tag{1.2}$$

admits a unique solution $u \in W^1_2(0, T; H) \cap L_\infty(0, T; V)$. The solution can be chosen as a weakly continuous function $u\colon [0, T] \to V$, and for this function one has the energy inequality

$$\int_s^t \|\dot{u}(\tau)\|^2_H \,\mathrm{d}\tau + \mathscr{E}(u(t)) \leqslant \mathscr{E}(u(s)) + \int_s^t \langle f(\tau), \dot{u}(\tau)\rangle_H \,\mathrm{d}\tau \tag{1.3}$$

for all $0 \leqslant s \leqslant t \leqslant T$.

P r o o f. For the existence and uniqueness of u we refer to [5], Theorem 6.1. In this reference the function $\mathscr{E}$ is assumed to be coercive. This property is used in order to obtain bounds for approximating solutions. Replacing [5], Part 2 of the proof of Theorem 6.1, by [5] Part 2 of the proof of Theorem 8.1, one obtains that H-ellipticity is sufficient for existence and uniqueness of the solution.

The fact that the solution can be chosen as a weakly continuous function $u\colon [0,T] \to V$ is a consequence of [5], Exercise 5.4. In fact, this weakly continuous function is nothing but the continuous representative $u\colon [0,T] \to H$ of the function $u \in W_2^1(0,T;H)$.

Finally, the energy inequality (1.3), which is stated in [5] only for $s=0$, is now a consequence of the fact that the function $u(\cdot+s)$ is a solution of (1.2), with the initial value $u(s) \in V$ and the right hand side $f(\cdot+s)$. □

Remark 1.2. The basic idea in the proof of [5], Theorem 6.1, for the construction of solutions is to apply a Ritz-Galerkin procedure, to solve approximating finite dimensional gradient systems, and then to take the limit.

This is in contrast to the theory of evolution equations with accretive operators, where solutions are constructed using the "implicit Euler method" involving Yosida approximations; see for example [1], Chapter 3.

In the following, a solution u will always be assumed to be the weakly continuous representative $u\colon [0,T] \to V$.

As a consequence of (1.3) we obtain that the energy $\mathscr{E}(u(t))$ is strictly decreasing in t, provided the force f is zero and the solution is not stationary.

Our goal is to show that the porous medium equation

$$\begin{cases} \dot{u} - \Delta u^{[m]} = f, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

for a given power $m \in (0,\infty)$ and the initial value u_0 and the force f in appropriate spaces, has the structure of a gradient system. In (1.4) we have used the notation

$$r^{[m]} := \operatorname{sgn}(r)|r|^m$$

for $r \in \mathbb{R}$. It is common in the theory of the porous medium equation to simply use the notation r^m for the above introduced $r^{[m]}$. We avoid this abuse of notation because of apparent inconsistencies. E.g., for negative r and even m the "signed power" $r^{[m]}$ is negative (whereas, in standard notation, r^m is positive).

In view of the physical interpretation of u as a density it might seem unreasonable to allow negative u. It is known that the PME/FDE is positivity preserving, i.e., starting with a positive initial value u_0 and assuming the force f to be zero, the solution $u(t)$ remains positive for all times $t > 0$. In Section 3 we will show this property in our context.

For the following we assume that $\Omega \subseteq \mathbb{R}^n$ is open and such that for some $C > 0$ the Poincaré inequality

$$\|u\|_2 \leqslant C\|\nabla u\|_2 \tag{1.5}$$

holds for all $u \in C_c^\infty(\Omega)$. This is true if Ω is bounded. For a more detailed discussion of necessary and sufficient conditions for this hypothesis we refer to [10], where it is shown that it suffices that

$$\begin{aligned}\varrho'(\Omega) := \sup\{R > 0;\ &\text{there exists a ball } B \subseteq \mathbb{R}^n \text{ with radius } R,\\ &\text{such that } B \cap (\mathbb{R}^n \setminus \Omega) \text{ contains no interior point}\}\end{aligned}$$

is finite. Loosely speaking, the above condition says that Ω must not contain arbitrarily large balls.

As usual define $H_0^1(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ with respect to the $H^1(\Omega)$-norm $(\|\cdot\|_2^2 + \|\nabla\cdot\|_2^2)^{1/2}$. Due to the Poincaré inequality, on $H_0^1(\Omega)$ this norm is equivalent to the norm $\|\nabla\cdot\|_2$, and in the following we will use the latter norm. Then $H_0^1(\Omega)$ with the modified norm is a Hilbert space with the scalar product

$$(u, v) \mapsto \langle u, v\rangle_{H_0^1} := \langle \nabla u, \nabla v\rangle_{L_2} := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x.$$

The space $H^{-1}(\Omega)$ is defined to be the dual space of $H_0^1(\Omega)$. It will not be identified with $H_0^1(\Omega)$, but rather interpreted as a space of distributions, as stated in the following lemma.

Lemma 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be open and such that the Poincaré inequality* (1.5) *holds. Then the mapping $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isometric isomorphism (and in fact the Riesz mapping) between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In particular, $H^{-1}(\Omega)$ can be identified with the set of all distributions of the form $-\Delta v$ for $v \in H_0^1(\Omega)$.*

Proof. Let $w \in H^{-1}(\Omega)$. Then, by the Fréchet-Riesz representation theorem, there exists $v \in H_0^1(\Omega)$ such that for all $u \in H_0^1(\Omega)$ one has

$$w(u) = \langle v, u\rangle_{H_0^1} = \int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x.$$

Considering $u \in C_c^\infty(\Omega) \subseteq H_0^1(\Omega)$, we obtain that $w = -\Delta v$ in the sense of distributions. This shows that $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$ is surjective.

Let B denote the unit ball in $H_0^1(\Omega)$. Then the elementary identity

$$\|w\|_{H^{-1}} = \sup_{u\in B} w(u) = \sup_{u\in B}\langle v, u\rangle_{H_0^1} = \sup_{u\in B}\int_\Omega \nabla v \cdot \nabla u \,\mathrm{d}x = \|v\|_{H_0^1}$$

shows that $-\Delta$ is isometric. □

In the following let

$$G := (-\Delta)^{-1}$$

denote the inverse of $-\Delta\colon H_0^1(\Omega) \to H^{-1}(\Omega)$. Then the scalar product in $H^{-1}(\Omega)$ can be written as

$$\langle u, v\rangle_{H^{-1}} = \langle Gu, Gv\rangle_{H_0^1} = \int_\Omega \nabla Gu \cdot \nabla Gv \,\mathrm{d}x,$$

because G is an isometry.

