Ladislav Truksa

Application of Bessel coefficients in approximative expressing of collectives

Aktuárské vědy, Vol. 1 (1930), No. 1, 2–13

Persistent URL: http://dml.cz/dmlcz/144501

Na rozdíl od „Časopisu pro pěstování matematiky a fysiky" chceme pěstovati publikace, v nichž je stránka aplikace podstatná.

Domníváme se, že nový časopis bude moci doplniti tato důležitá naše publikační centra a zprostí je po případě článků, které pro svůj specielní význam jsou určeny pouze zlomku jich čtenářů a jinak je zatěžují.

Kromě úkolu, buditi zájem o tyto nové obory vědní v stále rostoucích řadách našich aktuárů, má sloužiti časopis také úkolu, seznamovati cizí odbornou veřejnost s naší vědeckou produkcí. Z tohoto důvodu bude časopis přinášeti články a pojednání v cizích jazycích. Stojíme na stanovisku, že lidé pracující u nás odborně, musí ovládati nejvýznamnější cizí jazyky a doufáme tudíž, že umožňujíce cizině dověděti se o naší produkci, nezabraňujeme zároveň domácím pracovníkům seznámiti se s prací našich badatelů. Chceme všímati si také bedlivě otázek, souvisejících s výchovou a vzděláním osob, provozujících prakticky u nás funkci aktuára. Chceme dále upozorniti mladší pracovníky našeho oboru, na jichž součinnost přímo apelujeme, na důležité knihy a pojednání, vyšlé v cizích jazycích.

Pokud se vnější stránky vydávání časopisu týče, odpovídá poslání Jednoty čsl. matematiků a fysiků, že vzala na sebe úkol vydávati časopis, jehož hlavním předmětem jsou aplikace matematiky na nové obory vědní. Za články časopisu, pokud jsou podepsány, odpovídají jejich autoři, jinak redakce.

Zveme k součinnosti všecky pracovníky oborů, pro něž je určen, a žádáme je, aby podporovali Časopis v prvé řadě tvořivou činností, při nejmenším však aspoň laskavým zájmem.

Dr. *E. Schoenbaum* za redakční kruh.

Application of Bessel coefficients in approximative expressing of collectives.

By, Dr. *L. Truksa.*

We meet very often in mathematical statistics with the application of method of moments and the method of least squares especially so in approximative expression of empiric frequency curves, of which we assume, that their form can be expressed by a known analytical expression including arbitrary constants. With respect to practical applicability the method of moments has the great advantage of simple way of calculating the arbitrary constants also in cases, where the use of method of least squares meets practically nearly insuperable obstacles, as in the case of determination of constants of certain Pearsons frequency curves. The application of both methods is equally easy in especial expression of frequency curves by means of series of orthogonal polynomials of type

$$y_x = \varphi_x (a_0 P_0(x) + a_1 P_1(x) + \ldots) \tag{1}$$

The polynomials $P_i(x)$ of degree i fulfil by method of moments the condition of orthogonality

$$\sum P_i(x) P_k(x)\, \varphi_x = 0 \text{ resp. } \int P_i(x)\, P_k(x)\, dx = 0,\ i \lessgtr k: \tag{2}$$

the coefficients a_i are given by relation

$$a_i = \frac{\Sigma y_x\, P_i(x)}{\Sigma\, P_i^2(x)\, \varphi_x} \text{ resp. } a_i = \frac{\int y_x P_i(x)\, dx}{\int P_i^2(x)\, \varphi_x\, dx} \tag{3}$$

If there is question of application of method of least squares the condition (2) must be replaced by relation

$$\sum P_i(x)\, P_k(x)\, \varphi_x^2 = 0 \text{ resp. } \int P_i(x)\, P_k(x)\, \varphi_x^2\, dx = 0,\ i \lessgtr k; \tag{2'}$$

the coefficients a_i by expression

$$a_i = \frac{\Sigma\, y_x\, P_i(x)\, \varphi_x}{\Sigma\, P_i^2(x)\, \varphi_x^2} \text{ resp. } a_i = \frac{\int y_x\, P_i(x)\, \varphi_x\, dx}{\int P_i^2(x)\, \varphi_x^2\, dx}. \tag{3'}$$

