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Hypergeometric orthogonal systems of polynomials.

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INTRODUCTION.

The orthogonal systems of polynomials have become a very important tool of mathematical analysis. They play an essential part in the solution of a whole set of difficult mathematical problems as e. g. the expansion of arbitrary functions in series of polynomials, the solution of integral and sum equations etc. Orthogonal polynomials occupy an important position in numerous fields of applied mathematics. We may refer especially to their application in the theory of probability, in mathematical statistics connected with it, in the calculus of graduation, the theory of interpolation and numerical integration and summation.

It is generally known that every orthogonal system of polynomials corresponds to a certain function $\Phi(x)$, which we call characteristic function[1]) and further to a certain finite or infinite interval of integration or summation (α, β). The principal condition of orthogonality of the polynomials $P_\lambda(x)$ is expressed by the integral:

$$\int_\alpha^\beta P_n(x)\, P_m(x)\, \Phi(x)\, dx = 0, \quad n \neq m; \quad \int_\alpha^\beta P_n^2(x)\, \Phi(x)\, dx \neq 0$$

or by the sum:

$$\sum_\alpha^\beta \mathfrak{P}_n(x, s)\, \mathfrak{P}_m(x, s)\, \Phi(x, s)\, \omega = 0, \quad n \neq m; \quad \sum_\alpha^\beta \mathfrak{P}_n^2(x, s)\, \Phi(x, s)\, \omega \neq 0,$$

where s denotes the number of terms of summation, $(s - 1)\,\omega$ is the interval of summation. There is not hitherto any established special term which would distinguish the two groups of polynomials. For the sake of brevity, the systems of the first group will in this paper be called integral systems, those of the second group summation systems. In some cases the summation systems change into integral systems in the limit $s \to \infty$, $\omega \to 0$ and are therefore more general in this respect. If in such a case, a special term for the integral systems is already established, we shall

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1) Belegungsfunktion, Gewicht, poid.

add only the adjective „generalized", while referring to the respective summation systems.

To every orthogonal system of polynomials corresponds further a definite system of polynomials. which satisfies the same functional equation as the original polynomials. We shall use the term: polynomials of the second kind to express the difference between these polynomials and the original polynomials, which we call polynomials of the first kind.

A. M. Legendre was the first to introduce the simplest orthogonal system of polynomials into mathematical analysis while applying the theory of least squares advanced in his treatises „Recherche sur l'attraction des sphéroides homogènes" (1785) and „Recherches sur la figure des planètes" (1784). This is a system of polynomials corresponding to the characteristic function $\Phi(x) = \text{const.}$ and to a finite interval of integration (± 1), which we call Legendre polynomials or spherical functions. A considerable part of the other known integral systems was deduced from the problem of the Gaussian numerical integration (Gauss's mechanical quadrature) of the product of the given and the characteristic function.[2]

Because of the close connection with the subject of this paper let us refer here, among the numerous integral systems of polynomials occuring in mathematical analysis, only to the integral system of hypergeometric or Jacobi's polynomials. These polynomials of the characteristic function

$$\Phi(x) = (\alpha + x)^n \, (\beta - x)^m$$

and of a finite interval of integration are the source of a whole group of orthogonal systems corresponding either to a simple specialization of the parameters of the characteristic function or to a degeneration of this function in limiting cases. The best known are the polynomials corresponding in a finite interval to the function

$$\Phi(x) = 1 \quad \text{and} \quad \Phi(x) = \frac{1}{\sqrt{1 - x^2}}$$

further in an infinite interval to the functions

$$\Phi(x) = x^a \, e^{-x}, \quad \Phi(x) = x^{-a} \, e^{1/x}, \quad \varphi(x) = e^{-x^2}.$$

It may be remarked further that the principal properties of the integral systems of orthogonal polynomials have been very clearly summarized by N. Abramesco in his paper: „Résumé des principales propriétés des polynomes orthogonaux."[3] Analogous properties hold good also for summation systems.

Up to a very recent time, the summation systems of polynomials

[2]) See e. g. C. Runge — Fr. A. Willers, Numerische und graphische Quadratur, Enc. der math. Wiss. II C-2.

[3]) Nouvelles Annales de Mathématiques, 1923.

were, compared to the systems of integration, considerably neglected in mathematical research. Nevertheless the simplest system of this group of polynomials corresponding to the characteristic function $\Phi(x, s) = \text{const.}$ and to the finite interval of summation $[\pm \frac{1}{2}(s-1)\omega]$ was deduced already at the beginning of the second half of the 19th century by Tchebychef in his treatise „Sur les fractions continues"[4] (1855). These polynomials change into the integral system of the above mentioned Legendre polynomials in the limit $s \to \infty$, $\omega \to 0$. They acquired special importance in the numerical calculus of the theory of interpolation and approximation.

In his paper „Sur l'interpolation des valeurs équidistantes" (1875), Tchebychef discussed a very general summation system of orthogonal polynomials. These are polynomials of a finite interval of summation and of the characteristic function:

$$\Phi(x) = \frac{\Gamma(x+\alpha)\,\Gamma(m-x+\beta)}{\Gamma(x)\,\Gamma(m-x)}.$$

This paper is devoted to this system of polynomials and to the systems derived from it. As far as I know, nobody except Tchebychef has ever discussed them up till now. The term „Jacobi's generalized polynomials", which I am using, is derived from the fact that the above mentioned polynomials of Jacobi are the limiting case of this summation system of polynomials.

Hypergeometric or Jacobi's polynomials are, as it is well-known, a special case of a function very important in mathematical analysis, i. e. the Gauss hypergeometric series. Also Jacobi's generalized polynomials are in the same relation to the hypergeometric series of the third order, which was introduced into mathematical analysis by J. Thomae in his paper „Über die höheren hypergeometrischen Reihen".[5] This hypergeometric series of the third order is reduced in the limiting case into the above mentioned series of Gauss and fulfils the same function in the solution of the hypergeometric difference equation of the second order as the ordinary hypergeometric series in the solution of the Gauss differential equation.

A further important summation system of orthogonal polynomials was deduced by C. V. L. Charlier in his paper „Über die zweite Form des Fehlergesetzes".[6] These are polynomials of the characteristic function:

$$\Phi(x) = \frac{m^x e^{-m}}{x!}$$

and of the infinite interval of summation $(0, \infty)$. To this system accedes

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[4]) Oeuvres II/12.

[5]) Math. Annalen Bd II.