Proposition 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open, $m \in (0, \infty)$, and define $\mathscr{E}\colon L_{m+1}(\Omega) \to \mathbb{R}$ by*

$$\mathscr{E}(u) := \frac{1}{m+1} \int_\Omega |u|^{m+1} \,\mathrm{d}x. \tag{1.6}$$

Then $\mathscr{E}$ is continuously differentiable and convex, and one has

$$\mathscr{E}'(u)v = \int_\Omega u^{[m]}\, v \,\mathrm{d}x$$

for all $u, v \in L_{m+1}(\Omega)$. Moreover, $\mathscr{E}'\colon L_{m+1}(\Omega) \to L_{(m+1)/m}(\Omega)$ maps bounded sets to bounded sets.

For a proof of this result we refer to [5], Theorem 4.3, or [6], Theorem 3.12 (page 24), Theorem 2.14 (page 53) and Theorem 4.7 (page 71).

In analogy to other gradient systems, where the functional $\mathscr{E}$ often has an interpretation as a physical energy, we will call $\mathscr{E}$ energy in the following, although there is no physical energy related to $\mathscr{E}$.

Proposition 1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be open and such that the Poincaré inequality* (1.5) *holds. Then:*

(a) *For all*

$$m \in (0,1] \cap \Big[\frac{n-2}{n+2}, \infty\Big)$$

the space $L_{m+1}(\Omega)$ embeds continuously and densely into $H^{-1}(\Omega)$.

(b) *Assume additionally that the measure of Ω is finite. Then the embedding $L_{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is continuous and dense for*

$$m \in (0,\infty) \cap \Big[\frac{n-2}{n+2}, \infty\Big).$$

Note that for bounded Ω the continuous embeddings above (except for $m = (n-2)/(n+2)$) are even compact, a fact that we will not use.

P r o o f. Let us first note that due to the Poincaré inequality, $H^1_0(\Omega)$ is continuously embedded into $L_2(\Omega)$. Thus, due to the Sobolev embedding theorem, $H^1_0(\Omega)$ is continuously (and densely) embedded into $L_p(\Omega)$

(i) for all $p \in [2, 2n/(n-2)]$ if $n \geqslant 3$ and all $p \in [2, \infty)$ if $n = 1, 2$,
(ii) and additionally for all $p \in (1, 2)$, if Ω has finite measure.

Standard dualization arguments show that

$$L_q(\Omega) \hookrightarrow H^{-1}(\Omega)$$

continuously, where q is chosen conjugate to some admissible p above, the embedding being injective and dense by the denseness and the injectivity of the embeddings above. In particular,

(i) $q \in (1, 2] \cap [2n/(n+2), 2]$ for arbitrary Ω and
(ii) $q \in (1, \infty) \cap [2n/(n+2), \infty)$, if Ω has finite measure.

The assertion follows by the substitution $m = q - 1$. □

Remark 1.6. (a) The spaces $L_{m+1}(\Omega)$ and $H^{-1}(\Omega)$ are separable and reflexive, and both of the spaces are continuously embedded into the space of distributions $\mathcal{D}(\Omega)'$. This implies that their intersection $L_{m+1} \cap H^{-1}(\Omega) := L_{m+1}(\Omega) \cap H^{-1}(\Omega)$ (with norm given by $\|u\| := \|u\|_{m+1} + \|u\|_{H^{-1}}$) is a Banach space. It is separable and reflexive because it is isomorphic to the closed subspace $\{(u, u); u \in L_{m+1} \cap H^{-1}(\Omega)\}$ of the product space $L_{m+1}(\Omega) \times H^{-1}(\Omega)$. The embedding $L_{m+1} \cap H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ is dense, because $C_c^\infty(\Omega)$ is contained in $L_{m+1} \cap H^{-1}(\Omega)$ and dense in $H^{-1}(\Omega)$.

(b) The restriction of $\mathscr{E}$ to $L_{m+1} \cap H^{-1}(\Omega)$ is continuously differentiable, and

$$(\mathscr{E}|_{L_{m+1} \cap H^{-1}})' \colon L_{m+1} \cap H^{-1}(\Omega) \to (L_{m+1} \cap H^{-1}(\Omega))'$$

maps bounded sets to bounded sets. This is immediate from Proposition 1.4 and the continuity of the injection $L_{m+1} \cap H^{-1}(\Omega) \hookrightarrow H^{-1}(\Omega)$. The (obvious) coerciveness of the functional $\mathscr{E}$ implies the $H^{-1}(\Omega)$-ellipticity of its restriction to $L_{m+1} \cap H^{-1}(\Omega)$.

In the following we will use the notation $\mathscr{E}$ also for its restriction to $L_{m+1} \cap H^{-1}(\Omega)$.

The following proposition is the key observation which makes it possible to consider the PME/FDE as a gradient system in $H^{-1}(\Omega)$.

Proposition 1.7. *Assume that the Poincaré inequality* (1.5) *holds for* Ω. *Then the gradient* $\nabla_H\mathscr{E}$ *of* $\mathscr{E}\colon\ L_{m+1}\cap H^{-1}(\Omega)\to\mathbb{R}$,

$$\mathscr{E}(u) := \frac{1}{m+1}\int_\Omega |u|^{m+1}\,\mathrm{d}x,$$

in the Hilbert space $H := H^{-1}(\Omega)$ *is given by*

$$D(\nabla_H\mathscr{E}) := \{u\in L_{m+1}\cap H^{-1}(\Omega);\ -\Delta u^{[m]}\in H^{-1}(\Omega)\},$$
$$\nabla_H\mathscr{E}(u) = -\Delta u^{[m]}.$$

P r o o f. By definition, a function $u\in L_{m+1}\cap H^{-1}(\Omega)$ belongs to $D(\nabla_H\mathscr{E})$ if and only if there exists $w\in H^{-1}(\Omega)$ such that

$$\langle w,v\rangle_{H^{-1}} = \int_\Omega u^{[m]}\,v\,\mathrm{d}x$$

for all $v\in L_{m+1}\cap H^{-1}(\Omega)$, and in this case w is the gradient $\nabla_H\mathscr{E}(u)$ at u. Now the calculation

$$\langle w,v\rangle_{H^{-1}} = \langle Gw,Gv\rangle_{H_0^1} = \int_\Omega \nabla Gw\cdot\nabla Gv\,\mathrm{d}x = \int_\Omega Gwv\,\mathrm{d}x,$$

valid for all $v\in L_{m+1}\cap H^{-1}(\Omega)$, yields that $Gw = u^{[m]}$, or equivalently $w = -\Delta u^{[m]}$. □

Next we state the main existence and uniqueness theorem of this article.