To the expansion of type (1) deduced by method of moments belongs also the well-known Charlier's series B[1])

$$y_x = \frac{e^{-m}\, m^x}{x!} (a_0 + a_1\, G_1\,(x) + a_2\, G_2(x) + \ldots) \tag{4}$$

expressing the so called arithmetical distribution of frequencies, which is a certain generalisation of Poisson's law of small chances,

$$\psi(m, x) = \frac{e^{-m}\, m^x}{x!} = \lim_{\substack{n \to \infty \\ np = m}} \binom{n}{x} p^x\, (1 - p)^{n-x}.$$

I am going to deduce in this article the expansion of type (1) of the characteristical function $\psi(m, x)$ of this series by application of method of least squares and to compare in concrete cases results following from this new expression on the one hand and from the original Charlier's series on the other hand. An analogical case is given by Ch. Jordan[2]) for the characteristical function $\sqrt{\psi(m, x)}$.

The calculation of system of polynomials $P_i(x)$ and coefficients a_i in our case is by far more difficult; it is necessary to use Bessel coefficients of immaginary argument.

I.

Let the frequency curve be defined by

$$y_x = \psi\,(m, x)\,[a_0\, K_0(m, x) + a_1\, K_1(m, x) + \ldots] \tag{5}$$

in limits $0, \infty$. For the determination of polynomials $K_i(m, x)$ of degree i and of coefficients a_i we use the principle of least squares

$$\sum_{x=0}^{\infty} \{y_x - \psi(m, x)\,[a_0\, K_0\,(m, x) + a_1\, K_1\,(m, x) + \ldots]\}^2 = \text{Min}$$

[1]) C. V. L. Charlier: Über die zweite Form des Fehlergesetzes (1905).
[2]) Ch. Jordan: Statistique mathématique (1927).

It is known from the theory of extrems of functions of more variables, that this condition is not fulfilled, unless the partial derivatives of the left - hand expression with respect to a_i are equal to zero, i. e.

$$\sum_{x=0}^{\infty}\{y_x - \psi(m, x)\,[a_0\, K_0(m, x) + a_1\, K_1\,(m, x) + \ldots]\}\, K_i(m, x)\, \psi\,(m, x) = 0 \quad (6)$$

where $i = 0, 1, 2, \ldots$ If the polynomials $K_n(m, x)$ fulfil the orthogonal relation

$$\sum_{x=0}^{\infty} K_r\,(m, x)\, K_s\,(m, x)\, \psi^2\,(m, x) = 0 \quad r \lessgtr s \qquad (7)$$

the equations (6) are reduced to the simple forme of

$$\sum_{x=0}^{\infty} \{y_x\, \psi(m, x)\;\; K_i(m, x)\} - a_i \sum_{x=0}^{\infty} K_i^2(m, x)\, \psi^2(m, x) = 0,$$

where from it follows

$$a_i = \frac{\sum\limits_{x=0}^{\infty} y_x\, K_i(m, x)\, \psi(m, x)}{\sum\limits_{x=0}^{\infty} K_i^2(m, x)\, \psi^2(m, x)}. \qquad (8)$$

Let us introduce the following denotation of i-th-moment of function $\psi^2(m, x)$

$$M_i = \sum_{x=0}^{\infty} x^i\, \psi^2(m, x)\,.$$

The polynomial $K_i(m, x)$ which fulfills the condition of orthogonality (7) can be expressed apart from a constant factor by the determinant

$$D_n(m, x) = \begin{vmatrix} 1 & M_0 & M_1 & M_{n-1} \\ x & M_1 & M_2 & M_n \\ x^2 & M_2 & . & . \\ . & . & . & . \\ x^n & M_n & M_{n+1} & M_{2n-1} \end{vmatrix} \qquad (9)$$

what can be proved in an easy way. Let us multiply both sides of the equation (9) by $x^i\, \psi^2(m, x)$ and carry out the summation with respect to x in the limit from 0 to ∞. We obtain the values of the moments $M_i, M_{i+1}, \ldots M_{i+n}$ in the first column of the determinant. Consequently, two columns of the determinant are identical — as far as $i < n$ — and the value of the determinant is equal to zero. The same holds, when we multiply by the factor $R_i(x)\, \psi^2(m, x)$ [$R_i(x)$ being an arbitrary polynomial of degree i] viz.