[6]) Arkiv för Mat., Astron. och Fysik II/15, 1905.

finally the system of the same interval of summation and of the characteristic function

$$\Phi(x) = \left[\frac{m^x e^{-m}}{x!}\right]^2,$$

the principal properties of which are listed in my article: „Application of Bessel coefficients in approximative expressing of collectives.“[7])

The generalized Legendre polynomials, the polynomials of Charlier and some other systems of summation, which — as we have explained above — generalize the group of hypergeometric integral polynomials can be deduced, as will be shown below, from the systems of Jacobi's generalized polynomials. For the definition of this summation system of polynomials contained in the 1st part of this paper, we shall use — analogously as Tchebychef does — an extension of the well-known expression of Jacobi's polynomials $J_\lambda(p, q, x)$ in the form:

$$J_\lambda(p, q, x) = \frac{x^{1-q}(1-x)^{q-p}}{(q, \lambda)} \frac{d}{dx}[x^{q+\lambda-1}(1-x)^{p-q+\lambda}],$$

if we substitute in principle the i-th power of the variable x, in this expression by the corresponding factorial product:

$$(x, i) = x(x+1)(x+2)\dots(x+i-1) = (x+i-1, -i) = \binom{x+i-1}{i} i!$$

$$(x, 0) = 1$$

From the principal properties of the system of polynomials under consideration we shall deduce besides the orthogonality investigated already by Tchebychef, their relation to the hypergeometric series of the third order, further their functional equation, the respective hypergeometric difference equation, the expression of the polynomials in form of a determinant and by continued fraction.

By application of Jacobi's generalized polynomials we shall deduce a very general definite approximative series and a series of interpolation respectively, of the form

$$\Phi(x)[a_0 P_0(x) + a_1 P_1(x) + \dots + a_n P_n(x)],$$

in which the coefficients a_i will be determined by the method of moments. This series — convenient especially for the approximative expression of frequency functions — is reduced into Charlier's series of the type A and B in limiting cases.

In part two, while investigating the properties of the characteristic function, we shall summarize, above all, the remarkable views of Professor G. Pólya on the deduction of the characteristic function from the concrete problem of the theory of probability and on the application of this function in mathematical statistics. At the same time we shall

[7]) Aktuárské vědy, I/1, Praha 1929.

refer to the close connection of the characteristic function with the frequency curves of K. Pearson.

The general form of the characteristic function will enable us to give the theoretical deduction of the approximative expression of this function on the base of a greater number of practical calculations in the manner proposed by Professor K. Pearson in his paper „On the method of ascertaining limits of the actual number of marked members in a population of a given size from a sample".[8])

In the following, some cases of the characteristic function corresponding to special values of the arbitrary constants which appear in it will be considered and especially the degeneration of the function in limiting cases will be investigated.

In part III we shall deduce the respective summation systems of orthogonal polynomials in special cases of the characteristic function and refer briefly to the above mentioned hypergeometric integral systems resulting from them. In the first place the above mentioned generalized polynomials of Legendre will be considered as the simplest case. A further special case are the generalized polynomials of Tchebychef which — as far as I know — have not been mentioned hitherto in the literature of the subject. They are connected with the integral system of Tchebychef's orthogonal polynomials of the characteristic function $1 : \sqrt{1 - x^2}$ and of the interval of integration ± 1. Another important limiting case of Jacobi's generalized polynomials form the polynomials, the characteristic function of which is the binomial frequency function. We might call them Hermite's generalized polynomials, for in the limit $\omega \to 0$ they are reduced into the well-known polynomials of Hermite. The polynomials called generalized polynomials of Laguerre or Kummer are a summation system of special importance. These polynomials are reduced for the limiting value of the variable parameter into the polynomials of Charlier, in another limiting case, in which at the same time $\omega \to 0$, we obtain the integral system of Laguerre's polynomials from them.

It is not the aim of this paper to consider in detail the applications of the orthogonal polynomials mentioned above in the different fields of applied mathematics. Nevertheless we shall not miss the opportunity to refer frequently to their practical application especially in concrete problems of mathematical statistics, numerical summation and integration etc.

As to mathematics I generally used the elementary method. The single deductions are given all in considerable detail. The reason for that is, in the first place, the circumstance that in this paper a greater stress is laid on the constantly growing penetration of the orthogonal systems into numerous parts of applied mathematics, in which only

[8]) Biometrika, Vol. XX A, 1928.

a general preliminary mathematical training is required, rather than to their great importance in pure mathematics. Nevertheless, especially in the first part of the paper a number of suggestions will be found which lead to interesting questions of purely theoretical value. We may cite as an example the reduction of the hypergeometric difference equation of the second order into the hypergeometric differential equation of Gauss.

PART I.

Definition and principal properties of Jacobi's generalized polynomials. Relation to the hypergeometric series of the third order. Orthogonality. Functional equation. Hypergeometric difference equation. Determinant expression. Expression by aid of continued fraction. Interpolation and approximation of functions known only in a finite number of equidistant values of the argument. Polynomials of the second kind. Application of polynomials in numerical summation. The integral system of polynomials deduced from Jacobi's generalized polynomials.

Definition and principal properties of Jacobi's generalized polynomials.

Let the function $\Phi_0(n, m, x)$ of a real variable x be defined in s equidistant values of the argument x in the points

$$-a = -\frac{s-1}{2}\omega,\ -a+\omega, \ldots, a-\omega,\ a = \frac{s-1}{2}\omega$$

by the expression

$$\Phi_0(n, m, x) = \frac{F_n\left(\frac{s-1}{2}\omega + n\omega + x\right) F_m\left(\frac{s-1}{2}\omega + m\omega - x\right)}{F_{n+m+1}(n+m+s\,\omega)} = \tag{1}$$

$$= \omega^n \begin{pmatrix} \frac{s-1}{2} + n + \frac{x}{\omega} \\ \frac{s-1}{2} + \frac{x}{\omega} \end{pmatrix} \omega^m \begin{pmatrix} \frac{s-1}{2} + m - \frac{x}{\omega} \\ \frac{s-1}{2} - \frac{x}{\omega} \end{pmatrix} : \omega^{n+m+1} \binom{n+m+s}{s-1}.$$

Let the parameters n, m satisfy the following inequalities

$$-1 < n,\quad m < -s-1$$

$$\frac{n+1}{m+1} > 0. \tag{2}$$

By applying the known relation

$$\binom{-x}{i} = (-)^i \binom{x+i-1}{i} \tag{3}$$

we can express $\Phi_0(n, m, x)$ also in the form

$$\Phi_0(n, m, x) = \binom{-\overline{n+1}}{\frac{s-1}{2} + \frac{x}{\omega}}\binom{-\overline{m+1}}{\frac{s-1}{2} - \frac{x}{\omega}} : \omega\binom{-\overline{n+m+2}}{s-1}. \qquad (1')$$

On the above mentioned suppositions about the parameters n, m, the function $\Phi_0(n, m, x)$ takes on in s chosen points of the interval $\pm\, a$ only positive values greater than zero, which is immediately evident from the expression (1) and (1′).