Theorem 1.8. *Let* $\Omega\subseteq\mathbb{R}^n$ *be open and such that the Poincaré inequality* (1.5) *holds on* Ω, *and let* $m\in(0,\infty)$. *Then for all* $T>0$, $f\in L_2(0,T;H^{-1}(\Omega))$ *and* $u_0\in L_{m+1}\cap H^{-1}(\Omega)$ *the PME/FDE gradient system*

$$\begin{cases}\dot u + \nabla_H\mathscr{E}(u) = f,\\ u(0,\cdot) = u_0\end{cases}$$

admits a unique solution $u\in W_2^1(0,T;H^{-1}(\Omega))\cap L_\infty(0,T;L_{m+1}\cap H^{-1}(\Omega))$. *The solution can be chosen as a weakly continuous mapping* $u\colon\ [0,T]\to L_{m+1}\cap H^{-1}(\Omega)$, *and for this mapping one has the energy inequality*

$$\int_s^t \|\dot u(\tau)\|^2_{H^{-1}}\,\mathrm{d}\tau + \mathscr{E}(u(t)) \leqslant \mathscr{E}(u(s)) + \int_s^t \langle f(\tau),\dot u(\tau)\rangle_{H^{-1}}\,\mathrm{d}\tau$$

for all $0\leqslant s\leqslant t\leqslant T$. *(Recall that* $L_{m+1}\cap H^{-1}(\Omega)$ *reduces to* $L_{m+1}(\Omega)$ *provided the embedding* $L_{m+1}(\Omega)\hookrightarrow H^{-1}(\Omega)$ *holds; cf. Proposition* 1.5.*)*

P r o o f. Taking into account Proposition 1.7 and recalling Proposition 1.4 together with Remark 1.6 (b) we obtain the assertion as an immediate consequence of Theorem 1.1. □

Remark 1.9. (a) The property that $u(t) \in D(\nabla_H \mathscr{E})$, or equivalently, that $u(t)^{[m]} \in H_0^1(\Omega)$, is a weak replacement for $u(t)|_{\partial\Omega} = 0$

(b) Note that elements of the domain of $\nabla_H \mathscr{E}$ are regular distributions and thus have a reasonable physical interpretation.

(c) For a more detailed analysis it might be helpful to get a better understanding of the structure of $D(\nabla_H \mathscr{E})$. Note that by the invertibility of $-\Delta\colon H_0^1 \to H^{-1}$ we obtain that

$$
\begin{aligned}
D(\nabla_H \mathscr{E}) &= \{u \in L_{m+1} \cap H^{-1}(\Omega);\ -\Delta u^{[m]} \in H^{-1}(\Omega)\} \\
&= \{u \in L_{m+1} \cap H^{-1}(\Omega);\ u^{[m]} \in H_0^1(\Omega)\}.
\end{aligned} \tag{1.7}
$$

Moreover, recall that $u \mapsto u^{[m]}$ is the duality map from $L_{m+1} \to L_{(m+1)'} = L_{(m+1)/m}$. This continuous nonlinear map is bijective and we have

$$\langle u, u^{[m]} \rangle_{L_{m+1}, L_{(m+1)/m}} = \|u\|_{m+1}^{m+1} \quad \text{and} \quad \|u^{[m]}\|_{(m+1)/m} = \|u\|_{m+1}^{m},$$

and the inverse of $u \mapsto u^{[m]}$ is given by $v \mapsto v^{[1/m]}$. (For details on duality mappings we refer to [6], Chapter II, § 4.) If Ω and m are such that the embedding $L_{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ holds (cf. Proposition 1.5), this allows to rewrite

$$D(\nabla_H \mathscr{E}) = \{u \in L_{m+1}(\Omega);\ u^{[m]} \in H_0^1(\Omega)\} = \{v^{[1/m]};\ v \in L_{(m+1)/m}(\Omega) \cap H_0^1(\Omega)\}.$$

Note that $L_{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ if and only if $H_0^1(\Omega) \hookrightarrow L_{(m+1)/m}(\Omega)$ (see the proof of Proposition 1.5), and then

$$D(\nabla_H \mathscr{E}) = \{v^{[1/m]};\ v \in H_0^1(\Omega)\}.$$

To illustrate the connection to other common notions of solutions we recall the concept of a weak solution of the PME/FDE. Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, with Lipschitz boundary, $T > 0$, $Q_T := (0,T) \times \Omega$, $f \in L_1(Q_T)$ and $u_0 \in L_1(\Omega)$. A function $u \in L_1(Q_T)$ is called a *weak solution* of the initial boundary value problem

$$
\begin{cases}
\dfrac{\partial}{\partial t} u = \Delta u^{[m]} + f \text{ on } Q_T, \\
u(0,\cdot) = u_0, \quad u|_{[0,T]\times\partial\Omega} = 0,
\end{cases}
$$

if

(1) $u^{[m]} \in L_1(0,T; W_{1,0}^1(\Omega))$ and

(2) u satisfies

$$\int_{Q_T} \Big(\nabla u^{[m]} \cdot \nabla \eta - u \frac{\partial}{\partial t} \eta\Big) \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega u_0(x) \eta(0,x) \,\mathrm{d}x + \int_{Q_T} f\eta \,\mathrm{d}x \,\mathrm{d}t \tag{1.8}$$

for all (test functions) $\eta \in C^1(\overline{Q_T})$ with $\eta|_{[0,T]\times\partial\Omega} = 0$ and $\eta(T,\cdot) = 0$.

We refer to [11], Definition 5.4, for this notion of a weak solution.