$$\sum_{x=0}^{\infty} R_i(x)\, D_n(m, x)\, \psi^2(m, x) = 0, \quad i < n.$$

By replacing the polynomial $R_i(x)$ by $D_i(m, x)$ of the same degree the following orthogonal relation

$$\sum_{x=0}^{\infty} D_i(m, x)\, D_n(m, x)\, \psi^2(m, x) = 0 \quad i \leqq n$$

is obtained. It is identical with the relation (7) which defines the polynomials $K_i(m, x)$.

We obtain from the fundamental definition of Bessel coefficients $I_n(x)$ of immaginary argument[3])

$$I_n(x) = i^{-n} J_n(ix) = i^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{1}{2} i x\right)^{n+2r}}{r!\,(n+r)!} = \sum_{x=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!\,(n+r)!}$$

for the first two moments M_0, M_1 the expressions

$$M_0 = e^{-2m} \sum_{x=0}^{\infty} \frac{m^{2x}}{(x!)^2} = J_0(2im)\, e^{-2m} = e^{-2m} I_0\,(2m),$$
$$(10)$$
$$M_1 = e^{-2m} \sum_{x=0}^{\infty} x\, \frac{m^{2x}}{(x!)^2} = -\,ime^{-2m} J_s(2\,im) = me^{-2m}\, I_1(2\,m).$$

By using the well known property of Bessel coefficients

$$\frac{d}{dx}\{x^n I_n(x)\} = x^n I_{n-1}(x), \quad \frac{d}{dx} I_0(x) = I_1(x)$$

and relation

$$M_{i+1} = \frac{m}{2} \frac{d}{dm} M_i + M_i\, m$$

all the higher moments can be expressed by moments M_0 and M_1.

Let some initial values be given here

$$\begin{aligned}
M_2 &= m^2 e^{-2m} I_0\,(2\,m) = m^2\, M_0 \\
M_3 &= e^{-2m} \left[(m^2\, I_0(2\,m) + m^3\, I_1(2\,m)\right] = m^2\,(M_0 + M_1) \qquad (11)\\
M_4 &= e^{-2m} \left[(m^4 + m^2)\, I_0(2\,m) + 2\,m^3\, I_1(2\,m)\right] = M_3 + m^2\,(M_2 + M_1). \\
M_5 &= e^{-2m} \left[(4\,m^4 + m^2)\, I_0(2m) + (m^5 + 3\,m^3)\, I_1(2m)\right] = M_4 + \\
&\quad + m^2\,(M_3 + 2\,M_2 + M_1) \\
M_6 &= e^{-2m} \left[(m^6 + 11\,m^4 + m^2)\, I_0(2m) + (6\,m^5 + 4\,m^3)\, I_1(2\,m)\right] = \\
&= M_5 + m^2\,(M_4 + 3\,M_3 + 3\,M_2 + M_1).
\end{aligned}$$

If we now define the polynomial $K_n(m, x)$ as follows

$$K_n(m\; x) = \frac{(-1)^n}{D_n} D_n(m, x) \qquad (12)$$

where D_n is equal to the value of the subdeterminant appertaining to the element x^n in formula (9), the following values of the initial members of the system of polynomials result in consequence of (10) and (11)

---

[3]) See f. i. Whittaker and Watson, A course of Modern Analysis, 1920.

$$K_0(m, x) = 1,$$

$$K_1(m, x) = x - m\,\frac{I_1(2m)}{I_0(2m)} = x - A_1, \qquad (13)$$

$$K_2(m, x) = x^2 - x\,\frac{I_0^2\,(2m)}{I_0^2(2m) - I_1^2(2m)} + m\,\frac{I_0(2m)\,I_1(2m)}{I_0^2(2m) - I_1^2(2m)} - m^2$$

$$= x^2 - B_2 x + A_2$$

$$K_3\,(m, x) = x^3 - C_3 x^2 + B_3 x - A_3.$$

If we denote by

$$j_3 = 2m^5\,I_1(2m)\,[I_0^2(2m) - I_1^2(2m)] - m^4\,I_0(2m)\,I_1^2(2m)$$

the common denominator of the coefficients A_3, B_3, C_3, then these coefficients can be expressed as follows