If n is equal to a positive integer c or to zero, the function $F_c[\frac{1}{2}(s - 1)\omega + c\omega + x]$ is evidently a polynomial of the degree c

$$F_c\left(\frac{s-1}{2}\omega + c\omega + x\right) = \frac{1}{(1, c)}\left(\frac{s-1}{2}\omega + c\omega + x\right)\left(\frac{s-3}{2}\omega + c\omega + x\right)\cdots$$

$$\cdots\left(\frac{s+1}{2}\omega + x\right) = \frac{1}{(1, c)}\left(\frac{s+1}{2}\omega + x,\ c/\omega\right).$$

By application of the gamma-function it is possible to define the function $\Phi_0(n, m, x)$ in all points of the interval $\pm\, a$ by:

$$\Phi_0(n, m, x) =$$

$$= \frac{\Gamma\left(\frac{s+1}{2} + n + \frac{x}{\omega}\right)\Gamma\left(\frac{s+1}{2} + m - \frac{x}{\omega}\right)}{\Gamma(n+1)\,\Gamma\left(\frac{s+1}{2} + \frac{x}{\omega}\right)\Gamma(m+1)\,\Gamma\left(\frac{s+1}{2} - \frac{x}{\omega}\right)} : \omega\frac{\Gamma(n+m+s+1)}{\Gamma(n+m+2)\,\Gamma(s)} \qquad (1'')$$

Function $\Phi_0(n, m, x)$ will be discussed in detail in part II of this paper.

Starting from the function

$$\Phi_\lambda(n, m, x) =$$

$$= \frac{F_{n+\lambda}\left(\frac{s-1}{2}\omega + n\omega + x\right)F_{m+\lambda}\left(\frac{s-1}{2}\omega + m\omega + \lambda\omega - x\right)}{F_{n+m+2\lambda+1}\left(\overline{n+m+\lambda+s\,\omega}\right)}, \qquad (4)$$

which is reduced into the function $\Phi_0(n, m, x)$ for $\lambda = 0$, we define now the function

$$\Pi_\lambda(n, m, x) = \frac{(m+1, \lambda)}{2^\lambda} F_{n+m+2\lambda+1}\left(\overline{n+m+\lambda+s\,\omega}\right)\underset{\omega}{\Delta^\lambda}\Phi_\lambda(n, m, x) \qquad (5)$$

and the Jacobi's generalized polynomials

$$\mathfrak{J}_\lambda(n, m, x) = \frac{\Pi_\lambda(n, m, x)}{\Phi_0(n, m, x)}\cdot\frac{1}{F_{n+m+1}\left(\overline{n+m+s\,\omega}\right)}. \qquad (6)$$

on the supposition, that $\lambda < s$ is a positive integer.

By applying the well-known formula of the difference calculus

$$\underset{\omega}{\Delta^{\lambda}}[\varphi(x)\psi(x)] = \sum_{i=0}^{\lambda}\binom{\lambda}{i}\underset{\omega}{\Delta^{i}}\varphi(x+\overline{\lambda-i}\,\omega)\underset{\omega}{\Delta^{\lambda-i}}\psi(x)$$

we obtain the following expression for $\Pi_\lambda(n, m, x)$

$$\Pi_\lambda(n, m, x) =$$

$$= \frac{(m+1,\lambda)}{2^\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}(-)^{\lambda-i}F_{n+\lambda-i}\left(\frac{s-1}{2}\omega + \overline{n+\lambda-i}\,\omega + x\right)\times$$

$$\times F_{m+i}\left(\frac{s-1}{2}\omega + \overline{m+i}\,\omega - x\right) = \frac{(m+1,\lambda)}{2^\lambda}\sum_{i=0}^{\lambda}\binom{\lambda}{i}(-)^{i}\times$$

$$\times F_{n+i}\left(\frac{s-1}{2}\omega + n\omega + x\right)F_{m+\lambda-i}\left(\frac{s-1}{2}\omega + m\omega - x\right).$$

With regard to the evident relations

$$F_{n+\lambda-i}\left(\frac{s-1}{2}\omega + \overline{n+\lambda-i}\,\omega + x\right) =$$

$$= \frac{\omega^{\lambda-i}}{(n+1,\lambda-i)}F_n\left(\frac{s-1}{2}\omega + n\omega + x\right)\cdot\left(\frac{s+1}{2} + n + \frac{x}{\omega}, \lambda - i\right),$$

$$F_{m+i}\left(\frac{s-1}{2}\omega + \overline{m+i}\,\omega - x\right) =$$

$$= \frac{\omega^{i}}{(m+1,i)}F_m\left(\frac{s-1}{2}\omega + m\omega - x\right)\cdot\left(\frac{s+1}{2} + m - \frac{x}{\omega}, i\right)$$

follows for the polynomials $\mathfrak{J}_\lambda(n, m, x)$ the value

$$\mathfrak{J}_\lambda(n, m, x) = \frac{(1,\lambda)\,\omega^\lambda}{2^\lambda\,(n+1,\lambda)}\sum_{i=0}^{\lambda}(-)^{\lambda-i}\binom{n+\lambda}{i}\binom{m+\lambda}{\lambda-i}\times$$

$$\times\left(\frac{s+1}{2} + n + \frac{x}{\omega}, \lambda - i\right)\cdot\left(\frac{s+1}{2} + m - \frac{x}{\omega}, i\right) =$$

$$= \frac{(1,\lambda)\,\omega^\lambda}{2^\lambda\,(n+1,\lambda)}\sum_{i=0}^{\lambda}(-)^{i}\binom{n+\lambda}{\lambda-i}\binom{m+\lambda}{i}\left(\frac{s+1}{2} - i + \frac{x}{\omega}, i\right)\cdot$$

$$\cdot\left(\frac{s+1}{2} - \lambda + i - \frac{x}{\omega}, \lambda - i\right). \tag{6'}$$

To simplify this expression we use the expansion of the polynomial $(x-a, i)$ in the series

$$(x-a,i)=\sum_{k=0}^{i}(-)^k\binom{i}{k}\,(x,i-k)\,(a-k+1,k)\ \text{ and}$$

$$(x-a,-i)=\sum_{k=0}(-)^k\binom{i}{k}(x,-i+k)\,(a,k),\qquad \text{respectively}$$

which can be easily deduced e. g. from the well-known Newton interpolation formula.

If we apply these relations to the polynomials

$$\left(\frac{s+1}{2}+m-\frac{x}{\omega},i\right)=(-)^i\left(\frac{x}{\omega}-m-\frac{s-1}{2}-i,i\right)=$$

$$=(-)^i\left(\frac{x}{\omega}+n+\frac{s+1}{2}+\lambda-i-\overline{s+n+m+\lambda},i\right),$$

$$\left(\frac{s+1}{2}-\lambda+i-\frac{x}{\omega},\lambda-i\right)=(-)^{\lambda-i}\left(\frac{x}{\omega}-\frac{s+1}{2}+\lambda-i,-\lambda+i\right)=$$

$$=(-)^{\lambda-i}\left(\frac{x}{\omega}+\frac{s-1}{2}-i-\overline{s-\lambda},-\lambda+i\right),$$

we obtain:

$$\left(\frac{s+1}{2}+m-\frac{x}{\omega},i\right)=(-)^i\sum_{k=0}^{i}(-)^k\binom{i}{k}\times$$

$$\times\left(\frac{x}{\omega}+n+\frac{s+1}{2}+\lambda-i,i-k\right)(s+n+m+\lambda-k+1,k),$$

$$\left(\frac{s+1}{2}-\lambda+i-\frac{x}{\omega},\lambda-i\right)=(-)^{\lambda-i}\sum_{k=0}^{\lambda-i}(-)^k\binom{\lambda-i}{k}\times$$

$$\times\left(\frac{x}{\omega}+\frac{s-1}{2}-i,-\lambda+i+k\right)(s-\lambda,k).\qquad(7)$$