Proposition 1.10. *Assume that Ω is bounded and has a Lipschitz boundary. Let $T > 0$, $f \in L_2(0,T;H^{-1}(\Omega)) \cap L_1(0,T;L_1(\Omega))$, $u_0 \in L_{m+1} \cap H^{-1}(\Omega)$ and let u be the gradient system solution of the PME/FDE according to Theorem* 1.8, *with weakly continuous $u\colon [0,T] \to L_{m+1} \cap H^{-1}(\Omega)$.*

Then u is a weak solution of the initial boundary value problem for the PME/FDE.

Proof. (i) First we show property (1) of weak solutions. We know that $u \in W_2^1(0,T;H^{-1}(\Omega))$, and the fact that u is a solution of the gradient system implies that $u(t) \in D(\nabla_H \mathscr{E})$ for $t \in [0,T]$ a.e.,

$$-\Delta u^{[m]} = \nabla_H \mathscr{E}(u) = f - \dot{u} \in L_2(0,T;H^{-1}(\Omega)),$$

and therefore

$$u^{[m]} = G(-\Delta u^{[m]}) \in L_2(0,T;H_0^1(\Omega)) \subseteq L_1(0,T;W_{1,0}^1(\Omega)).$$

(ii) As a first step for the proof of property (2) of weak solutions let $\eta \in C^1(\overline{Q_T})$ be such that $\eta|_{[0,T]\times\partial\Omega} = 0$ and $\eta(T,\cdot) = 0$, and assume additionally that also $(\partial/\partial t)\eta \in C^1(\overline{Q_T})$. It is well-known that $C^1(\overline{\Omega}) \cap C_0(\Omega) \subseteq H_0^1(\Omega)$, with continuous inclusion, and our hypotheses on η imply that the function $[0,T] \ni t \mapsto \eta(t,\cdot)$ belongs to $C^1([0,T];C^1(\overline{\Omega}) \cap C_0(\Omega)) \subseteq C^1([0,T];H_0^1(\Omega))$. It is not too difficult to show that then the function $[0,T] \ni t \mapsto \int_\Omega u(t,x)\eta(t,x)\,\mathrm{d}x = \langle u(t),\eta(t)\rangle_{H^{-1},H_0^1}$ belongs to $W_2^1(0,T)$, with the weak derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(t,x)\eta(t,x)\,\mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t),\eta(t)\rangle_{H^{-1},H_0^1}\\ &= \langle \dot{u}(t),\eta(t)\rangle_{H^{-1},H_0^1} + \Big\langle u(t), \frac{\partial}{\partial t}\eta(t)\Big\rangle_{H^{-1},H_0^1}\\ &= \langle \dot{u}(t),\eta(t)\rangle_{H^{-1},H_0^1} + \int_\Omega u(t,x)\frac{\partial}{\partial t}\eta(t,x)\,\mathrm{d}x.\end{aligned}$$

Integrating and using the fact that u is a solution of the PME/FDE we obtain that

$$\begin{aligned}\int_0^T\!\!\int_\Omega u(x,t)\frac{\partial}{\partial t}\eta(t,x)\,\mathrm{d}x\,\mathrm{d}t &= \langle u(T),\eta(T)\rangle_{H^{-1},H_0^1} - \langle u(0),\eta(0)\rangle_{H^{-1},H_0^1}\\ &\quad - \int_0^T \langle \Delta u^{[m]}(t),\eta(t)\rangle_{H^{-1},H_0^1}\,\mathrm{d}t - \int_{Q_T} f\eta\,\mathrm{d}x\,\mathrm{d}t\\ = -\int_\Omega u_0(x)\eta(0,x)\,\mathrm{d}x &+ \int_{Q_T}\nabla u^{[m]}\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t - \int_{Q_T} f\eta\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

This shows equation (1.8) for the present case.

(iii) Next, let $\eta \in C^1(\overline{Q_T})$ be such that $\eta|_{[0,T]\times\partial\Omega} = 0$, and assume that there exists $\delta > 0$ such that $\eta(t,\cdot) = 0$ for all $t \in (T-\delta, T]$. Performing suitable convolutions in the t-direction one can show that there exists a sequence (η_k) consisting of functions as in part (ii) such that η_k and the first derivatives of η_k converge uniformly on $\overline{Q_T}$ to η and the corresponding first derivatives of η. Using the validity of (1.8) for η replaced by η_k, as established in part (ii), we conclude the validity of (1.8) for η.

(iv) Finally, let $\eta \in C^1(\overline{Q_T})$ be such that $\eta|_{[0,T]\times\partial\Omega} = 0$ and $\eta(T,\cdot) = 0$. Let $\alpha \in C^1(\mathbb{R})$ be an increasing function, $\alpha|_{(-\infty,1/2]} = 0$, $\alpha|_{[1,\infty]} = 1$. For $k \in \mathbb{N}$ define α_k and η_k by

$$\alpha_k(t) := \alpha(k(T-t)), \quad \eta_k(t,x) := \alpha_k(t)\eta(t,x).$$

Then evidently the functions η_k are of the kind considered in step (iii), and therefore (1.8) holds for η replaced by η_k. Letting $k \to \infty$ in (1.8) with η_k, the only term for which it is not evident that it converges to the corresponding term with η is the second, $\int_{Q_T} u(\partial/\partial t)\eta_k \,\mathrm{d}x\,\mathrm{d}t$. In fact, because of $(\partial/\partial t)\eta_k(t,x) = \alpha_k'(t)\eta(t,x) + \alpha_k(t)(\partial/\partial t)\eta(t,x)$, we only have to show that

$$\int_{Q_T} u(t,x)\eta(t,x)\alpha_k'(t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \langle u(t), \eta(t)\rangle_{H^{-1},H_0^1}\alpha_k'(t)\,\mathrm{d}t$$

tends to zero as $k \to \infty$. Now, $u\colon [0,T] \to H^{-1}(\Omega)$ is continuous, $[0,T] \ni t \mapsto \eta(t,\cdot) \in H_0^1(\Omega)$ is continuous with $\eta(T,\cdot) = 0$, and therefore $\langle u(t), \eta(t)\rangle_{H^{-1},H_0^1} \to 0$ as $t \to T$. Using that $\int_{\mathbb{R}} \alpha_k'(t)\,\mathrm{d}t = 1$ and that $\operatorname{spt}\alpha_k' \subseteq [T-1/k, T]$ we conclude that the desired convergence holds. □