$$A_3 = \{-2m^8\,I_0(2m)\,[I_0^2(2m) - I_1^2(2m)] + m^7\,I_1(2m)\,[2I_0^2(2m) + - I_1^2(2m)]\} : j_3$$

$$B_3 = \{m^6\,I_0(2m)\,[2I_0^2(2m) - I_1^2(2m)] - 2m^7\,I_1(2m)\,[I_0^2(2m) + - I_1^2(2m)]\} : j_3$$

$$C_3 = \{2m^6\,I_0(2m)\,[I_0^2(2m) - I_1^2(2m)] + m^5\,I_1(2m)\,[2I_0^2(2m) + - 3I_1^2(2m)] - m^4\,I_0(2m)\,I_1^2(2m)\} : j_3.$$

Using the known values of A_1, A_2, we can reduce by a short calculation A_3, B_3, C_3 to the following simple form, which is suitable for the numerical process

$$A_3 = \frac{m^4}{m^2 - A_2} - \frac{m^4}{A_1}$$

$$B_3 = \frac{2m^4}{A_1(m^2 - A_2)} - m^2$$

$$C_3 = 1 + \frac{m^2}{A_1} + \frac{m^2}{m^2 - A_2}.$$

A control of these values — if we suppose the values of M_i in the formulas (11) being correct — may be carried out by calculating the sum

$$\sum_{x=0}^{\infty} x^i K_3(m, x)\,\psi^2(m, x),$$

which for $i = 1, 2, 3$, must be equal to zero. We obtain in this way f. i. for $i = 0$

$$M_3 - M_2 - \frac{m^2 M_2}{A_1} - \frac{m^2 M_2}{m^2 - A_2} + \frac{2m^4 M_1}{A_1(m^2 - A_2)} - m^2 M_1 + - \frac{m^2 M_2}{m^2 - A_2} + \frac{m^4 M_0}{A_1} = m^2 M_1 - m^2 M_1 - \frac{m^4 M_0}{A_1} + + \frac{m^4 M_0}{A_1} - \frac{2m^2 M_2}{m^2 - A_2} + \frac{2m^2 M_2}{m^2 - A_2} = 0$$

Leaving aside the calculation of polynomials of higher degree and examination of the common recurrent relation for the three adjoining polynomials, because for practical use the first three polynomials are mostly sufficient, let us determine immediately the coefficients

$$a_i = \frac{\alpha_i}{\beta_i} = \frac{\sum\limits_{x=0}^{\infty} y_x\, K_i(m, x)\, \psi(m, x)}{\sum\limits_{x=0}^{\infty} K_i^2(m,x)\ \ \psi^2(m, x)}\,.$$

The denominator β_i can be obtained
f. i. from the relation

$$\beta_i = \sum_{x=0}^{\infty} x^i K_i(m, x)\, \psi^2(m, x)$$

or by directly substituting the values (12) into the expression

$$\beta_i = \sum_{x=0}^{\infty} K_i^2(m, x)\, \psi^2(m, x),$$

from which it is easy to deduce

$$\beta_i = \frac{D_{i+1}}{D_i}.$$

The first five values of the determinants D_i are:

$$D_0 = 1$$
$$D_1 = e^{-2m}\, I_0(2m)$$
$$D_2 = e^{-4m}\, m^2\, [I_0^2(2m) - I_1^2(2m)]$$
$$D_3 = e^{-6m}\, m^4 \{2m\, [I_0^2(2m) - I_1^2(2m)] - I_0(2m)\, I_1(2m)\}$$
$$D_4 = e^{-8m}\, m^8 \{[8m\, I_0(2m)\, I_1(2m) + 4m^2\, I_1^2(2m) - 4m^2\, I_0^2(2m) + - I_1^2(2m)]\, [I_0^2(2m) - I_1^2(2m)] - 3I_0^2(2m)\, I_1^2(2m)\}$$