We made the choice of values $s+n+m+\lambda$, and $s-\lambda$ with regard to the polynomial $\left(\frac{s+1}{2}+n+\frac{x}{\omega},\lambda-i\right)$ and $\left(\frac{s+1}{2}-i+\frac{x}{\omega},i\right)$ respectively in the formula (6′) in order to be able to express the product

$$\left(\frac{s+1}{2}+n+\frac{x}{\omega},\lambda-i\right)\left(\frac{x}{\omega}+n+\frac{s+1}{2}+\lambda-i,i-k\right)$$

and

$$\left(\frac{s+1}{2}-i+\frac{x}{\omega},i\right)\left(\frac{x}{\omega}+\frac{s-1}{2}-i,-\lambda+i+k\right)\ \text{respectively}$$

by a single symbol independent of i:

$$\left(\frac{s+1}{2}+n+\frac{x}{\omega}, \lambda-k\right) \text{ and } \left(\frac{x}{\omega}+\frac{s-1}{2}, -\lambda+k\right).$$

Inserting the value (7) in the expression (6′) of the polynomials $\mathfrak{J}_\lambda(n, m, x)$ we obtain

$$\begin{aligned}\mathfrak{J}_\lambda(n, m, x) &= \frac{(1, \lambda)(-)^\lambda}{2^\lambda (n+1, \lambda)} \omega^\lambda \sum_{k=0}^{\lambda} \sum_{i=k}^{\lambda} (-)^k \binom{i}{k} \binom{n+\lambda}{i} \binom{m+\lambda}{\lambda-i} \\ &\times \left(\frac{s+1}{2}+n+\frac{x}{\omega}, \lambda-k\right)(s+n+m+\lambda-k+1, k) = \\ &= \frac{(1, \lambda)(-)^\lambda}{2^\lambda(n+1, \lambda)} \omega^\lambda \sum_{k=0}^{\lambda} (-)^k \sum_{i=k}^{\lambda-k} \binom{\lambda-i}{k} \binom{n+\lambda}{\lambda-i} \binom{m+\lambda}{i} \times \\ &\times \left(\frac{x}{\omega}+\frac{s-1}{2}, -\lambda+k\right)(s-\lambda, k).\end{aligned} \tag{6''}$$

Using the known relations

$$\binom{a}{i}\binom{i}{b} = \binom{a}{i-b}\binom{a+b-i}{b}; \quad \sum_{i=0}^{\lambda} \binom{a}{i}\binom{b}{\lambda-i} = \binom{a+b}{\lambda}$$

we now carry out the summation

$$\sum_{i=k}^{\lambda} \binom{i}{k}\binom{n+\lambda}{i}\binom{m+\lambda}{\lambda-i} = \sum_{i=k}^{\lambda} \binom{n+\lambda}{k}\binom{n+\lambda-k}{i-k}\binom{m+\lambda}{\lambda-i} =$$

$$= \binom{n+\lambda}{k} \sum_{i=0}^{\lambda-k} \binom{n+\lambda-k}{i}\binom{m+\lambda}{\lambda-k+i} = \binom{n+\lambda}{k}\binom{n+m+2\lambda-k}{\lambda-k}$$

and for the sake of control also the summation

$$\sum_{i=0}^{\lambda-k} \binom{\lambda-i}{k}\binom{n+\lambda}{\lambda-i}\binom{m+\lambda}{i} = \binom{n+\lambda}{k} \sum_{i=0}^{\lambda-k} \binom{n+\lambda-k}{\lambda-i-k}\binom{m+\lambda}{i} =$$

$$= \binom{n+\lambda}{k}\binom{n+m+2\lambda-k}{\lambda-k}.$$

It is obvious that the two sums are identical. After a short further transformation, the expression for $\mathfrak{J}_\lambda(n, m, x)$ assumes the simple form

$$\left.\begin{aligned}
\mathfrak{J}_\lambda(n, m, x) &= \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(n+m+\lambda+1, \lambda-k)}{(n+1, \lambda-k)} \times \\
&\times \left(x + n\omega + \frac{s+1}{2}\omega, \overline{\lambda - k/\omega}\right)(\overline{s+n+m+\lambda-k+1\,\omega}, k/\omega) = \\
&= \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(n+m+\lambda+1, \lambda-k)}{(n+1, \lambda-k)} \times \\
&\times \left(x + \frac{s-1}{2}\omega, -\overline{\lambda - k/\omega}\right)(s - \lambda\omega, k/\omega) = \\
&= \frac{1}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \frac{(1, \lambda)}{(1, \lambda-k)} \frac{(n+m+\lambda+1, k)}{(n+1, k)} \times \\
&\times \binom{\frac{s-1}{2} + \frac{x}{\omega}}{k} (\overline{s - \lambda\,\omega}, \overline{\lambda - k/\omega})\, \omega^k.
\end{aligned}\right\} \quad (8)$$

For special value, e. g. $\lambda = 0, 1, 2$, there follow from this expression the following polynomials

$$\left.\begin{aligned}
&\mathfrak{J}_0(n, m, x) = 1 \\
&\mathfrak{J}_1(n, m, x) = -\frac{n+m+2}{2(n+1)} x + \frac{(n-m)(s-1)}{4(n+1)} \omega \\
&\mathfrak{J}_2(n, m, x) = \\
&\quad = \frac{(n+m+3)(n+m+4)}{4(n+1)(n+2)} \left(x + \frac{s-1}{2}\omega\right)\left(x + \frac{s-3}{2}\omega\right) - \\
&- \frac{2(n+m+3)}{4(n+1)}(s-2)\omega\left(x + \frac{s-1}{2}\omega\right) + \tfrac{1}{4}(s-2)(s-1)\omega^2.
\end{aligned}\right\} \quad (9)$$