2. Asymptotic behaviour for the PME/FDE

In this section we assume that $L_{m+1}(\Omega)$ embeds into $H^{-1}(\Omega)$ (cf. Proposition 1.5), and we assume the validity of the Poincaré inequality (1.5), as before. We define $\mathscr{E}$ as in (1.6) and consider the autonomous PME/FDE gradient system

$$\begin{cases} \dot{u} + \nabla_H \mathscr{E}(u) = 0, \\ u(0) = u_0 \end{cases} \tag{2.1}$$

for a given initial value $u_0 \in L_{m+1}(\Omega)$. Let $u\colon [0,\infty) \to L_{m+1}(\Omega)$ be the unique solution of the above abstract initial value problem according to Theorem 1.8. Our goal is to show that the solution tends to zero as t goes to infinity, and we will derive some $L_{m+1}(\Omega)$-decay estimates. In contrast to other investigations of asymptotics in the literature we need not distinguish between $m < 1$, $m = 1$, and $m > 1$, a priori.

Remark 2.1. The function $u\colon [0,\infty) \to L_{m+1}(\Omega)$ is weakly continuous by Theorem 1.1. Thus, by the uniform convexity of $L_{m+1}(\Omega)$, in order to prove continuity of $u\colon [0,\infty) \to L_{m+1}(\Omega)$ at some $t_0 \in [0,\infty)$ it suffices to show that $t \mapsto \|u(t)\|_{m+1}$ is continuous at t_0, or equivalently, that the function $g\colon [0,\infty) \to [0,\infty)$, defined by

$$g(t) := \mathscr{E}(u(t)) = \frac{1}{m+1}\|u(t)\|_{m+1}^{m+1},$$

is continuous at t_0.

From the energy inequality (1.3) we obtain that g is decreasing, and thus g (and hence $u\colon [0,\infty) \to L_{m+1}(\Omega)$) is continuous on the complement of a countable subset of $[0,\infty)$, the jump points of the monotone function g. We do not know if these jump points can occur at all. If the solution has more regularity properties, then it can be shown that the energy inequality is in fact an equality; cf. [11], Proposition 8.9.

Proposition 2.2. *If u is a solution of the gradient system* (2.1), *then for almost all $t > 0$ one has*

$$\|\dot u(t)\|_{H^{-1}} \geqslant \sqrt{\lambda_1}\|u(t)\|_{m+1}^{m} = \sqrt{\lambda_1}(m+1)^{m/(m+1)}\mathscr{E}(u(t))^{m/(m+1)},$$

where λ_1 is such that $1/\sqrt{\lambda_1}$ is the optimal embedding constant of $H_0^1(\Omega)$ into $L_{(m+1)/m}(\Omega)$.

P r o o f. Since u is a solution we have for almost all $t > 0$ that

$$\dot u(t) = -\Delta u(t)^{[m]},$$

and for these t the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L_{(m+1)/m}(\Omega)$ implies that

$$\begin{aligned}\|\dot u(t)\|_{H^{-1}} &= \|-\Delta u(t)^{[m]}\|_{H^{-1}} = \|u(t)^{[m]}\|_{H_0^1} \geqslant \sqrt{\lambda_1}\|u(t)^{[m]}\|_{(m+1)/m}\\ &= \sqrt{\lambda_1}\|u(t)\|_{m+1}^{m} = \sqrt{\lambda_1}(m+1)^{m/(m+1)}\mathscr{E}(u(t))^{m/(m+1)}.\end{aligned}$$

□

Inserting the previous estimate into the energy estimate (1.3) we directly derive the following proposition.

Proposition 2.3. *Let $u_0 \in L_{m+1}(\Omega)$, and let $u\colon [0,\infty) \to L_{m+1}(\Omega) \cap H^{-1}(\Omega)$ be the solution of the gradient system* (2.1). *Then for all $0 \leqslant s \leqslant t$ one has the estimate*

$$0 \leqslant \mathscr{E}(u(t)) \leqslant \mathscr{E}(u(s)) - \lambda_1 (m+1)^{2m/(m+1)} \int_s^t \mathscr{E}(u(\tau))^{2m/(m+1)} \, \mathrm{d}\tau.$$

In the following we discuss the consequences for the asymptotics of the solutions of the porous medium gradient system.

First of all we see that $t \mapsto \mathscr{E}(u(t))$ is not only decreasing, but also goes to zero as t tends to infinity, otherwise one would get a contradiction to Proposition 2.3. In particular, this implies that the L_{m+1}-norm (and hence also the H^{-1}-norm) of the solution tends to zero. Our next goal is to derive a decay rate for the solution.

We need the following proposition.

Proposition 2.4. *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be monotone increasing. Let $g\colon [0,\infty) \to [0,\infty)$ be a decreasing function satisfying the integral inequality*

$$g(t) \leqslant g(s) - \int_s^t \Phi(g(\tau)) \, \mathrm{d}\tau \tag{2.2}$$

for all $t \geqslant s \geqslant 0$, and let $f\colon [0,\infty) \to [0,\infty)$ be a solution of the integral equation

$$f(t) = g(0) - \int_0^t \Phi(f(\tau)) \, \mathrm{d}\tau.$$

Then $f(t) \geqslant g(t)$ for all $t \geqslant 0$.

Proof. Note that the equality for f implies that $f(t) = f(s) - \int_s^t \Phi(f(\tau)) \, \mathrm{d}\tau$ for all $t \geqslant s \geqslant 0$.

Define $h := g - f$. Then the monotonicity of g and the continuity of f imply that $h(t-) \geqslant h(t) \geqslant h(t+)$ for all $t \geqslant 0$ (where $h(t-)$ and $h(t+)$ denote the left-hand and right-hand limits of h at t, respectively).

Assume that there exists $t > 0$ such that $h(t) = g(t) - f(t) > 0$. Let

$$s_0 := \sup\{\tau \in [0,t);\, h(\tau) \leqslant 0\}.$$

Then $s_0 < t$ because $h(t-) \geqslant h(t) > 0$, and for all $s \in (s_0, t]$ one has $h(s) > 0$, i.e., $g(s) > f(s)$. From $0 \geqslant h(s_0-) \geqslant h(s_0) \geqslant h(s_0+) \geqslant 0$ one obtains that $h(s_0) = 0$. (This also holds if $s_0 = 0$.)