By forming the concerned ratios $D_{i+1} : D_i$ and using some of the coefficients A, B, deduced above, we get

$$\beta_0 = M_0$$
$$\beta_1 = \frac{M_2}{B_2} \qquad (14)$$
$$\beta_2 = M_1(m^2 - A_2)$$
$$\beta_3 = m^2 M_1 \left(2m^2 - \frac{m^2}{m^2 - A_2}\right) + m^2 M_2 \left(3 - \frac{2m^2}{A_1}\right)$$

To the end of calculating of numerators α_i let us form the product

$$Y_x = y_x\ \psi(m, x)$$

and let us denote the moments of this function about the origin of coordinates

$$\mu_k = \sum_{x=0}^{\infty} Y_x\, x^k.$$

Then it is evident:

$$\alpha_0 = \mu_0, \ \alpha_1 = \mu_1 - \mu_0 A_1, \ \alpha_2 = \mu_2 - \mu_1 B_2 + \mu_0 A_2. \qquad (15)$$
$$\alpha_3 = \mu_3 - \mu_2 C_3 + \mu_1 B_3 - \mu_0 A_3$$

Consequently after having used the calculated values the expansion (5) gets the form

$$y_x = \psi(m, x)\left[\frac{\mu_0}{M_0} + (x - A_1) B_2 \frac{\mu_1 - \mu_0 A_1}{M_2} + \ldots\right]. \qquad (16)$$

We have yet to choose the parameter m. In Charlier's series m is chosen as follows

$$m = \frac{\sum_{x=0}^{\infty} x y_x}{\sum_{x=0}^{\infty} y_x},$$

hence it follows that the arithmetical means of functions $\psi(m, x)$ and y_x are identical and $a_1 = 0$. By an analogical choice of m in our case

$$m = \frac{\sum_{x=0}^{\infty} x\, y_x\, \psi(m, x)}{\sum_{x=0}^{\infty} y_x\, \psi(m, x)}, \qquad (17)$$

as suggests Ch. Jordan in the above-mentioned expansion for the characteristic function $\sqrt{\psi(m, x)}$, we could achieve the same simplification, but even approximative determination of m by this method is very difficult, because the unknown m appears also in summation on the right-hand side of the equation (17). Further if m is to be determined more precisely then to one decimal, it is impossible to use with advantage the tables of values $\psi(m, x)$, which are tabulated in regular intervals 0.1[4]). With regard to the already said, it appears expedient to put for m the same value as in the Charlier's series, viz.

$$m = \frac{\sum_{x=0}^{\infty} x y_x}{\sum_{x=0}^{\infty} y_x}. \qquad (18)$$

To expedite the numerical application of the formula (16) we tabulate the coefficients β_i and constants $A, B, \ldots$ which appear in polynomials $K_n(m, x)$ for different values of m.

The constants A_1, A_2, B_2, β_0, β_1, β_2 in the following table 1. were calculated for $m = 0.5$, 1, 1.5, ... 5.5 by using the values of $I_0(2m)$,

[4]) See f. i. K. Pearson, Tables for Statisticians and Biometricians, 1924.

$I_1(2m)$[5]) published in „Proceedings of the Royal Society, Vol. L XIV, 1898, London. The constants belonging to higher values of m could be determined f. i. by use of tables: E. Anding, Sechsstellige Tafeln der Bessel'schen Funktionen imaginären Argumentes, 1911. It is necessary however to limit the calculation to a relatively small number of decimals as the ratio $I_1(2m) : I_0(2m)$ which determine the accuracy of the result, is given only to 6 decimals. In the second part of this article these tables were also used for determination of the necessary constants in a concrete case.

Tab. 1.