Displacing the interval $\pm \frac{1}{2}(s-1)\omega$ so that the extreme value $-\frac{1}{2}(s-1)\omega$ coincides with the point $x = 0$, or substituting the variable x by the new variable

$$z = x + \frac{s-1}{2}\omega,$$

we obtain the function $\Phi_0(n, m, z)$ and the polynomials $\mathfrak{J}_\lambda(n, m, z)$ corresponding to the interval $(0, \overline{s-1\,\omega})$. We shall make use of this modification later for the investigation of some extreme cases of the function $\Phi_0(n, m, z)$ and of the polynomials $\mathfrak{J}_\lambda(n, m, z)$. With regard to (4) and (8) it is evident that

$$\begin{aligned}
\Phi_\lambda(n, m, z) = F_{n+\lambda}(n\omega + z)\, F_{m+\lambda}(\overline{m+\lambda+s-1\,\omega} - z) : \\
: F_{n+m+2\lambda+1}(\overline{n+m+\lambda+s\,\omega}).
\end{aligned} \quad (4')$$

$$\mathfrak{J}_\lambda(n, m, z) = \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(n+m+\lambda+1, \lambda-k)}{(n+1, \lambda-k)} \times$$
$$\times (\overline{z+n+1\,\omega, \lambda-k/\omega})\,(\overline{s+n+m+\lambda\,\omega, -k/\omega}) =$$
$$= \frac{(-)^\lambda}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \binom{\lambda}{k} \frac{(n+m+\lambda+1, \lambda-k)}{(n+1, \lambda-k)} \times$$
$$\times (z, -\lambda-k/\omega)\,(\overline{s-\lambda\,\omega, k/\omega}) = \tag{8'}$$
$$= \frac{1}{2^\lambda} \sum_{k=0}^{\lambda} (-)^k \frac{(1, \lambda)}{(1, \lambda-k)} \frac{(n+m+\lambda+1, k)}{(n+1, k)} \binom{\frac{z}{\omega}}{k} \times$$
$$\times (\overline{s-\lambda\,\omega, \lambda-k/\omega})\,\omega^k.$$

The relation to the hypergeometric series of the third order.

By a simple transformation of the expression (8′) it is possible to obtain an especially remarkable modification of the expression of polynomials $\mathfrak{J}_\lambda(n, m, z)$:

$$\mathfrak{J}_\lambda(n, m, z) = \left(\frac{\omega}{2}\right)^\lambda (s-1)(s-2)\dots(s-\lambda)\left[1 + \frac{(n+m+\lambda+1)(-\lambda)}{1\,(n+1)} \times \right.$$
$$\times \frac{\frac{z}{\omega}}{s-1} + \frac{(n+m+\lambda+1)(n+m+\lambda+2)}{1\,.\,2} \frac{(-\lambda)(-\lambda+1)}{(n+1)(n+2)} \times$$
$$\left. \times \frac{\left(\frac{z}{\omega}\right)\left(\frac{z}{\omega}-1\right)}{(s-1)(s-2)} + \dots\right] \tag{10}$$

and

$$\mathfrak{J}_\lambda(n, m, z) = \left(\frac{\omega}{2}\right)^\lambda (s+n+m+\lambda, -\lambda)\left[1 + \frac{n+m+\lambda+1}{n+1} \frac{-\lambda}{1} \times \right.$$
$$\times \frac{\frac{z}{\omega}+n+1}{s+n+m+1} + \frac{(n+m+\lambda+1)(n+m+\lambda+2)}{(n+1)(n+2)} \times \tag{10'}$$
$$\left. \times \frac{(-\lambda)(-\lambda+1)}{1\,.\,2} \frac{\left(\frac{z}{\omega}+n+1\right)\left(\frac{z}{\omega}+n+2\right)}{(s+n+m+1)(s+n+m+2)} + \dots\right]$$

respectively.

The series in the brackets shows a complete analogy to the expression of ordinary Jacobi's polynomials by a finite hypergeometric series.

Using the notation

$$n+m+\lambda+1=\alpha,\quad -\lambda=\beta,\quad n+1=\gamma,\quad 1-\frac{z}{\omega}=\xi,\qquad 1-s=\delta.$$

$$\frac{z}{\omega}+n=\eta,\; s+n+m+1=\delta',$$

we obtain in the brackets the series[9])

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}\,\frac{\xi}{\delta}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,.\,\gamma\,(\gamma+1)}\;\frac{\xi(\xi+1)}{\delta(\delta+1)}+\dots$$

and the series

$$1+\frac{\alpha\beta}{1\,.\,\gamma}\,\frac{\eta}{\delta'}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\,.\,2\,\gamma(\gamma+1)}\;\frac{\eta(\eta+1)}{\delta'(\delta'+1)}+\dots\ \text{respectively.}$$

Using Nörlund's[10]) notation of the hypergeometric series of the third order

$$F(\alpha,\beta,\gamma;\,a,b)=1+\frac{\alpha\,\beta\,\gamma}{1\,.\,a\,.\,b}+\frac{\alpha(\alpha+1)\,\beta(\beta+1)\,\gamma(\gamma+1)}{1\,.\,2\,a(a+1)\,b(b+1)}+\dots$$

we can express the series (10) and (10′) by the symbol

$$\frac{\Gamma(s)}{\Gamma(s-\lambda)}\left(\frac{\omega}{2}\right)^{\lambda}F\left(n+m+\lambda+1,-\lambda,-\frac{z}{\omega};\,1-s,\;n+1\right)$$

and

$$\left(\frac{\omega}{2}\right)\frac{\Gamma(s+n+m+\lambda+1)}{\Gamma(s+n+m+1)}\;F\left(n+m+\lambda+1,-\lambda,\frac{z}{\omega}+n+1;\right.$$
$$\left.n+1,\,s+n+m+1\right)\quad\text{respectively.}$$

---

9) J. Thomae first studied this series and the series of even more general form

$$1+\frac{aa'a''\dots a^{(n)}}{1\;\beta'\beta''\dots\beta^{(n)}}+\frac{a(a+1)\,a'(a'+1)\dots a^{(n)}(a^{(n)}+1)}{1\,.\,2\;\beta'(\beta'+1)\dots\beta^{(n)}(\beta^{(n)}+1)}+\dots$$

in the treatise: Ueber die höheren hypergeom. Reihen, insbesondere über die Reihe

$$1+\frac{a_0a_1a_2}{1b_1b_2}\,x+\frac{a_0(a_0+1)\,b_1(a_1+1)\,a_2(a_2+1)}{1\,.\,2\,b_1(b_1+1)\,b_2(b_2+1)}\;x^2+\dots$$

(Mathem. Annalen, Bd II 1870) and later in the treatises: Integration der Differenzengleichung

$$(n+\varkappa+1)\,(n+\lambda+1)\;\Delta^2\varphi(n)+(a+bn)\;\Delta\varphi(n)+c\varphi(n)=0$$

(Math.-Phys. Zeitschrift 16, 1871) and „Ueber die Funktionen, welche durch die Reihen

$$1+\frac{pp'p''}{1\,q'q''}+\dots$$

dargestellt werden (Journal f. reine u. angew. Math. 87, 1879). Thomae calls these series hypergeometric series of higher order.

10) Sur une classe de fonctions hypergéométriques, Oversigt danske Vidsk. Selskab Forhandl., 1913.