Applying the integral inequality (2.2) with $s = s_0$ and using the monotonicity of Φ we get the contradiction

$$f(t) < g(t) \leqslant g(s_0) - \int_{s_0}^t \Phi(g(\tau)) \, \mathrm{d}\tau \leqslant f(s_0) - \int_{s_0}^t \Phi(f(\tau)) \, \mathrm{d}\tau = f(t).$$

□

For given $u_0 \in L_{m+1}(\Omega)$ we define the functions $G_m\colon [0,\infty) \to [0,\infty)$ ($m \in (0,\infty)$),

$$G_m(t) := \begin{cases} ((\mathscr{E}(u_0)^{(1-m)/(1+m)} \\ \quad -\lambda_1(1-m)(m+1)^{-(1-m)/(1+m)}t)^+)^{(1+m)/(1-m)} & \text{if } m < 1, \\ \mathscr{E}(u_0)\mathrm{e}^{-2\lambda_1 t} & \text{if } m = 1, \\ (\lambda_1(m-1)(m+1)^{(m-1)/(m+1)}t \\ \quad +\mathscr{E}(u_0)^{-(m-1)/(m+1)})^{-(m+1)/(m-1)} & \text{if } m > 1. \end{cases}$$

The plus in the case $m < 1$ indicates that we take the positive part, and therefore $G_m(t) = 0$ for all

$$t \geqslant t_{\max} := \frac{(m+1)^{(1-m)/(1+m)}}{\lambda_1(1-m)}\mathscr{E}(u_0)^{(1-m)/(1+m)}.$$

Theorem 2.5 (Asymptotic behaviour). *Assume that the Poincaré inequality* (1.5) *holds for* Ω *and let* $m > 0$ *be such that the embedding* $L_{m+1}(\Omega) \hookrightarrow H^{-1}(\Omega)$ *holds, let* $u_0 \in L_{m+1}(\Omega)$ *and let* $u\colon [0,\infty) \to L_{m+1}(\Omega)$ *be the unique solution of* (2.1). *Then* $\mathscr{E}(u(t))$ *is dominated by the function* G_m *defined above, i.e.,* $\mathscr{E}(u(t)) \leqslant G_m(t)$ *for all* $t \geqslant 0$.

Proof. Recall the function $g\colon [0,\infty) \to [0,\infty)$, $g(t) := \mathscr{E}(u(t))$, from Remark 2.1.

Let $\Phi\colon [0,\infty) \to [0,\infty)$ be defined by

$$\Phi(r) := \lambda_1(m+1)^{2m/(m+1)}\, r^{2m/(m+1)}.$$

It is straightforward to check that G_m is a solution of the integral equality

$$\begin{aligned} G_m(t) &= \mathscr{E}(u_0) - \lambda_1(m+1)^{2m/(m+1)} \int_0^t G_m(\tau)^{2m/(m+1)}\,\mathrm{d}\tau \\ &= \mathscr{E}(u_0) - \int_0^t \Phi(G_m(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Moreover, the function g is monotone decreasing as the energy of a gradient system, and Proposition 2.3 implies that the integral inequality (2.2) holds for all $0 \leqslant s \leqslant t$. Thus the assertion is a consequence of Proposition 2.4. $\square$

Remark 2.6. Theorem 2.5 shows that the energy $\mathscr{E}$ of solutions of the PME/FDE with $f = 0$ tends to zero as t goes to infinity as follows:

(1) polynomially with decay rate $-(m+1)/(m-1)$ provided $m > 1$,

(2) exponentially with decay rate $-2\lambda_1$ provided $m = 1$,
(3) in finite time for $m < 1$.

Many types of these results are known. We refer to [9], [7], [11] for general information and to [1], Proposition 5.13, for a version of (3).

Concerning the case $m = 1$ we note that in this (linear!) case the number λ_1 is the infimum of the spectrum of the negative Laplace operator, and our estimate reproduces the known asymptotic decay of the norm with rate $-\lambda_1$.

Remark 2.7. Apart from the decay due to the energy inequality (1.3), the central estimate of our derivation of the asymptotic behaviour is the elementary identity

$$\|\mathscr{E}'(v)\|_{m/(m+1)} = (m+1)^{m/(m+1)}\mathscr{E}(v)^{m/(m+1)}$$

holding for all arguments $v \in L_{m+1}(\Omega)$ of our C^1-functional $\mathscr{E}$. Note that this formula implies that for all $0 < \theta \leqslant m/(m+1)$ there exists $C > 0$ and a neighbourhood U of zero such that the *Łojasiewicz-Simon inequality*

$$|\mathscr{E}(v) - \mathscr{E}(0)|^{1-\theta} \leqslant C\|\mathscr{E}'(v)\|_{m/(m+1)}$$

holds for all $v \in U$. Even though [5], Theorem 12.2, does not directly apply, we derive similar (and slightly better) decay rates in our approach.

3. Order and positivity preservation

The goal of this section is to present a proof of the following result; the hypothesis that $L_{m+1}(\Omega)$ embeds into $H^{-1}(\Omega)$ will no longer be needed in the present section.

Theorem 3.1 (Comparison principle). *Let $\Omega \subseteq \mathbb{R}^n$ be open and such that the Poincaré inequality* (1.5) *holds, $m > 0$. Let $u_{1,0}, u_{2,0} \in L_{m+1} \cap H^{-1}(\Omega))$, $u_{1,0} \leqslant u_{2,0}$, and let $u_1, u_2 \in W^1_{2,\mathrm{loc}}([0,\infty); H^{-1}(\Omega)) \cap L_\infty([0,\infty); L_{m+1} \cap H^{-1}(\Omega))$ denote the unique gradient system solutions (Theorem* 1.8*) of the PME/FDE*

$$\dot{u} - \Delta u^{[m]} = 0 \tag{3.1}$$

with initial values $u_{1,0}, u_{2,0}$, respectively. Then $u_1(t) \leqslant u_2(t)$ for all $t \geqslant 0$.

Remark 3.2 (Positivity preservation). Setting $u_{1,0} = 0$ in Theorem 3.1 we obtain that for initial values $u_0 \geqslant 0$ the unique solution u of (3.1) remains positive: $u(t) \geqslant 0$ for all $t \geqslant 0$.