m	A_1	A_2	B_2	β_0	β_1	β_2	m
0.5	0.2231 950	0.0287 373	1.2488 511	0.4657 596	0.0932 376	0.0230 014	0.5
1	0.6977 747	0.3598 915	1.9488 978	0.3085 083	0.1582 989	0.1377 957	1
1.5	1.2149 779	1.2826 949	2.9076 206	0.2430 004	0.1880 406	0.2855 872	1.5
2	1.7270 452	2.7906 029	3.9319 196	0.2070 019	0.2105 861	0.4323 615	2
2.5	2.2334 578	4.8140 303	4.9537 672	0.1835 408	0.2315 672	0.5886 480	2.5
3	2.7370 779	7.3309 651	5.9665 693	0.1666 574	0.2513 868	0.7613 376	3
3.5	3.2393 627	10.3412 537	6.9739 809	0.1537 377	0.2700 448	0.9505 792	3.5
4	3.7409 420	13.8476 464	7.9786 446	0.1434 318	0.2876 314	1.1548 883	4
4.5	4.2421 047	17.8519 239	8.9818 444	0.1349 595	0.3042 727	1.3729 284	4.5
5	4.7429 991	22.3549 951	9.9841 880	0.1278 333	0.3200 895	1.6037 019	5
5.5	5.2437 098	27.3573 205	10.9859 857	0.1217 302	0.3351 850	1.8464 484	5.5

II.

First of all we make a numerical application of the formula (16), under the supposition that the observed values of the frequencies P_x are identical with the theoretical probability, that a certain event of the probability p will occure x-times in n trials

$$P_x = \binom{n}{x} p^x (1 - p)^{n-x}.$$

For this purpose we choose an unfavourable case of a small number of trials ($n = 19$) and the probability $p = \frac{1}{2}$. Let us observe that for sufficiently large n and small p Ch. Jordan obtained very good result by means of the Charlier's series. We deduce the parameter m from the formula (18)

$$m = \sum_{x=0}^{\infty} x P_x = 9{\cdot}5.$$

The further necessary constants corresponding to the values of the function P_x are

---

[5]) W. Steadman Aldis, Tables for the Solution of the Equation $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right) y = 0.$

$\mu_0 = 0.10547$ $\mu_1 = 0.99004$ $\mu_2 = 9.63274$

$A_1 = 9.24652$ $A_2 = 85.365$ $B_2 = 18.9926$

$\alpha_0 = 0.10547$ $\alpha_1 = 0.0148$ $\alpha_2 = -0.1672$

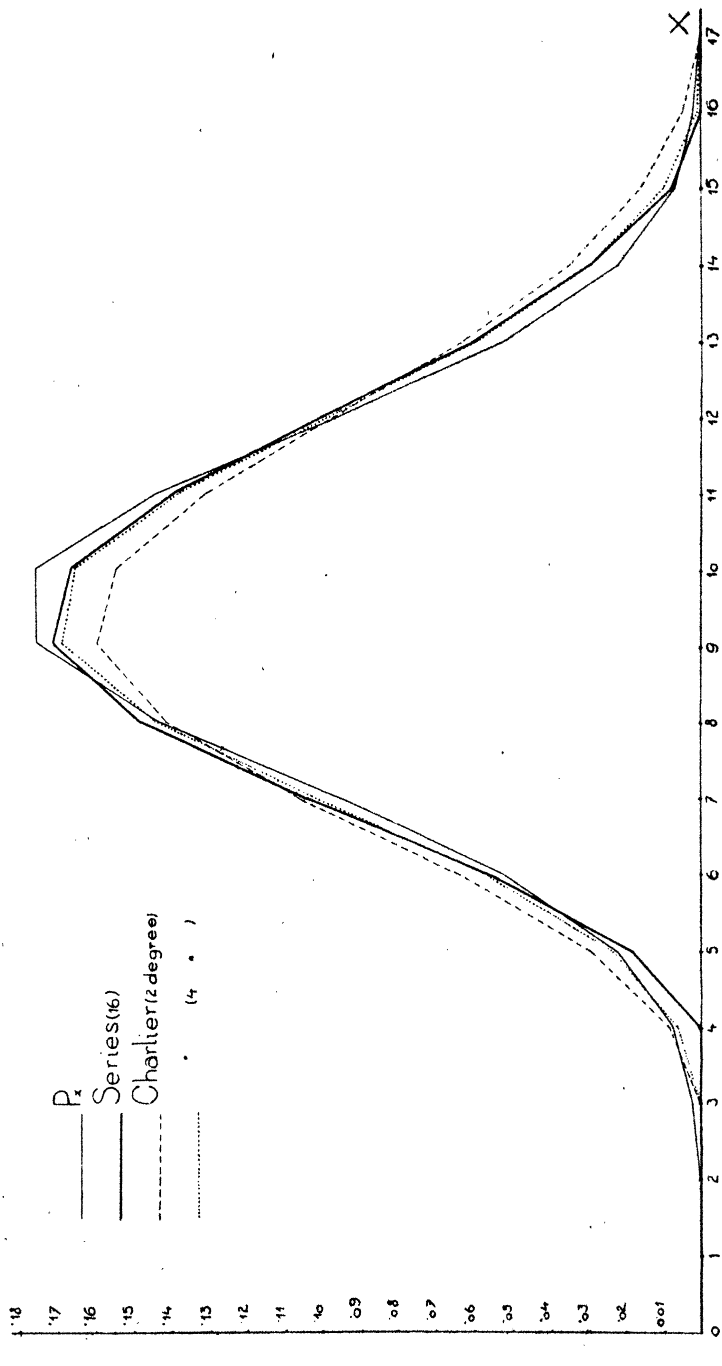
$\beta_0 = 0.0921446$ $\beta_1 = 0.438$ $\beta_2 = 4.162$