From the relation

$$F(\alpha, \beta, \gamma; a, b) =$$

$$= \frac{\Gamma(a)}{\Gamma(\gamma)\,\Gamma(a-\gamma)} \int_0^1 F(\alpha, \beta, b; y)\, y^{\gamma-1} (1-y)^{a-\gamma-1}\, dy$$

deduced in the papers of J. Thomae and in the above-quoted paper of Nörlund there results the new integral expression of the polynomials $\mathfrak{J}_\lambda(n, m\, z)$, in the form

$$\mathfrak{J}_\lambda(n, m, z) = \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma(s)}{\Gamma(s-\lambda)} \frac{\Gamma(n+1)}{\Gamma\left(-\frac{z}{\omega}\right)\Gamma\left(n+1+\frac{z}{\omega}\right)} \int_0^1 F(n+m+$$

$$+\lambda+1, -\lambda, 1-s; y)\, y^{\frac{-z}{\omega}-1} (1-y)^{n+\frac{z}{\omega}}\, dy, \tag{11}$$

$$= \left(\frac{\omega}{2}\right)^\lambda \frac{\Gamma(s+n+m+\lambda+1)}{\Gamma(s+n+m+1)} \frac{\Gamma(n+1)}{\Gamma\left(n+1+\frac{z}{\omega}\right)\Gamma\left(-\frac{z}{\omega}\right)} \times$$

$$\times \int_0^1 F(n+m+\lambda+1, -\lambda, n+1; y)\, y^{\frac{z}{\omega}+n} (1-y)^{\frac{-z}{\omega}-1}\, dy.$$

With regard to the admissibility of the permutations of the elements $\alpha, \beta, \gamma \ldots$ and the elements $a, b, c, \ldots$ in the series $F(\alpha, \beta, \gamma, \ldots a, b, c, \ldots)$ we can express the polynomials $\mathfrak{J}_\lambda(n, m, z)$ in several ways by analogous formulas.

We proceed now to the deduction of some important properties of the function $\Pi_\lambda(n, m, x)$ and of the polynomials $\mathfrak{J}_\lambda(n, m, x)$.

1. We shall prove first that $\Pi_\lambda(n, m, x)$ and $\mathfrak{J}_\lambda(n, m, x)$ form orthogonal systems with respect to the function $1 : \Phi_0(n, m, x)$ and $\Phi_0(n, m, x)$ respectively as characteristic functions. The respective conditions of orthogonality are expressed by the following relations

$$\sum_{-a}^{a} \Pi_\lambda \Pi_\mu \frac{\omega}{\Phi_0} = 0; \quad \sum_{-a}^{a} \mathfrak{J}_\lambda \mathfrak{J}_\mu \Phi_0 \omega = 0, \ \lambda \neq \mu. \tag{12}$$

To prove this property, we use the well-known summation formula

$$\sum_{\alpha} \varphi(x) \mathop{\Delta^\lambda}_{\omega} \psi(x)\, \omega = \left[\sum_{i=0}^{\lambda-1} (-)^i \mathop{\Delta^{\lambda-i-1}}_{\omega} \psi(x+i\omega) \mathop{\Delta^i}_{\omega} \varphi(x)\right]_\alpha^\beta \omega +$$

$$+ (-)^\lambda \sum_{\alpha}^{\beta} \psi(x+\lambda\omega) \mathop{\Delta^\lambda}_{\omega} \varphi(x)\, \omega \tag{13}$$

which results from the repeated application of the formula for partial summation. If $R(x)$ denotes an arbitrary function, we can express according to (13) the sum

$$\sum_{-a}^{a} \Im_\lambda(x)\,\Phi_0(x)\,R(x)\,\omega = \frac{(m+1,\lambda)\,F_{n+m+2\lambda+1}\,(\overline{n+m+s+\lambda\,\omega})}{2^\lambda\,F_{n+m+1}\,(\overline{n+m+s\,\omega})} \times$$

$$\times \sum_{-a}^{a} \underset{\omega}{\Delta^\lambda}\,\Phi_\lambda(x)\,R(x)\,\omega$$

in the form

$$a_\lambda \left[\sum_{i=0}^{\lambda-1} (-)^i \underset{\omega}{\Delta^{\lambda-i-1}}\,\Phi_\lambda(x+i\omega)\,\underset{\omega}{\Delta^i}\,R(x) \right]_{-a}^{a} \omega + (-)^\lambda a_\lambda \times$$

$$\times \sum_{-a}^{a} \Phi_\lambda(x+\lambda\omega)\,\underset{\omega}{\Delta^\lambda}\,R(x)\,\omega.$$

The expression in the brackets, however, is in the limits $\pm\,a$ equal to zero, because of

$$\underset{\omega}{\Delta^{\lambda-i-1}}\,\Phi(x+i\omega) = \sum_{r=0}^{\lambda-i-1} \binom{\lambda-i-1}{r} \underset{\omega}{\Delta^r}\,F_{n+\lambda}\left(x+\frac{s-1}{2}\,\omega + \right.$$

$$\left. + \overline{n+\lambda-r-1\omega}\right) \underset{\omega}{\Delta^{\lambda-r-i-1}}\,F_{m+\lambda}\left(\frac{s-1}{2}\,\omega + \overline{m+\lambda-i\,\omega} - x\right)$$

and the value under consideration is therefore given by the sum

$$\sum_{r,i} \gamma_{r,i}\,F_{n+\lambda-r}\left(x+\frac{s-1}{2}\,\omega + \overline{n+\lambda-r-1\,\omega}\right) F_{m+i+r+1}\left(\frac{s-1}{2}\,\omega + \right.$$

$$\left. + \overline{m+r+1\,\omega} - x\right) = \sum_{r,i} \gamma_{r,i} \begin{pmatrix} \dfrac{x}{\omega} + \dfrac{s-1}{2} + n+\lambda-r-1 \\ \dfrac{x}{\omega} + \dfrac{s-1}{2} - 1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \dfrac{s-1}{2} + m+r+1 - \dfrac{x}{\omega} \\ \dfrac{s-1}{2} - i - \dfrac{x}{\omega} - 1 \end{pmatrix} \omega^{n+m+\lambda+i+1},$$

which in the extreme points of the summation interval is equal to zero for all values of r, and i. There remains therefore for the original sum the expression

$$(-)^\lambda a_\lambda \sum_{-a}^{a} \Phi_\lambda(x+\lambda\omega)\,\underset{\omega}{\Delta^\lambda}\,R(x)\,\omega = \sum_{-a}^{a} \Im_\lambda(x)\,\Phi_0(x)\,R(x)\,\omega. \qquad (13')$$

If we choose e. g. a polynomial $p_i(x)$ of the i-th degree lower than λ for the function $R(x)$, this sum is evidently equal to zero, so that

$$\sum_{-a} \mathfrak{J}_\lambda(x)\, \Phi_0(x) p_i(x)\, \omega = 0 = \sum_{-a}^{a} \Pi_\lambda(x)\, p_i(x)\, \omega ; \quad 0 \leq i < \lambda.$$

If we then substitute an arbitrary polynomial $\mathfrak{J}_\mu(n, m, x)$ for the polynomial $p_i(x)$, we obtain the required condition of orthogonality

$$\sum_{-a}^{a} \mathfrak{J}_\lambda(x)\, \mathfrak{J}_\mu(x)\, \Phi_0(x)\, \omega = 0 = \sum_{-a}^{a} \Pi_\lambda(x)\, \Pi_\mu(x) \frac{\omega}{\Phi_0(x)}$$

This property is characteristic for the polynomials $\mathfrak{J}_\lambda(n, m, x)$ and is sufficient — disregarding any multiplicative constant — for the complete definition of these polynomials. We can be convinced of this by a simple consideration:

Let us suppose that there is another polynomial $\bar{\mathfrak{J}}_\lambda(x)$ of the degree λ of the same property.