We note that (at least for bounded Ω with sufficiently smooth boundary) these results are known for the PME/FDE (cf. [11], Proposition 6.1 (page 127)). However, we will show that general properties of gradient systems, instead of arguments involving classical constructions, can be applied to obtain the assertions.

Before we turn to the proof let us recall standard concepts to prove order and positivity preservation of gradient systems. We consider (1.1) in the setting as introduced at the beginning of Section 1 and assume additionally that $\mathscr{E}$ is convex and coercive. It is well-known that gradient systems of that kind are subgradient systems, i.e., for $f\colon I \to H$ given, u is a solution of the gradient system (1) if and only if u is a solution of the subgradient system

$$\dot{u}(t) + \partial\widetilde{\mathscr{E}}(u(t)) \ni f(t), \tag{3.2}$$

where

$$\widetilde{\mathscr{E}}(u) := \begin{cases} \mathscr{E}(u) & \text{if } u \in V, \\ \infty & \text{if } u \in H \setminus V, \end{cases}$$

and $\partial\widetilde{\mathscr{E}}\colon H \to 2^H$ denotes the subdifferential of convex analysis. A solution of the subgradient system (3.2) is a function

$u \in W^1_{2,\mathrm{loc}}(I;H)$, with $u(t) \in D(\partial\widetilde{\mathscr{E}}) := \{u \in H;\, \partial\widetilde{\mathscr{E}}(u) \neq \emptyset\}$ for almost all $t \in I$, and equation (3.2) holds for almost all $t \in I$.

For an exposition of the theory of subgradient systems we refer to [5], Lectures 13 and 14. The correspondence of solutions of gradient systems and subgradient systems relies on the existence and uniqueness results for subgradient systems (cf. [5], Theorem 14.1) and the fact that in the above setting we have

$$\begin{aligned} D(\partial\widetilde{\mathscr{E}}) &= D(\nabla_H \mathscr{E}), \\ \partial\widetilde{\mathscr{E}}(u) &= \{\nabla_H \mathscr{E}(u)\} \quad \text{for all } u \in D(\partial\widetilde{\mathscr{E}}). \end{aligned} \tag{3.3}$$

After this short general discussion we return to the case of autonomous systems (i.e., the case that the forcing term f in (1.1) and (3.2) is equal to zero). Let $u_0 \in \{u \in H;\, \widetilde{\mathscr{E}}(u) < \infty\}$ be given and let u be the unique solution of the subgradient system (3.2). Then we have the exponential formula

$$u(t) = \lim_{k\to\infty} (J_{t/k})^k u_0, \tag{3.4}$$

with local uniform convergence on $[0,\infty)$, where $J_h\colon H \to H$ $(h > 0)$ is the operator defined by

$$J_h g = \operatorname*{arg\,min}_{v\in H} \Big(\mathscr{E}(v) + \frac{\|v-g\|_H^2}{2h}\Big). \tag{3.5}$$

We refer to [5], Corollary 14.8, for this statement.

A set $C \subseteq H$ is called invariant under a (sub)gradient system if $u_0 \in C$ implies that $u(t) \in C$ for all $t > 0$ (where u is the solution with the initial value u_0). The following remark illustrates how in the general context conditions can be formulated for a set to be invariant.

Remark 3.3. Let $\widetilde{\mathscr{E}}\colon H \to \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous, convex, coercive function. Let C be a closed convex set contained in the closure of the effective domain $D_{\text{eff}}(\widetilde{\mathscr{E}}) := \{u \in H;\ \widetilde{\mathscr{E}}(u) < \infty\}$ in H. Then C is invariant under the subgradient system

$$\dot{u} + \partial\widetilde{\mathscr{E}}(u) \ni 0$$

if and only if

$$\widetilde{\mathscr{E}}(P_C u) \leqslant \widetilde{\mathscr{E}}(u)$$

for all $u \in H$, where $P_C\colon H \to C$ denotes the best approximation projection onto C.

For the proof we refer to [4], Proposition 4.5, and [5], Theorem 15.3 and Remark 15.4.

We mention that in many classical situations (p-Laplace evolution, weighted p-Laplace evolution) the Hilbert space H is an L_2-space, and the set C is often taken to be the standard positive cone $L_{2,+}$. Then the best approximation projection onto C is given by $u \mapsto u^+$, and the necessary and sufficient condition for positivity preservation reduces to the statement that $\widetilde{\mathscr{E}}(u^+) \leqslant \widetilde{\mathscr{E}}(u)$ for all $u \in L_2$. In order to apply the above criterion in the context of our treatment of the PME/FDE, one would have to calculate the best approximation projection from $H^{-1}(\Omega)$ onto certain convex sets. We have not been able to apply this procedure in our context.

Our method of proof to obtain order preservation is a combination of classical techniques and subgradient system arguments.

Proof of Theorem 3.1. Let $\mathbf{H} := H^{-1}(\Omega) \times H^{-1}(\Omega)$, and define $\mathbf{E}\colon \mathbf{H} \to \mathbb{R} \cup \{\infty\}$ by

$$\mathbf{E}(u_1, u_2) := \widetilde{\mathscr{E}}(u_1) + \widetilde{\mathscr{E}}(u_2),$$

with

$$\widetilde{\mathscr{E}}(u) := \begin{cases} \mathscr{E}(u) & \text{if } u \in L_{m+1} \cap H^{-1}(\Omega), \\ \infty & \text{if } u \in H^{-1}(\Omega) \setminus L_{m+1}(\Omega). \end{cases}$$

Given $\mathbf{u}_0 = (u_{1,0}, u_{2,0}) \in (L_{m+1} \cap H^{-1}(\Omega))^2$, it is easy to see that $\mathbf{u} = (u_1, u_2) \in W^1_{2,\text{loc}}([0,\infty); \mathbf{H})$ is a solution of the subgradient system

$$\dot{\mathbf{u}} + \partial\mathbf{E}(\mathbf{u}) \ni 0 \tag{3.6}$$

with an initial value $\mathbf{u}_0$ if and only if u_1 and u_2 are solutions of the subgradient system

$$\dot{u} + \partial\widetilde{\mathscr{E}}(u) \ni 0 \tag{3.7}$$

with initial values $u_{1,0}$ and $u_{2,0}$, respectively, and this holds if and only if u_1 and u_2 are solutions of the gradient system (3.1) with initial values $u_{1,0}$ and $u_{2,0}$, respectively.