$a_0 = 1.1446$ $a_1 = 0.0338$ $a_2 = -0.0402$

In the column (2) of the table 2. there are given the precise values of P_x, in the column 3 the values of the product $P_x \psi(9.5, x)$. The approximative values of the function P_x following from the Charlier's series by leaving out the polynomials of higher degree than 2. are given in the column 5, the analogical values from the formula (16) in the column 4. By neglecting the polynomials of higher degree than 3 or 4 respectively in the series of Charlier we obtain the approximative values P_x which are contained in the columns 6 and 7 respectively. The negative values which occur in the extreme approximative values were replaced by zero in the graphic representation (p. 11.). It is seen from the graphic representation, that in the given case the expression by means of the method of least squares describes more exactly the values P_x than that by means of the method of moments, if we retain the terms of the same degree (the second) in both developments. We have to take four terms of the Charlier's series if we will obtain, the same expres-

Tab. 2.

x	P_x	$P_x \psi(9{\cdot}5, x)$	Series (16) incl. polynomial of the 2. degree	Charlier's series incl. polynomial of the 2. degree	Charlier's series incl. polynomial of the 3. degree	Charlier's series incl. polynomial of the 4. degree	x
1	2	3	4	5	6	7	8
0	0.000 00	0.000 00	− 0.000	− 0.000	− 0.000	+ 0.000	0
1	04	000 00	− 0.001	− 001	− 001	+ 000	1
2	33	2	− 004	− 001	− 003	− 000	2
3	1 85	19	− 006	− 000	− 002	000	3
4	7 39	1 07	− 001	008	006	006	4
5	22 18	3 95	018	029	028	023	5
6	51 75	9 97	056	064	066	057	6
7	96 11	17 75	105	106	111	102	7
8	144 16	22 91	149	142	147	145	8
9	176 20	21 76	172	160	164	169	9
10	176 20	15 38	167	155	156	166	10
11	144 16	8 12	139	131	129	138	11
12	96 11	3 19	100	097	093	098	12
13	51 75	93	060	063	059	059	13
14	22 18	20	029	035	032	029	14
15	7 39	3	008	016	015	010	15
16	1 85	0	− 002	005	005	001	16
17	33	0	− 006	− 000	000	− 002	17
18	4	0	− 006	− 002	− 001	− 002	18
19	000 00	000 00	− 005	− 001	− 001	− 001	19


P_x
Series(16)
Charlier(2 degree)
" (4 ")
X
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
.18 .17 .16 .15 .14 .13 .12 .11 .10 .09 .08 .07 .06 .05 .04 .03 .02 .001

sion as from the series (16) where the term of the second degree is the highest respected.

On the other hand the calculation of the values of the Charlier's series is by far more simple than of those of the series (16), as the values of $G_i(m, x)\,\psi(m, x)$ can be obtained by forming simply the n-th differences of the function $\psi(m, x)$ and the product $P_x\,\psi(m, x)$ does not appear any more.

If the frequency curve approaches nearlier to the characteristic value $\psi(m, x)$, then of course, the difference in the results by the method of moments and least squares is not too evident. As illustration of that we give results of approximative expression of P_x, if $n = 32$, $p = \frac{1}{8}$, $m = 4$.