Then also the polynomial

$$(a\, \mathfrak{J}_\lambda(x) - b\bar{\mathfrak{J}}_\lambda(x)); \quad a, b \lesseqgtr 0$$

would possess the same orthogonal property. If we choose a and b so, that the coefficient of the power x^λ vanishes, this expression will have the degree $\lambda - 1$. But from the expression

$$\sum_{-a}^{a} (a\, \mathfrak{J}_\lambda(x) - b\bar{\mathfrak{J}}_\lambda(x))^2\, \Phi_0(x)\, \omega =$$

$$= a \sum_{-a}^{a} \mathfrak{J}_\lambda(x)\, \Phi_0(x)\, [a\, \mathfrak{J}_\lambda(x) - b\bar{\mathfrak{J}}_\lambda(x)]\, \omega -$$

$$- b \sum_{-a}^{a} \bar{\mathfrak{J}}_\lambda(x)\, \Phi_0(x)\, [a\, \mathfrak{J}_\lambda(x) - b\, \bar{\mathfrak{J}}_\lambda(x)]\, \omega$$

follows that, on the given supposition, it must be

$$\sum_{-a}^{a} (a\mathfrak{J}_\lambda(x) - b\bar{\mathfrak{J}}_\lambda(x))^2\, \Phi_0(x)\, \omega = 0.$$

Because $\Phi_0(x)$ is in the interval of summation constantly positive and different from zero, there follows from the mean-value theorem that $(a\mathfrak{J}_\lambda(x) - b\,\bar{\mathfrak{J}}_\lambda(x))$ is equal to zero in the whole interval of summation, i. e. it must be

$$a\, \mathfrak{J}_\lambda(x) = b\bar{\mathfrak{J}}_\lambda(x)$$

If $\mu = \lambda$, the sum (12) presents a very important value, i. e. the value

$$I_\lambda = \sum_{-a}^{a} \mathfrak{J}_\lambda^2(x)\, \Phi_0(x)\, \omega$$

which occurs especially in the approximative expression of arbitrary functions by the series

$$\Phi_0(x)\,[a_0\,\mathfrak{J}_0(x) + a_1\,\mathfrak{J}_1(x) + \ldots]$$

For the calculation of the value under consideration let us use the relations:

$$\Delta_{\omega}^{\lambda} x^{\lambda} = (1, \lambda); \quad \Delta_{\omega}^{\lambda}\mathfrak{J}_{\lambda}(x) = \frac{(-)^{\lambda}\,(1, \lambda)\,(n + m + \lambda + 1, \lambda)}{2^{\lambda}\,(n + 1, \lambda)}$$

and insert them in the equation (13′), then

$$I_{\lambda} = \sum_{-a}^{a} \mathfrak{J}_{\lambda}^{2}(x)\,\Phi_0(x)\,\omega =$$

$$= \frac{(1, \lambda)\,(n + m + \lambda + 1, \lambda)\,(m + 1, \lambda)\,F_{n+m+2\lambda+1}(\overline{n + m + s + \lambda\,\omega})}{2^{2\lambda}\,(n + 1, \lambda)\,F_{n+m+1}(n + m + s\,\omega)} \times$$

$$\times \sum_{-a}^{a} \Phi_{\lambda}(x + \lambda\omega)\,\omega \tag{14}$$

We can reduce the upper limit of summation in the sum on the right side of this equation to $\alpha' = \frac{1}{2}\,\overline{s - 1}\,\omega - \lambda\omega$, for the function $\Phi_{\lambda}(x + \lambda\omega)$ is constantly equal to zero for arguments greater than α'.

The sum of the function $\Phi_{\lambda}(x + \lambda\omega)$ in the limits $-a, \alpha'$ can be written in the form

$$\frac{\omega^{n+m+2\lambda}}{F_{n+m+2\lambda+1}(\overline{n + m + \lambda + s\,\omega})} \sum_{-a}^{\alpha'} \binom{\frac{x}{\omega} + \frac{s-1}{2} + n + \lambda}{\frac{x}{\omega} + \frac{s-1}{2}} \binom{\frac{s-1}{2} + m - \frac{x}{\omega}}{\frac{s-1}{2} - \lambda - \frac{x}{\omega}}$$

The function after the summation sign is, according to (3), equal to the expression

$$(-)^{s-1-\lambda} \binom{-\overline{n + \lambda + 1}}{\frac{s-1}{2} + \frac{x}{\omega}} \binom{-\overline{m + \lambda + 1}}{\frac{s-1}{2} - \lambda - \frac{x}{\omega}}$$

and its sum in limits $-a = -\frac{1}{2}\,(s - 1)\,\omega$, $\alpha' = \frac{1}{2}\,(s + 1)\,\omega - \lambda\omega$ can be carried out by aid of the known relation

$$\sum_{i=0}^{k} \binom{a}{i}\binom{b}{k - i} = \binom{a + b}{k}; \quad \sum_{i=0}^{k} \binom{-a}{i}\binom{-b}{k - i} = \binom{-\overline{a + b}}{k} \tag{15}$$

Putting

$$\frac{x}{\omega} + \frac{s-1}{2} = i$$

we obtain the value of the sum

$$(-)^{s-1-\lambda} \sum_{-a}^{a'} \binom{-\overline{n+\lambda+1}}{\frac{s-1}{2} + \frac{x}{\omega}} \binom{-m\overline{+\lambda+1}}{\frac{s-1}{2} - \lambda - \frac{x}{\omega}} =$$

$$= (-)^{s-1-\lambda} \sum_{i=0}^{s-\lambda-1} \binom{-\overline{n+\lambda+1}}{i} \binom{-\overline{m+\lambda+1}}{s-\lambda-1-i} =$$

$$= (-)^{s-1-\lambda} \binom{-\overline{n+m+2\lambda+2}}{s-\lambda-1} = \binom{n+m+s+\lambda}{s-\lambda-1} =$$

$$= \frac{F_{n+m+2\lambda+1}(\overline{n+m+s+\lambda\,\omega})}{\omega^{n+m+2\lambda+1}}.$$

If we now insert this value in the equation (14) we obtain the required sum

$$I_\lambda = \sum_{-a}^{a} \mathfrak{J}_\lambda^2(x)\, \Phi_0(x)\, \omega =$$

$$= \frac{(1,\lambda)\,(m+1,\lambda)\,(n+m+\lambda+1,\lambda)}{2^{2\lambda}\,(n+1,\lambda)} \; \frac{F_{n+m+2\lambda+1}(\overline{n+m+s+\lambda\omega})}{F_{n+m+1}(\overline{n+m+s\omega})} \tag{14'}$$