For $\alpha > 0$ we define the convex set

$$\begin{aligned} C_\alpha := \{\mathbf{u} = (u_1, u_2) \in (L_{m+1} \cap H^{-1}(\Omega))^2;\\ \|u_1\|_{m+1} \leqslant \alpha,\ \|u_2\|_{m+1} \leqslant \alpha,\ u_1 \leqslant u_2\}. \end{aligned}$$

Then in order to show the comparison principle it is sufficient to show that for all $\alpha > 0$ the set C_α is invariant under the subgradient system (3.6). In order to achieve this we will apply an argument involving the operator J_h from the exponential formula (3.4).

Let $\alpha > 0$. First we note that C_α is a closed subset of $\mathbf{H}$. Indeed, let $(\mathbf{u}^k)$ be a sequence in C_α, $\mathbf{u}^k \to \mathbf{u}$ in $\mathbf{H}$. Then $(\mathbf{u}^k)$ is bounded in $\mathbf{H}$ and $(L_{m+1}(\Omega))^2$; thus the reflexivity of $(L_{m+1} \cap H^{-1}(\Omega))^2$ implies that there exist $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2) \in L_{m+1} \cap H^{-1}(\Omega)$ and a subsequence $(\mathbf{u}^{k_j})$ such that $\mathbf{u}^{k_j} \to \tilde{\mathbf{u}}$ weakly in $(L_{m+1} \cap H^{-1}(\Omega))^2$. Then also $\mathbf{u}^{k_j} \to \tilde{\mathbf{u}}$ weakly in $\mathbf{H}$, and therefore $\mathbf{u} = \tilde{\mathbf{u}}$. Moreover, we also obtain that $u_i^{k_j} \to \tilde{u}_i = u_i$ weakly in $L_{m+1}(\Omega)$ $(i = 1, 2)$, and therefore $\mathbf{u} = \tilde{\mathbf{u}} \in C_\alpha$.

Let J_h be the operator associated with (3.7). Let $u, g \in H^{-1}(\Omega)$. By standard minimization arguments from convex analysis we know that

$$u = J_h g = \operatorname*{arg\,min}_{v \in H^{-1}} \Big(\widetilde{\mathscr{E}}(v) + \frac{\|v - g\|_{H^{-1}}^2}{2h}\Big) \tag{3.8}$$

if and only if $v \mapsto \widetilde{\mathscr{E}}(v) + \frac{1}{2}\|v - g\|_{H^{-1}}^2/h$ is subdifferentiable at u and

$$0 \in \partial\Big(v \mapsto \widetilde{\mathscr{E}}(v) + \frac{\|v - g\|_{H^{-1}}^2}{2h}\Big)(u)$$

(cf. [5], Lemma 13.10), and this is equivalent to

$$g - u \in h\partial\widetilde{\mathscr{E}}(u) = \{-h\Delta u^{[m]}\},$$

by (3.3).

If $g \in L_{m+1} \cap H^{-1}(\Omega)$, then (3.8) shows that $u \in L_{m+1} \cap H^{-1}(\Omega)$ and $\|u\|_{m+1} \leqslant \|g\|_{m+1}$.

Now, let $\mathbf{g} = (g_1, g_2) \in C_\alpha$, $u_j := J_h g_j$ for $j = 1, 2$. Then

$$g_j - u_j = -h\Delta u_j^{[m]} \quad (j = 1, 2). \tag{3.9}$$

Moreover, $u_1^{[m]}$, $u_2^{[m]} \in H_0^1(\Omega) \cap L_{(m+1)/m}(\Omega)$ (cf. (1.7)), and thus

$$w := (u_1^{[m]} - u_2^{[m]})^+ = (u_1^{[m]} - u_2^{[m]})\mathbf{1}_{[u_1 > u_2]} \in H_0^1(\Omega) \cap L_{(m+1)/m}(\Omega)$$

and

$$\nabla w = (\nabla u_1^{[m]} - \nabla u_2^{[m]})\mathbf{1}_{[u_1 > u_2]}$$

(see e.g. [12], page 47). Using w as an (admissible!) test function in (3.9) we obtain that

$$\begin{aligned}\int_\Omega (u_1 - u_2) w \,\mathrm{d}x &= -h \int_\Omega (\nabla u_1^{[m]} - \nabla u_2^{[m]}) \cdot \nabla w \,\mathrm{d}x + \int_\Omega (g_1 - g_2) w \,\mathrm{d}x \\ &= -h \int_{[u_1 > u_2]} |\nabla u_1^{[m]} - \nabla u_2^{[m]}|^2 \,\mathrm{d}x - \int_\Omega (g_2 - g_1) w \,\mathrm{d}x \\ &\leqslant 0,\end{aligned}$$

since the integrands in the integrals are $\geqslant 0$. On the other hand, we have

$$\int_\Omega (u_1 - u_2) w \,\mathrm{d}x = \int_{[u_1 > u_2]} (u_1 - u_2)(u_1^{[m]} - u_2^{[m]}) \,\mathrm{d}x$$

and since the integrand is strictly positive we conclude that $[u_1 > u_2]$ is a set of measure zero, i.e. $u_1 \leqslant u_2$.

Thus we have shown that $(u_1, u_2) \in C_\alpha$, and this shows that $J_h \times J_h$ leaves the set C_α invariant.

Let now $\mathbf{u}_0 = (u_{1,0}, u_{2,0}) \in C_\alpha$ and $t > 0$, and let $u_1(t), u_2(t)$ be the solutions of the PME/FDE (3.1) with initial values $u_{1,0}, u_{2,0}$ at time t. Applying the exponential formula (3.4) we obtain that

$$(u_1(t), u_2(t)) = \lim_{k\to\infty} ((J_{t/k})^k u_{1,0}, (J_{t/k})^k u_{2,0}) \quad \text{in } \mathbf{H}.$$

Recalling that C_α is invariant under $J_h \times J_h$ and that C_α is closed we conlude that $(u_1(t), u_2(t)) \in C_\alpha$. □

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Authors' address: Samuel Littig, Jürgen Voigt, TU Dresden, Fachrichtung Mathematik, Helmholtzstrasse 10, D-01062 Dresden, Germany, e-mail: `samuel.littig@tu-dresden.de`, `juergen.voigt@tu-dresden.de`.