Tab. 3.

x	P_x	$P_x\,\psi(4, x)$	Series (16) incl. polynomial of the 2. degree	Charlier's series incl. polynomial of the			x
				2. degree	3. degree	4. degree	
1	2	3	4	5	6	7	8
0	0.0139 4	0.0002 6	0.0133	0.0137	0.0134	0.0139	0
1	0637 2	46 7	633	641	637	637	1
2	1411 1	206 8	1415	1419	1423	1413	2
3	2015 8	393 8	2018	2015	2024	2016	3
4	2087 8	407 9	2085	2076	2081	2086	4
5	1670 2	261 0	1666	1661	1659	1669	5
6	1073 7	111 9	1075	1075	1069	1074	6
7	0569 7	33 9	573	577	573	571	7
8	0254 3	7 6	256	260	259	255	8
9	0096 9	1 3	96	99	100	97	9
10	0031 8	2	29	31	32	32	10
11	0009 1		7	8	9	9	11
12	0002 3		1	1	2	2	12
13	0000 5						13
14	0000 1						14

From the table 3. it is seen that the expression by means of series (16) stopped in the term of 2. degree is better than that by means of the Charlier's series with 2 or 3 terms, but it is not so good if we add the fourth term to the latter.

The corresponding constants of the formula (16) are

$\mu_0 = 0.14737 \qquad \mu_1 = 0.55615 \qquad \mu_2 = 2.37707$

$A_1 = 3.7409 \qquad B_2 = 7.9786 \qquad A_2 = 13.8476$

$\alpha_0 = 0.14737 \qquad \alpha_1 = 0.00485 \qquad \alpha_2 = -0.01951$

$\beta_0 = 0.1434 \qquad \beta_1 = 0.2876 \qquad \beta_2 = 1.1549$

$a_0 = 1.0277 \qquad a_1 = 0.01686 \qquad a_2 = -0.01689$

As a further example we give in the table 4. the expression of the function y_x, which indicates the number of the stormy days in the month of August in the years 1753—1857 in Lund[1]) by means of the series (16). The constants corresponding to the series are

$m = 2,\ \mu_0 = 18.869$	$\mu_1 = 29.098$	$\mu_2 = 69.618$
$A_1 = 1.727$	$A_2 = 2.791$	$B_2 = 3.932$
$\alpha_0 = 18.869$	$\alpha_1 = -3.489$	$\alpha_2 = 7.868$
$\beta_0 = 0.207$	$\beta_1 = 0.211$	$\beta_2 = 0.432$
$a_0 = 91.15$	$a_1 = -16.53$	$a_2 = 18.21$

The values in the column 5 given by Charlier on the page 30 of his mentioned book were calculated for $m = 2.133$. The expression by means of the series (16) is better fitting than that of Charlier.

Tab. 4.

x	y_x	$y_x\,\psi(2, x)$	Series (16) incl. polynomial of the 2. degree	Charlier's series incl. polynomial of the 2. degree	x
1	2	3	4	5	6
0	24	3.248	23.1	24.8	0
1	26	7.037	27.2	28.2	1
2	19	5.143	18.2	15.9	2
3	13	2.346	12.6	10.2	3
4	9	0.812	10.1	9.5	4
5	6	0.216	6.7	7.9	5
6	5	0.060	3.6	4.6	6
7	2	0.007	1.5	2.5	7
8	0	0.000	0.5	0.7	8
9	0	0.000	0.2	0.3	9
10	0	0.000	0.0	0.1	10
11	1	0.000	0.0	0.0	11

Über die Bedeutung und Anwendung der Nomogramme in der Versicherungsmatematik.

Dr. V. Havlík.

In der letzten Zeit scheint es noch fraglich zu sein, ob man in der Praxis der Versicherungsmathematik graphische Rechnungsmethoden und Rechentafeln gebrauchen könnte und sollte. Einige Versicherungsmathematiker weisen richtig darauf hin, dass in allen Wissenschaftszweigen, welche mit einer häufigen Anwendung in der täglichen Praxis

[1]) See L. Charlier: Vorlesungen über die Grundzüge der mathem. Statistik, p. 83 and 84.