By means of the reciprocal value of the root of this expression, the polynomials $\mathfrak{J}_\lambda(n, m, x)$ can be reduced to the normalized form $\mathring{\mathfrak{J}}_\lambda(n, m, x)$ satisfying the condition

$$\sum_{a}^{a} \mathring{\mathfrak{J}}^2{}_\lambda(x)\, \Phi_0(x)\, \omega = s\omega.$$

2. Functional equation of the polynomials $\mathfrak{J}_\lambda(n, m, x)$. An arbitrary polynomial of the degree r can be expressed as a sum of polynomials $\mathfrak{J}_i(x)$

$$\sum_{i=0}^{r} a_i \mathfrak{J}_i(x),$$

for the number of constants a_i is exactly the same as the number of given coefficients of powers of the variable x in polynomial under consideration, i. e. $\overline{r+1}$. E. g. for the polynomial $x\mathfrak{J}_\lambda(x)$ the relation

$$x\mathfrak{J}_\lambda(x) = a_{\lambda+1}\mathfrak{J}_{\lambda+1}(x) + a_\lambda \mathfrak{J}_\lambda(x) + \ldots + a_0 \mathfrak{J}_0(x) \tag{16}$$

holds good.

By successive multiplication of this equation with the product $\mathfrak{J}_{\lambda-\iota}(x) \cdot \Phi_0(x)$ and by carrying out the summation according to x in the limits $\pm a$ we see that

$$a_0 = a_1 = a_2 = \dots = a_{\lambda-2} = 0$$

We obtain the coefficient $a_{\lambda+1}$ by equating the coefficients of the highest power of the variable x:

$$\frac{(-)^\lambda (n+m+\lambda+1, \lambda)}{2^\lambda (n+1, \lambda)} = a_{\lambda+1} \frac{(-)^{1+\lambda}(n+m+\lambda+2, \lambda+1)}{2^{\lambda+1}(n+1, \lambda+1)},$$

hence the value

$$a_{\lambda+1} = -\frac{2(n+m+\lambda+1)(n+\lambda+1)}{(n+m+2\lambda+1)(n+m+2\lambda+2)}$$

results for $a_{\lambda+1}$.

The remaining constants a_λ, $a_{\lambda-1}$ can be calculated, if e. g. we substitute the following values for x

$$x_1 = -\frac{s-1}{2}\omega \quad \text{and} \quad x_2 = \frac{s-1}{2}\omega$$

in the equation (16).

Using the expression (6′) of the polynomials $\mathfrak{J}_\lambda(x)$ after the substitution we obtain the two required equations

$$\left(-a_\lambda - \frac{s-1}{2}\omega\right)\frac{\omega^\lambda}{2^\lambda}(s-\lambda, \lambda) = a_{\lambda+1}\frac{\omega^{\lambda+1}}{2^{\lambda+1}}(s-\lambda-1, \lambda+1) +$$
$$+ a_{\lambda-1}\frac{\omega^{\lambda-1}}{2^{\lambda-1}}(s-\lambda+1, \lambda-1),$$

$$(-)^\lambda\left(-a_\lambda + \frac{s-1}{2}\omega\right)\frac{\omega^\lambda}{2^\lambda}(s-\lambda, \lambda)\frac{(m+1, \lambda)}{(n+1, \lambda)} =$$
$$= a_{\lambda+1}\frac{(-)^{\lambda+1}\omega^{\lambda+1}}{2^{\lambda+1}}(s-\lambda-1, \lambda+1)\frac{(m+1, \lambda+1)}{(n+1, \lambda+1)} +$$
$$+ a_{\lambda-1}\frac{\omega^{\lambda-1}(-)^{\lambda-1}}{2^{\lambda-1}}(s-\lambda+1, \lambda-1)\frac{(m+1, \lambda-1)}{(n+1, \lambda-1)},$$

which after the reduction by common factors assume the simple form:

$$-\left(a_\lambda + \frac{s-1}{2}\omega\right)\frac{\omega}{2}(s-\lambda) = a_{\lambda+1}\tfrac{1}{4}(s-\lambda-1)(s-\lambda)\omega^2 + a_{\lambda-1}$$

$$-\left(-a_\lambda + \frac{s-1}{2}\omega\right)\frac{m+\lambda}{2(n+\lambda)}(s-\lambda)\omega =$$
$$+ a_{\lambda+1}\frac{(m+\lambda)(m+\lambda+1)}{4(n+\lambda)(n+\lambda+1)}(s-\lambda-1)(s-\lambda)\omega^2 + a_{\lambda-1}.$$

By an easy calculation we obtain the required coefficients

$$a_\lambda = \frac{(n+m)(s+1) - 2(\lambda+1)(n+m+\lambda)}{2(n+m+2\lambda)(n+m+2\lambda+2)} \cdot (n-m)\,\omega =$$

$$= \frac{(n+m)(s-2\lambda-1) - 2\lambda(\lambda+1)}{2(n+m+2\lambda)(n+m+2\lambda+2)}(n-m)\,\omega$$

$$a_{\lambda-1} = -\frac{(s+m+n+\lambda)(m+\lambda)\lambda(s-\lambda)}{2(n+m+2\lambda)(n+m+2\lambda+1)}\omega^2$$

(To be continued.)

Das jährliche mathematische Risiko der Versicherungen, bei welchen zwei von einander verschiedene Ereignisse die vorzeitige Auflösung herbeiführen können.

Von *Hans Koeppler*, Berlin.

Unterliegen die drei Wahrscheinlichkeiten p_1, p_2 und p_3 der Bedingung

$$p_1 + p_2 + p_3 = 1$$

und sind eine große Zahl s Beobachtungen angestellt worden, so besteht nach dem Satze von Bernoulli die Wahrscheinlichkeit

$$P(\sigma_1, \sigma_2) = \frac{1}{2\pi\sqrt{s^2 p_1 p_2 p_3}} e^{-\frac{1}{2sp_1p_2p_3}\left\{p_2(1-p_2)\sigma_1^2 + p_1(1-p_1)\sigma_2^2 + 2p_1p_2\sigma_1\sigma_2\right\}},$$

daß die Abweichung $\pm\sigma_1$ von der wahrscheinlichen Ereignißzahl sp_1 und die Abweichung $\pm\sigma_2$ von der wahrscheinlichen Ereignißzahl sp_2 stattfinden wird. Setzen wir

$$\frac{1-p_2}{2sp_1p_3} = a_{11}, \quad \frac{1-p_1}{2sp_2p_3} = a_{22} \quad \text{und} \quad \frac{1}{sp_3} = 2a_{12},$$

sowie

$$\frac{1}{4s^2p_1p_2p_3} = \frac{(1-p_1)(1-p_2)}{4s^2p_1p_2p_3^2} - \frac{1}{4s^2p_3^2} = a_{11}a_{22} - a_{12}^2 = \Delta,$$

so können wir der Wahrscheinlichkeit auch die bekannte allgemeine Form geben: