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NON-DECOMPOSABLE NAMBU BRACKETS

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Abstract. It is well-known that the Fundamental Identity (FI) implies that Nambu brackets are decomposable, i.e., given by a determinantal formula. We find a weaker alternative to the FI that allows for non-decomposable Nambu brackets, but still yields a Darboux-like Theorem via a Nambu-type generalization of Weinstein's splitting principle for Poisson manifolds.

1. Introduction

Recall the definition of an almost Nambu-Poisson structure.

Definition 1.1. An **almost n-Nambu-Poisson manifold** $(M;\pi)$ is a d-dimensional manifold M with an n-multi-vector field

$$\pi = \frac{1}{n!}\pi^{i_1\ldots i_n}\partial_{i_1}\wedge\cdots\wedge\partial_{i_n} \in \Gamma(\textstyle\bigwedge^n TM)\ , \tag{1.1}$$

with corresponding n-bracket $\{\cdot,\ldots,\cdot\}\colon [C^\infty(M)]^{\times n} \to C^\infty(M)$ defined as

$$\{f_1,\ldots,f_n\} = \pi(df_1\wedge\cdots\wedge df_n) = \pi^{i_1\ldots i_n}\frac{\partial f_1}{\partial x^{i_1}}\cdots\frac{\partial f_n}{\partial x^{i_n}}\ , \qquad f_1,\ldots,f_n\in C^\infty(M)\,, \tag{1.2}$$

which is $\mathbb{R}$-multi-linear, totally skewsymmetric, and has the Poisson property (i.e., Leibniz rule with respect to each entry).

The main question that we would like to discuss in this paper is: "Which integrability conditions should one impose on the n-multi-vector field π?" The case $n=1$ is just a vector field π, which has no non-trivial* integrability conditions. Moreover, the $n=1$ case is already manifestly decomposable — in fact, it is what we call *decomposable Darboux*, cf. definition 12.5. For $n=2$, the bi-vector field π should satisfy the Jacobi identity, and $(M;\pi)$ becomes a Poisson manifold. The

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*We will for simplicity not discuss the case where M is a supermanifold, and/or where π is Grassmann-odd. Recall that a Grassmann-odd vector field is not automatically in involution with itself.

sixty-four-thousand-dollar question is what should replace the Jacobi identity for $n \geq 3$? Nambu himself left this question unanswered in his seminal 1973 paper [16].

Twenty years later, in 1993, Takhtajan suggested to use the fundamental identity (4.3) as the missing integrability condition [20], cf. Section 4. We call such a structure a **fundamental Nambu-Poisson structure.** Takhtajan also conjectured[†] (and it was proven in 1996 by Gautheron [10]) that the multi-vector field π then necessarily must be decomposable, i.e., the n-bracket is given as a determinant, cf. Theorem 14.5. This is surprisingly rigid and in contrast to what happens in the $n = 2$ Poisson case, where only the rank 2 case is decomposable. Technically speaking, the culprit is the fundamental algebraic identity (5.1), cf. Section 5, which is an unavoidable consequence of the fundamental identity, cf. Proposition 5.2. More generally, a non-degenerately weighted fundamental algebraic identity (8.1) necessarily implies pointwise decomposability, cf. Theorem 14.2, a result often attributed to a 1996 paper [1] by Alekseevsky and Guha, although it was basically already known to Weitzenböck [23] in 1923.

One of the consequences of decomposability is as follows. Recall that the Cartesian product $M_1 \times M_2$ of two Poisson manifolds $(M_1; \pi_1)$ and $(M_2; \pi_2)$ is again a Poisson manifold $(M_1 \times M_2; \pi_1 + \pi_2)$ by simply adding the two Poisson-bivectors $\pi_i \in \Gamma(\bigwedge^2 TM_i)$ together, $i \in \{1, 2\}$. On the other hand, the Cartesian product $(M_1 \times M_2; \pi_1 + \pi_2)$ of two n-Nambu-Poisson manifolds $(M_1; \pi_1)$ and $(M_2; \pi_2)$, where π_1 and π_2 are both n-multi-vector fields, that satisfy the fundamental identity, is *almost never* an n-Nambu manifold itself for $n \geq 3$, if one requires the fundamental identity to hold.

One may ponder what decomposability means from a physics perspective? First a disclaimer. We have nothing new to say about the interesting and vast topic of quantum Nambu brackets [16, 8]. Thus we are only discussing classical physics, i.e., the part of physics that does not dependent on Planck's constant $\hbar$. Also we have nothing new to say about Nambu-type Hamiltonian dynamics and equations of motion. Here we will only make a general comment about kinematics. The decomposability issue does not affect Nambu structures formulated on a world-volume V, as in membrane theory, e.g., the recent Bagger-Lambert-Gustavsson (BLG) theory [4, 11, 6], because there the world-volume V is of fixed low dimension, and one would not be interested in forming Cartesian products of world volumes. Rather, the issue arises in a field theoretic context with Nambu structures in the target space. In the simplest Darboux case, one would formally have infinitely many n-tuples of canonical field variables $\phi^i(x)$, $i = 1, \ldots, n$, formally labeled by a continuous space-time index $x \in V$, i.e., one is taking an infinite Cartesian product of Nambu structures.

Motivated by such considerations, we will abandon the fundamental identity in this paper, and take another route. We are seeking a new definition of n-Nambu-Poisson manifolds, that (as a consequence of yet-to-be-found conditions)

[†]Takhtajan likely made the conjecture shortly after the publication of Ref. [20], see Ref. [8] and Remark 6 in Ref. [20].

1. includes the decomposable case (where the n-bracket is given as a determinant, and where the fundamental identity is satisfied) as a special case;
2. is stable under forming Cartesian products;
3. has a Darboux Theorem (in the form of a Weinstein splitting Theorem [22]).

Item 1 and 2 imply that one must allow n-multi-vector fields π on Darboux form

$$\pi = \sum_{m=1}^{r} \partial_{(m-1)n+1} \wedge \ldots \wedge \partial_{mn} , \tag{1.3}$$

which are by definition non-decomposable when $r > 1$, cf. Section 12.

Another obstacles is related to the fact that not even a *pointwise* Darboux Theorem (as opposed to the usual neighborhood Darboux Theorem) holds for $n \geq 3$.

Perhaps the first idea is to replace the fundamental identity with a non-degenerately weighted generalized Poisson identity (10.1), cf. Section 10. However, this seems not to be a feasible route for odd $n \geq 5$, and it is definitely excluded for $n = 3$. In fact, we prove in the $n = 3$ case, that a non-degenerately weighted generalized Poisson identity (10.1) implies pointwise decomposability, cf. Theorem 15.2.

We have investigated various integrability and algebraic conditions in this paper. In the end, we choose to define a Nambu-Poisson structure as follows.

Definition 1.2. A **Nambu-Poisson structure** is an almost Nambu-Poisson structure that satisfied

1. the nested integrability property (11.2),
2. and the fundamental algebraic hyper-identity (6.1).

The algebraic condition 2 in definition 1.2 help ensure a pointwise Darboux Theorem, while condition 1 is the actual integrability condition. From these two assumptions we prove a Weinstein splitting principle, cf. Theorem 14.4. This is our main result.

Finally, we investigate in Appendix A if one may generalize Moser's trick [14] for symplectic 2-forms to n-pre-multi-symplectic forms with $n \geq 3$. This seems not to be generally possible, essentially because the flat map $\flat$ is almost never surjective for $n \geq 3$. However, for a limited result, see Theorem A.9.

2. Basic formalism

The **sharp map** $\sharp\colon \Gamma(\bigwedge^{n-1} T^*M) \to \Gamma(TM)$ takes a differential $n-1$ form

$$\alpha = \frac{1}{(n-1)!} \alpha_{i_1 \ldots i_{n-1}} dx^{i_1} \wedge \ldots \wedge dx^{i_{n-1}} \in \Gamma(\bigwedge^{n-1} T^*M) \tag{2.1}$$

into a vector field $\sharp(\alpha)^j \partial_j = \sharp(\alpha) = i_\alpha \pi$ with vector field components $\sharp(\alpha)^j = \alpha_{i_1 \ldots i_{n-1}} \pi^{i_1 \ldots i_{n-1} j}$.

Definition 2.1. The **rank** of a multi-vector $\pi_{|_p} \in \bigwedge^n TM$ in a point $p \in M$ is the dimension of the image of the sharp map, $\mathrm{rank}(\pi_{|_p}) := \dim(\mathrm{Im}(\sharp_{|_p}))$.

The rank is lower semi-continuous as a function of the point $p \in M$.

Definition 2.2. A multi-vector $\pi_{|_p} \in \bigwedge^n TM$ is called **non-degenerate** in a point $p \in M$ if the sharp map $\sharp_{|_p}\colon \bigwedge^{n-1} T_p^* M \to T_p M$ is surjective, i.e., if $\mathrm{rank}(\pi_{|_p}) = d := \dim(M)$.

Definition 2.3. An n-multi-vector field $\pi \in \Gamma(\bigwedge^n TM)$ is called **invertible** if there exists an n-form $\omega \in \Gamma(\bigwedge^n T^*M)$ such that $J := \sharp \circ \flat\colon TM \to TM$ is a pointwise invertible map, where $\flat$ denotes the flat map, cf. Appendix A.

An invertible n-multi-vector field is always non-degenerate.

Definition 2.4. A function $f \in C^\infty(M)$ is called a **Casimir function** if $i_{df}\pi = 0$. The **center** $Z(M) := \{f \in C^\infty(M) \mid i_{df}\pi = 0\}$ is the subalgebra of all Casimir functions.

Definition 2.5. A **Hamiltonian vector field** is

$$X_{\vec{f}} := \{f_1, \ldots, f_{n-1}, \cdot\} = \sharp(df_1 \wedge \ldots \wedge df_{n-1}), \quad f_1, \ldots, f_{n-1} \in C^\infty(M), \tag{2.2}$$

and the $(n-1)$-tuple $\vec{f} := (f_1, \ldots, f_{n-1}) \in [C^\infty(M)]^{\times(n-1)}$ is called a **Hamiltonian**.

3. Pre-combing the n-bracket locally

In this Section we consider an arbitrary almost Nambu-Poisson structure $(M; \pi)$ without imposing any integrability conditions at all.

Lemma 3.1 (Pre-Combing in a Neighborhood)**.** *Let $\pi \in \Gamma(\bigwedge^n TM)$ be an n-multi--vector field with $n \geq 2$. If the multi-vector $\pi_{|_p} \neq 0$ is non-vanishing in a point $p \in M$, then there exists a local coordinate system $(x^1, \ldots, x^d)$ in a neighborhood U of the point $p \in M$, such that the Hamiltonian vector field*

$$X_{(x^1,\ldots,x^{n-1})} \equiv \frac{\partial}{\partial x^n}, \tag{3.1}$$

or equivalently, the corresponding n-bracket $\{\cdot, \ldots, \cdot\}$ fulfills

$$\forall k \in \{1, \ldots, d\} \quad : \quad \{x^1, \ldots, x^{n-1}, x^k\} \equiv \delta^k_n. \tag{3.2}$$

Proof of Lemma 3.1. One can choose local coordinates $(x^1, \ldots, x^d)$ in a neighborhood W, such that $\{x^1, \ldots, x^n\}_{|_p} \neq 0$, i.e., such that $X_{(x^1,\ldots,x^{n-1})|_p} \neq 0$. One may always stratify locally a non-vanishing vector field $X_{(x^1,\ldots,x^{n-1})}$ by choosing new coordinates $(y^1, \ldots, y^d)$ in a smaller neighborhood $V \subseteq W$, such that $X_{(x^1,\ldots,x^{n-1})} = \partial/\partial y^n$. There must exist a subset of $n-1$ new coordinates $(y^{i_1},\ldots,y^{i_{n-1}})$ with indices $i_1,\ldots,i_{n-1} \in \{1,\ldots,d\}$ such that the Jacobian

$$\det\left(\left.\frac{\partial x^j}{\partial y^{i_k}}\right|_p\right)_{1\leq j,k\leq n-1} \neq 0 \tag{3.3}$$

is non-vanishing. Note that the $n-1$ indices $i_1,\ldots,i_{n-1} \neq n$ must all be different from index n, since

$$\forall j \in \{1,\ldots,n-1\} \ : \ \frac{\partial x^j}{\partial y^n} = \{x^1,\ldots,x^{n-1},x^j\} = 0\,, \tag{3.4}$$

because the n-bracket $\{\cdot,\ldots,\cdot\}$ is totally antisymmetric. By relabeling the y-coordinates and perhaps shrinking to a smaller neighborhood $U \subseteq V$, one may assume that the Jacobian

$$\det\left(\frac{\partial x^j}{\partial y^k}\right)_{1\leq j,k\leq n-1} \neq 0 \tag{3.5}$$

is non-vanishing in the whole neighborhood U. It is easy to check that the mixed coordinate system $(x^1,\ldots,x^{n-1},y^n,\ldots,y^d)$ has the sought-for properties (3.1) and (3.2). □

Corollary 3.2 (Pre-Combing in a Point). *Let $\pi \in \Gamma(\bigwedge^n TM)$ be an n-multi-vector field with $n \geq 2$. If the multi-vector $\pi_{|_p} \neq 0$ is non-vanishing in a point $p \in M$, then there exist local coordinates $(x^1,\ldots,x^n,y^{n+1},\ldots,y^d)$ such that*

$$\{x^1,\ldots,x^n\}_{|_p} = 1\,, \tag{3.6}$$

and such that

$$\forall j \in \{1,\ldots,n\} \forall k \in \{n+1,\ldots,d\} \ : \ \{x^1,\ldots,\hat{x}^j,\ldots,x^n,y^k\}_{|_p} = 0 \tag{3.7}$$

in the point $p \in M$.

Proof of Corollary 3.2. By Lemma 3.1, there exist local coordinates $(x^1,\ldots,x^n,$ $y^{n+1},\ldots,y^d)$ such that $\{x^1,\ldots,x^n\}=1$, and such that

$$\forall k \in \{n+1,\ldots,d\} \ : \ \{x^1,\ldots,x^{n-1},y^k\} = 0\,. \tag{3.8}$$

Define new y-coordinates

$$y'^k := y^k - \sum_{i=1}^{n}(-1)^{n-i}\{x^1,\ldots,\hat{x}^i,\ldots,x^n,y^k\}(x^j - x^j_{|_p}) \tag{3.9}$$

for $k \in \{n+1,\ldots,d\}$. It is easy to check that the mixed coordinate system $(x^1,\ldots,x^n,y'^{n+1},\ldots,y'^d)$ has the sought-for property (3.7). □

4. Fundamental identity

The **fundamental identity function** $FI\colon [C^{\infty}(M)]^{\times(2n-1)} \to C^{\infty}(M)$ is defined by nested n-brackets as follows

$$
\begin{aligned}
(4.1)\quad & FI(f_1,\ldots,f_{n-1},g_1,\ldots,g_n) \\
& := X_{\vec{f}}\{g_1,\ldots,g_n\} - \sum_{i=1}^{n}\{g_1,\ldots,g_{i-1},X_{\vec{f}}[g_i],g_{i+1},\ldots,g_n\}\,.
\end{aligned}
$$

Definition 4.1. The **fundamental identity** is [20]

$$
FI(f_1,\ldots,f_{n-1},g_1,\ldots,g_n) = 0\,, \tag{4.2}
$$

or explicitly,

$$
X_{\vec{f}}\{g_1,\ldots,g_n\} = \sum_{i=1}^{n}\{g_1,\ldots,g_{i-1},X_{\vec{f}}[g_i],g_{i+1},\ldots,g_n\}\ , \tag{4.3}
$$

or equivalently,

$$
[X_{\vec{f}},X_{\vec{g}}] = \sum_{i=1}^{n-1} X_{(g_1,\ldots,g_{i-1},X_{\vec{f}}[g_i],g_{i+1},\ldots,g_{n-1})}\ , \tag{4.4}
$$

or equivalently, that Hamiltonian vector fields preserve the multi-vector field π,

$$
\mathcal{L}_{X_{\vec{f}}}\pi = 0\ , \tag{4.5}
$$

or equivalently,

$$
X_{\vec{f}}\{g_1,\ldots,g_n\} = \frac{1}{(n-1)!}\sum_{\sigma\in S_n}(-1)^{\sigma}\{X_{\vec{f}}[g_{\sigma(1)}],g_{\sigma(2)},\ldots,g_{\sigma(n)}\}\ . \tag{4.6}
$$

The fundamental identity (4.3) was introduced in 1993 by Takhtajan [20].[‡]

5. Fundamental algebraic identity

Definition 5.1. The **fundamental algebraic identity** is

$$
\sum_{i=1}^{n}\{h_1,f_1,\ldots,f_{n-2},g_i\}\{g_1,\ldots,g_{i-1},h_2,g_{i+1},\ldots,g_n\} = -(h_1\leftrightarrow h_2)\,, \tag{5.1}
$$

or equivalently,

$$
\sum_{\sigma\in S_n}(-1)^{\sigma}\{h_1,f_1,\ldots,f_{n-2},g_{\sigma(1)}\}\{g_{\sigma(2)},\ldots,g_{\sigma(n)},h_2\} = -(h_1\leftrightarrow h_2)\,. \tag{5.2}
$$

[‡]The fundamental identity in the $n=3$ case was considered in 1992 by Sahoo and Valsakumar [19] under the name 5-*point identity*, presumably because it has $2n-1=5$ entries. If one forgets about Leibniz rule, and think of $(C^{\infty}(M);\{\cdot,\ldots,\cdot\})$ as an infinite dimensional n-Lie algebra, the fundamental identity (4.3) was actually already introduced in 1985 by Filippov [9].

The fundamental identity function (4.1) satisfies Leibniz rule in each of its last n entries $g_1, \ldots, g_n$, but it does *not* satisfy Leibniz rule in each of its first $n-1$ entries $f_1, \ldots, f_{n-1}$ if $n \geq 3$. In general, lack of Leibniz rule induces additional algebraic constraints. Concretely,

Proposition 5.2. *The fundamental identity* (4.3) *implies the fundamental algebraic identity* (5.1).

Proof of Proposition 5.2. Replace the entry $f_{n-1} = h_1 h_2$ in the fundamental identity (4.3) with a product of functions. □

The fundamental algebraic identity (5.1) is trivial for $n = 2$.

Remark 5.3. The following tests are often useful in practice.

- To check if the fundamental algebraic identity (5.1) holds, it is enough to test it locally, using only local coordinate functions $x^1, \ldots, x^d$ as entries.
- If the fundamental algebraic identity (5.1) holds, to check if the fundamental identity (4.3) also holds, it is enough to test it locally, using only local coordinate functions $x^1, \ldots, x^d$ as entries.

Similar practical tests exist for other identities below, although we will not always go into details.

6. Fundamental algebraic hyper-identity

Definition 6.1. The **fundamental algebraic hyper-identity** is said to be satisfied if the fundamental algebraic identity

$$\sum_{i=1}^{n} \{h_1, f_1, \ldots, f_{n-2}, g_i\}\{g_1, \ldots, g_{i-1}, h_2, g_{i+1}, \ldots, g_n\} = -(h_1 \leftrightarrow h_2) \tag{6.1}$$

holds for all $\mathbb{R}$-linearly dependent function tuples $(f_1, \ldots, f_{n-2}, g_1, \ldots, g_n, h_1, h_2)$, i.e., function tuples so that

$$\exists (a_1, \ldots, a_{n-2}, b_1, \ldots, b_n, c_1, c_2) \in \mathbb{R}^{2n} \backslash \{\vec{0}\} : \quad \sum_{i=1}^{n-2} a_i f_i + \sum_{j=1}^{n} b_j g_j + \sum_{k=1}^{2} c_k h_k = 0\,. \tag{6.2}$$

Remark 6.2. We mention the following practical test.

- To check if the fundamental algebraic hyper-identity (6.1) holds, it is enough to test it locally, using only local coordinate functions $x^1, \ldots, x^d$ as entries, where at least two entries are the same.

7. Weighted fundamental identity

Definition 7.1. A **weighted fundamental identity** is

$$X_{\vec{f}}\{g_1, \ldots, g_n\} = \sum_{i=1}^{n} \lambda_i \{g_1, \ldots, g_{i-1}, X_{\vec{f}}[g_i], g_{i+1}, \ldots, g_n\}\,, \tag{7.1}$$

with weight functions $\lambda_i \in C^\infty(M)$.

A weighted fundamental identity (7.1) implies via symmetrization a **scaled fundamental identity**

$$X_{\vec{f}}\{g_1,\ldots,g_n\} = \lambda\sum_{i=1}^{n}\{g_1,\ldots,g_{i-1},X_{\vec{f}}[g_i],g_{i+1},\ldots,g_n\} \tag{7.2}$$

with scale function $\lambda = \frac{1}{n}\sum_{i=1}^{n}\lambda_i \in C^\infty(M)$. A scaled fundamental identity (7.2) implies an algebraic identity

$$(\lambda-1)X_{\vec{f}}[h_1]X_{\vec{g}}[h_2] = -(h_1\leftrightarrow h_2)\,, \tag{7.3}$$

which can easily be seen by replacing the entry $g_n = h_1h_2$ in the scaled fundamental identity (7.2) with a product of functions. The algebraic identity (7.3) implies that $(\lambda-1)\{f_1,\ldots,f_n\}^2 = 0$, which immediately leads to the following alternatives:

$$\forall p\in M:\quad \lambda_{|_p} = 1 \quad\vee\quad \pi_{|_p} = 0\,. \tag{7.4}$$

Conclusion: *There is nothing gained in terms of generality by introducing weights λ_i in the fundamental identity.*

8. Weighted fundamental algebraic identity

Definition 8.1. A **weighted fundamental algebraic identity** is

$$\sum_{i=1}^{n}\lambda_i\{h_1,f_1,\ldots,f_{n-2},g_i\}\{g_1,\ldots,g_{i-1},h_2,g_{i+1},\ldots,g_n\} = -(h_1\leftrightarrow h_2)\,, \tag{8.1}$$

with weight functions $\lambda_i \in C^\infty(M)$ that are non-degenerate, i.e.,

$$\forall p\in M\,\exists i\in\{1,\ldots,n\}\colon \lambda_{i|_p}\neq 0\,. \tag{8.2}$$

Proposition 8.2. *A weighted fundamental identity* (7.1) *implies a weighted fundamental algebraic identity* (8.1) *with the same weights.*
Proof of Proposition 8.2. Replace the entry $f_{n-1} = h_1h_2$ in the weighted fundamental identity (7.1) with a product of functions. □

The fundamental algebraic identity (5.1) is a special case of the weighted fundamental algebraic identity (8.1) with constant weights $\lambda_1 = \cdots = \lambda_n = 1$. Conversely, the weighted fundamental algebraic identity (8.1) with non-vanishing average $\frac{1}{n}\sum_{i=1}^{n}\lambda_i \neq 0$ becomes a fundamental algebraic identity (5.1) via symmetrization. In Corollary 14.3, we prove that *there is nothing gained in terms of generality by introducing weights λ_i in the fundamental algebraic identity.*

Remark 8.3 (Normalization)**.** The non-degeneracy condition (8.2) implies that locally in a sufficiently small neighborhood $U\subseteq M$, it is possible to assume that

$$\lambda_{1|_U} = 1 \tag{8.3}$$

by relabeling and rescaling of the weighted fundamental algebraic identity (8.1).

Definition 8.4. A **weighted fundamental algebraic hyper-identity** is said to be satisfied if a weighted fundamental algebraic identity

$$\sum_{i=1}^{n} \lambda_i \{h_1, f_1, \ldots, f_{n-2}, g_i\}\{g_1, \ldots, g_{i-1}, h_2, g_{i+1}, \ldots, g_n\} = -(h_1 \leftrightarrow h_2) \tag{8.4}$$

holds for all $\mathbb{R}$-linearly dependent function tuples $(f_1, \ldots, f_{n-2}, g_1, \ldots, g_n, h_1, h_2)$, cf. eq. (6.2).

9. Generalized Poisson structure

Definition 9.1. The **generalized Poisson identity** [7, 6] is

$$\sum_{\sigma \in S_{2n-1}} (-1)^{\sigma} \{f_{\sigma(1)}, \ldots, f_{\sigma(n-1)}, \{f_{\sigma(n)}, \ldots, f_{\sigma(2n-1)}\}\} = 0 \,. \tag{9.1}$$

The **generalized algebraic Poisson identity** is

$$\begin{aligned} &\sum_{\sigma \in S_{2n-2}} (-1)^{\sigma} \{h_1, f_{\sigma(1)}, \ldots, f_{\sigma(n-1)}\}\{f_{\sigma(n)}, \ldots, f_{\sigma(2n-2)}, h_2\} \\ &\qquad = -(h_1 \leftrightarrow h_2) \ . \end{aligned} \tag{9.2}$$

Proposition 9.2. *The generalized Poisson identity* (9.1) *implies the generalized algebraic Poisson identity* (9.2)*.*

Proof of Proposition 9.2. Replace the entry $f_{2n-1} = h_1 h_2$ in the generalized Poisson identity (9.1) with a product of functions. □

Remark 9.3. For even n, the generalized Poisson identity (9.1) is equivalent to involution

$$(\pi, \pi)_{SN} = 0 \tag{9.3}$$

with respect to the Schouten-Nijenhuis antibracket $(\partial_i, x^j)_{SN} = \delta_i^j$. For odd n, the involution condition (9.3) is trivially satisfied because of the symmetry property of the Schouten-Nijenhuis antibracket.

Remark 9.4. The fundamental identity (4.6) implies

$$\begin{aligned} &\sum_{\sigma \in S_{2n-2}} (-1)^{\sigma} \{f_{\sigma(1)}, \ldots, f_{\sigma(n-1)}, \{f_{\sigma(n)}, \ldots, f_{\sigma(2n-2)}, g_1\}\} \\ &\qquad = n \sum_{\sigma \in S_{2n-2}} (-1)^{\sigma} \{\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}, f_{\sigma(n+1)}, \ldots, f_{\sigma(2n-2)}, g_1\} \,, \end{aligned} \tag{9.4}$$

which, in turn, implies the generalized Poisson identity (9.1). The identity (9.4) implies the algebraic identity

$$\begin{aligned} &\sum_{\sigma \in S_{2n-3}} (-1)^{\sigma} \{h_1, f_{\sigma(1)}, \ldots, f_{\sigma(n-1)}\}\{f_{\sigma(n)}, \ldots, f_{\sigma(2n-3)}, g_1, h_2\} \\ &\qquad = -(h_1 \leftrightarrow h_2) \ , \end{aligned} \tag{9.5}$$

which can easily be seen by replacing the entry $f_{2n-2} = h_1 h_2$ in the identity (9.4) with a product of functions. The algebraic identity (9.5) implies the generalized algebraic Poisson identity (9.2), and for n odd, the two algebraic identities (9.2) and (9.5) are equivalent. Finally, consider the 180° cyclic permutation

$$\tau \;:=\; (n,\ldots,2n{-}2,1,\ldots,n{-}1) \;\in\; S_{2n-2}\;, \tag{9.6}$$

of permutation parity $(-1)^{\tau}=-(-1)^{n}$. The parity implies that the generalized algebraic Poisson identity (9.2) is trivially satisfied for even n.

Remark 9.5. For completeness, let us also mention the algebraic identity [13]

$$(i_{\alpha}\pi)\wedge(i_{\beta}\pi)=0\,,\qquad \alpha,\beta\in\Gamma(T^{*}M)\,, \tag{9.7}$$

or equivalently,

$$\sum_{\sigma\in S_{2n-2}}(-1)^{\sigma}\{h_1,f_{\sigma(1)},\ldots,f_{\sigma(n-1)}\}\{f_{\sigma(n)},\ldots,f_{\sigma(2n-2)},h_2\} = (-1)^{n}(h_1\leftrightarrow h_2)\,, \tag{9.8}$$

or equivalently,

$$\sum_{\sigma\in S_{2n-3}}(-1)^{\sigma}\{h_1,f_{\sigma(1)},\ldots,f_{\sigma(n-1)}\}\{f_{\sigma(n)},\ldots,f_{\sigma(2n-3)},g_1,h_2\} = (-1)^{n}(h_1\leftrightarrow h_2)\,, \tag{9.9}$$

which are equivalent to the algebraic identities (9.2) and (9.5) when n is odd.

10. Weighted generalized Poisson structures

Definition 10.1. A **weighted generalized Poisson identity** is

$$\sum_{\sigma\in S_{2n-1}}(-1)^{\sigma}\mu(\sigma)\{f_{\sigma(1)},\ldots,f_{\sigma(n-1)},\{f_{\sigma(n)},\ldots,f_{\sigma(2n-1)}\}\}=0\,, \tag{10.1}$$

with weight functions $\mu\colon M\times S_{2n-1}\to\mathbb{R}$ that are non-degenerate, i.e.,

$$\forall p\in M \;:\; \mu_{|_p}\;\neq\;0\,, \tag{10.2}$$

and $\mu_{|_p}:S_{2n-1}\to\mathbb{R}$ is symmetric in its first $n{-}1$ (and its last n) entries, respectively. Moreover, it is always assumed that $\mu(\sigma)\in C^{\infty}(M)$ is a smooth function for each permutation $\sigma\in S_{2n-1}$.

The generalized Poisson structure (9.1) is a special case of a weighted generalized Poisson structure (10.1) with constant weights $\mu{=}1$. Conversely, a weighted generalized Poisson structure (10.1) with non-vanishing average $\frac{1}{(2n-1)!}\sum_{\sigma\in S_{2n-1}}\mu(\sigma)\neq 0$ becomes a generalized Poisson structure (9.1) by total antisymmetrization.

Definition 10.2. A **weighted generalized algebraic Poisson identity** is

$$\sum_{\sigma\in S_{2n-2}}(-1)^{\sigma}\mu(\sigma)\{h_1,f_{\sigma(1)},\ldots,f_{\sigma(n-1)}\}\{f_{\sigma(n)},\ldots,f_{\sigma(2n-2)},h_2\} = -(h_1\leftrightarrow h_2)\,. \tag{10.3}$$

with weight functions $\mu\colon M\times S_{2n-2}\to\mathbb{R}$ that are non-degenerate, i.e.,

$$\forall p\in M \;:\; \mu_{|_p}\;\neq\;0\,, \tag{10.4}$$

and $\mu_{|_p} : S_{2n-2} \to \mathbb{R}$ is symmetric in its first (and last) $n-1$ entries, respectively. Moreover, it is always assumed that $\mu(\sigma) \in C^\infty(M)$ is a smooth function for each permutation $\sigma \in S_{2n-2}$.

Remark 10.3 (Associated Weighted Generalized Algebraic Poisson Identities)**.** Consider some $k \in \{1, \dots, 2n-1\}$. By replacing the entry $f_k = h_1 h_2$ in the weighted generalized Poisson identity (10.1), one derives

$$\sum_{\substack{\sigma \in S_{2n-1} \\ \sigma(2n-1)=k}} (-1)^\sigma \mu(\sigma) \{h_1, f_{\sigma(1)}, \dots, f_{\sigma(n-1)}\}\{f_{\sigma(n)}, \dots, f_{\sigma(2n-2)}, h_2\}$$

$$= -(h_1 \leftrightarrow h_2)\,, \tag{10.5}$$

which is of the form of a weighted generalized algebraic Poisson identity (10.3).

Remark 10.4 (Normalization)**.** The non-degeneracy conditions (10.2) (or (10.4)) imply that locally in a sufficiently small neighborhood $U \subseteq M$, it is possible to assume that

$$\mu_{|_{U \times \{\mathrm{id}\}}} = 1 \tag{10.6}$$

by relabeling and rescaling of the weighted identities (10.1) (or (10.3)), respectively.

11. Integrability

Definition 11.1. Given $2n-2$ functions $f_1, \dots, f_{2n-2} \in C^\infty(M)$, the **nested Hamiltonian distribution** is

$$\Delta_2(f_1, \dots, f_{2n-2}) := \operatorname{span}_{C^\infty(M)} \left\{ X_{(X_{(f_{\sigma(1)}, \dots, f_{\sigma(n-1)})}[f_{\sigma(n)}], f_{\sigma(n+1)}, \dots, f_{\sigma(2n-2)})} \middle| \sigma \in S_{2n-2} \right\}. \tag{11.1}$$

The **nested integrability property** is

$$\forall f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1} \in C^\infty(M) \ : \ [X_{\vec{f}}, X_{\vec{g}}] \ \in \ \Delta_2(\vec{f}, \vec{g})\,. \tag{11.2}$$

The **Casimir integrability property** is

$$\forall f_1, \dots, f_n \in C^\infty(M) : \tag{11.3}$$

$$\{f_1, \dots, f_n\} \in Z(M) \Rightarrow \begin{cases} \text{The } n \text{ Hamiltonian vector fields} \\ X_{(\hat{f}_1, f_2, \dots, f_n)}, X_{(f_1, \hat{f}_2, f_3, \dots, f_n)}, \dots, X_{(f_1, \dots, f_{n-1}, \hat{f}_n)} \\ \text{are in involution}\,. \end{cases}$$

Remark 11.2. The fundamental identity (4.4) implies the nested integrability property (11.2), which, in turn, implies the Casimir integrability property (11.3), and, with abuse of language, a weighted generalized Poisson identity (10.1), where the weight functions $\mu(\sigma)$ may depend the input functions $f_1, \dots, f_{2n-1}$.

On the other hand, the generalized Poisson identity (9.1) (or a weighted generalized Poisson identity (10.1)) does *not* necessarily have the nested integrability property (11.2) or the Casimir integrability property (11.3). We can now prove a neighborhood version of Corollary 3.2.

Lemma 11.3 (Combing with the Casimir Integrability Property)**.** *Let $\pi \in \Gamma(\bigwedge^n TM)$ be an n-multi-vector field that has the Casimir integrability property* (11.3) *with $n \geq 2$. If the multi-vector $\pi_{|_p} \neq 0$ is non-vanishing in a point $p \in M$, then there exists a local coordinate system $(x^1, \ldots, x^n, y^{n+1}, \ldots, y^d)$ in a neighborhood U of the point $p \in M$ such that*

$$\{x^1, \ldots, x^n\} = 1 \,, \tag{11.4}$$

and such that

$$\forall j \in \{1, \ldots, n\} \forall k \in \{n{+}1, \ldots, d\} \;:\; \{x^1, \ldots, \hat{x}^j, \ldots, x^n, y^k\} = 0 \tag{11.5}$$

in the whole neighborhood U.

Proof of Lemma 11.3. One may assume the $\pi_{|_p} \neq 0$. By Lemma 3.1, there exists a local coordinate system $(x^1, \ldots, x^d)$ such that $\{x^1, \ldots, x^n\} = 1$ in a neighborhood V. By the Casimir integrability property (11.3), the n Hamiltonian vector fields

$$X_{(\hat{x}^1, x^2, \ldots, x^n)}, X_{(x^1, \hat{x}^2, x^3, \ldots, x^n)}, \ldots, X_{(x^1, \ldots, x^{n-1}, \hat{x}^n)} \tag{11.6}$$

are in involution and linearly independent. By Frobenius Theorem, there exists a coordinate system $(y^1, \ldots, y^d)$ in a neighborhood $U \subseteq V$ such that

$$\forall j \in \{1, \ldots, n\} \;:\; X_{(x^1, \ldots, \hat{x}^j, \ldots, x^n)} = \frac{\partial}{\partial y^j} \,. \tag{11.7}$$

Since

$$\forall i, j \in \{1, \ldots, n\} \;:\; \frac{\partial x^i}{\partial y^j} = \{x^1, \ldots, \hat{x}^j, \ldots, x^n, x^i\} = (-1)^{n-j} \delta^i_j \,, \tag{11.8}$$

the Jacobian

$$\det \left(\frac{\partial x^j}{\partial y^k} \right)_{1 \leq j,k \leq n} \neq \; 0 \tag{11.9}$$

is non-vanishing in the whole neighborhood U. The mixture $(x^1, \ldots, x^n, y^{n+1}, \ldots, y^d)$ is therefore a coordinate system. It is easy to check that eq. (11.5) is satisfied. □

12. Decomposability and Darboux coordinates

Definition 12.1. An n-multi-vector field $\pi \in \Gamma(\bigwedge^n TM)$ is called **(globally) decomposable** if there exist n (globally defined) vector fields $X_1, \ldots, X_n \in \Gamma(TM)$ such that

$$\pi = X_1 \wedge \ldots \wedge X_n \,. \tag{12.1}$$

In other words, a **decomposable n-bracket** is the same as a **determinant n-bracket**

$$\{f_1, \ldots, f_n\} = \det(X_i[f_j]) \,. \tag{12.2}$$

Definition 12.2. An n-multi-vector $\pi_{|_p} \in \bigwedge^n T_pM$ is said to be **decomposable in a point** $p \in M$, if there exist n vectors $X_{1|_p}, \ldots, X_{n|_p} \in T_pM$, such that

$$\pi_{|_p} = X_{1|_p} \wedge \ldots \wedge X_{n|_p} \,. \tag{12.3}$$

Definition 12.3. A **Darboux coordinate system** $(x^1, \ldots, x^{nr}, y^{nr+1}, \ldots, y^d)$ in a local neighborhood U, where $r \in \{0, 1, 2, \ldots, [d/n]\}$, satisfies

$$\pi_{|_U} = \sum_{m=1}^{r} \partial_{(m-1)n+1} \wedge \ldots \wedge \partial_{mn} \tag{12.4}$$

in the whole neighborhood U.

The rank, $\mathrm{rank}(\pi_{|_U}) = nr$, of the n-multi vector field π is then a multiplum of the order n, corresponding to that **canonical coordinates** $(x^1, \ldots, x^{nr})$ come in n-tuples. The y-coordinate functions $y^{nr+1}, \ldots, y^d$ are local Casimir functions in U.

Definition 12.4. A **Weinstein split coordinate system** $(x^1, \ldots, x^{nr}, y^{nr+1}, \ldots, y^d)$ in a local neighborhood U around a point $p \in M$, where $r \in \{0, 1, 2, \ldots, [d/n]\}$, satisfies

$$\pi_{|_U} = \sum_{m=1}^{r} \partial_{(m-1)n+1} \wedge \ldots \wedge \partial_{mn} + \pi^{(y)} \tag{12.5}$$

where the remainder $\pi^{(y)} \in \Gamma(\bigwedge^n TM_{|_U})$ is independent of the x-coordinates $(x^1, \ldots, x^{nr})$ in the whole neighborhood U, and where $\pi^{(y)}_{|_p}$ has vanishing rank, $\mathrm{rank}(\pi^{(y)}_{|_p}) = 0$, in the point $p \in M$.

In particular, a Weinstein split coordinate patch $(U; \pi_{|_U}) = (U^{(x)}; \pi^{(x)}) \times (U^{(y)}; \pi^{(y)})$ is a product $U = U^{(x)} \times U^{(y)}$ of a Darboux patch $(U^{(x)}; \pi^{(x)})$ and a patch $(U^{(y)}; \pi^{(y)})$ with vanishing rank in at least one point.

Definition 12.5. An n-multi-vector field $\pi \in \Gamma(\bigwedge^n TM)$ is said to be **decomposable Darboux**, if for all points $p \in M$ with $\pi_{|_p} \neq 0$, there exist local coordinates $(x^1, \ldots, x^d)$ in a local neighborhood U around $p \in M$ such that

$$\pi_{|_U} = \partial_1 \wedge \ldots \wedge \partial_n \,. \tag{12.6}$$

13. Decomposable and Darboux cases

Proposition 13.1 (Decomposable $\Rightarrow$ Fundamental Algebraic Identity)**.** *A multi--vector $\pi_{|_p}$ that is decomposable in a point $p \in M$ must satisfy the fundamental algebraic identity* (5.2) *in $p \in M$.*

Proof of Proposition 13.1. This follows from the Schouten identity[§]

$$\sum_{\sigma \in S_n} (-1)^{\sigma}\ \varepsilon^{k_1 i_1 \cdots i_{n-2} j_{\sigma(1)}}\ \varepsilon^{j_{\sigma(2)} \cdots j_{\sigma(n)} k_2} = -(k_1 \leftrightarrow k_2)\,. \tag{13.1}$$

□

Proposition 13.2 (Darboux $\Rightarrow$ Fundamental Algebraic Hyper-Identity)**.** *A multi--vector $\pi_{|_p}$ on Darboux form in a point $p \in M$ must satisfy the fundamental algebraic hyper-identity* (6.1) *in $p \in M$.*

Proof of Proposition 13.2. One only has to consider non-zero contributions to eq. (6.1). A non-zero contribution $\pi^{k_1 i_1 \cdots i_{n-2} j_{\sigma(1)}}\ \pi^{j_{\sigma(2)} \cdots j_{\sigma(n)} k_2}$ must have indices $k_1, i_1, \ldots, i_{n-2}, j_{\sigma(1)}$ that belong to the same canonical n-tuple, and similarly, the indices $j_{\sigma(2)}, \ldots, j_{\sigma(n)}, k_2$ must belong to the same canonical n-tuple. So one may assume that all the $2n$ indices fit within no more than 2 canonical n-tuples. If all the indices belong to the same canonical n-tuple, the claim follows from the Schouten identity (13.1). Now assume that n indices belong to one tuple and n indices belong to a different tuple. By hyper-assumption, two indices must be the same. But this can only happen inside a tuple. But then the contribution vanish by skew-symmetry. □

Proposition 13.3 (Decomposable Darboux $\Rightarrow$ Fundamental Identity)**.** *A decomposable Darboux multi-vector field $\pi \in \Gamma(\bigwedge^n T^*M)$ must satisfy the fundamental identity* (4.3)*.*

Proof of Proposition 13.3. This follows from the pointwise observation (Proposition 13.1), and the fact that the Levi-Civita ε symbol is x-independent. □

14. Weinstein splitting principle

In this section we prove converse statements to Propositions 13.1, 13.2 and 13.3.

Lemma 14.1 (Combing with the Weighted Fundamental Algebraic Hyper-Identity)**.** *Let $\pi \in \Gamma(\bigwedge^n TM)$ be an n-multi-vector field that satisfies a non-degenerately weighted fundamental algebraic hyper-identity* (8.4) *with $n \geq 2$.*

[§]Proof of the Schouten identity (13.1): One only has to consider non-zero contributions to eq. (13.1). In particular, one may assume that all indices take values inside $\{1, \ldots, n\}$ (where the Levi-Civita ε symbol can be non-zero) rather than $\{1, \ldots, d\}$. If there are repetitions among $j_1, \ldots, j_n \in \{1, \ldots, n\}$, they must cancel out in the alternating sum. Hence one may assume that $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$. It follows that $j_{\sigma(1)} = k_2$, and hence that $k_1 \neq k_2$. Moreover, there must exists $\ell \in \{2, \ldots, n\}$ such that $j_{\sigma(\ell)} = k_1$. This contribution is canceled by a corresponding term in the second sum where $k_1 \leftrightarrow k_2$ and $\sigma(1) \leftrightarrow \sigma(\ell)$ are both interchanged. □

1. *If the multi-vector* $\pi_{|_p} \neq 0$ *is non-vanishing in a point* $p \in M$*, then there exists a local coordinate system* $(x^1, \ldots, x^n, y^{n+1}, \ldots, y^d)$ *in a neighborhood* U *of the point* $p \in M$ *such that*

$$\{x^1, \ldots, x^n\}_{|_p} = 1\,, \tag{14.1}$$

and

$$\begin{gathered}\{x^{i_1}, \ldots, x^{i_k}, y^{i_{k+1}}, \ldots, y^{i_n}\}_{|_p} = 0\,,\\ 1 \leq i_1 < \ldots < i_k \leq n < i_{k+1} < \ldots < i_n \leq d\,, \quad 1 \leq k < n\,,\end{gathered} \tag{14.2}$$

in the point $p \in M$*.*

2. *If furthermore the multi-vector* $\pi_{|_p}$ *satisfies a non-degenerately weighted fundamental algebraic identity* (8.1) *or a non-degenerately weighted generalized algebraic Poisson identity* (10.3) *in* $p \in M$*, then* (14.2) *holds for* $k = 0$ *as well, i.e.,*

$$\{y^{i_1}, \ldots, y^{i_n}\}_{|_p} = 0\;, \qquad n < i_1 < \ldots < i_n \leq d\,. \tag{14.3}$$

In particular, the multi-vector $\pi_{|_p} = \partial_{1|_p} \wedge \ldots \wedge \partial_{n|_p}$ *is decomposable in* $p \in M$*.*

Proof of part 1 of Lemma 14.1. One may assume the $\pi_{|_p} \neq 0$. By Corollary 3.2, there exist local coordinates $(x^1, \ldots, x^n, y^{n+1}, \ldots, y^d)$ such that $\{x^1, \ldots, x^n\}_{|_p} = 1$, and such that

$$\{x^{i_1}, \ldots, x^{i_{n-1}}, y^{i_n}\}_{|_p} = 0\,, \qquad 1 \leq i_1 < \ldots < i_{n-1} \leq n < i_n \leq d\,, \tag{14.4}$$

which is just eq. (14.2) with $k = n-1$, i.e., when there is precisely one y-coordinate y^{i_n} present on the left-hand side of eq. (14.2). We would like to prove eq. (14.2) for *any* number k of x-coordinates, where $k \in \{1, \ldots, n-1\}$. So assume that $k \geq 1$. Then there is at least one x-coordinate x^{i_1} on the left-hand side of eq. (14.2). Since $k < n$, there must also be an x-coordinate x^ℓ, $\ell \in \{1, \ldots, n\}$, that is *not* present on the left-hand side of eq. (14.2). It is possible to normalize the weight $\lambda_{1|_p} = 1$ due to Remark 8.3. Choose functions $h_1 = h_2 = x^{i_1}$; $g_1 = x^\ell$; $f_1, \ldots, f_{n-2} \in \{x^1, \ldots, x^n\} \backslash \{x^{i_1}, x^\ell\}$; and $g_2, \ldots, g_n \in \{x^{i_2}, \ldots, x^{i_k}, y^{i_{k+1}}, \ldots, y^{i_n}\}$ in the weighted fundamental algebraic hyper-identity (8.4) . This proves eq. (14.2) for $k \in \{1, \ldots, n-1\}$. □

Proof of part 2 of Lemma 14.1. Finally, consider the case $k = 0$. Let us assume a weighted generalized algebraic Poisson identity (10.3). (The case of a weighted fundamental algebraic identity (8.1) is very similar.) Choose functions $f_1, \ldots, f_{2n-2}, h_1, h_2 \in \{x^1, \ldots, x^n, y^{i_1}, \ldots, y^{i_n}\}$ in the weighted generalized algebraic Poisson identity (10.3). Make sure that the weight in front of the term $\{x^1, \ldots, x^n\}_{|_p} \{y^{i_1}, \ldots, y^{i_n}\}_{|_p}$ is non-vanishing. □

Theorem 14.2 (Non-Deg. Weighted Fund. Alg. Identity ⇒ Pointwise Decomposable [23, 1])**.** *If $n \geq 3$, a non-degenerately weighted fundamental algebraic identity* (8.1) *implies that the multi-vector $\pi_{|_p}$ is decomposable in the corresponding point $p \in M$.*

Proof of Theorem 14.2. A non-degenerately weighted fundamental algebraic identity implies all the assumptions of Lemma 14.1. □

Corollary 14.3. *Let $\pi \in \Gamma(\bigwedge^n TM)$ be an n-multi-vector with $n \geq 3$. The following conditions are equivalent.*

1. *A non-degenerately weighted fundamental algebraic identity* (8.1) *is satisfied in $p \in M$.*
2. *The multi-vector $\pi_{|_p}$ is decomposable.*
3. *The fundamental algebraic identity* (5.1) *is satisfied in $p \in M$.*

Theorem 14.4 (Weinstein Splitting Principle)**.** *If $n \geq 2$, the nested integrability property* (11.2) *and the fundamental algebraic hyper-identity* (6.1) *imply that for every point $p \in M$ there exists a Weinstein split coordinate system in a local neighborhood U of $p \in M$.*

Proof of Theorem 14.4. This proof essentially follows Nakashima's proof of Theorem 14.5, cf. Ref. [15] and Ref. [21], which use Weinstein splitting principle [22]. One may assume the $\pi_{|_p} \neq 0$. By Lemma 11.3, there exists a local coordinate system $(x^1, \ldots, x^d)$ such that $\{x^1, \ldots, x^n\} = 1$, and such that

$$\forall j \in \{1, \ldots, n\} \forall k \in \{n+1, \ldots, d\} \ : \ \{x^1, \ldots, \hat{x}^j, \ldots, x^n, y^k\} = 0 \tag{14.5}$$

in the whole neighborhood U. Now continue the proof pointwise as in the proof of the first part of Lemma 14.1 to establish eq. (14.2) for each point $p \in U$. Next use the nested integrability property (11.2) to the commutator

$$(-1)^{n-j} \frac{\partial}{\partial x^j} \{y^{i_1}, \ldots, y^{i_n}\} = [X_{(x^1, \ldots, \hat{x}^j, \ldots, x^n)}, X_{(y^{i_1}, \ldots, \ldots, y^{i_{n-1}})}][y^{i_n}] \tag{14.6}$$

to deduce that the n-bracket $\{y^{i_1}, \ldots, y^{i_n}\}$ cannot depend on the coordinates x^j, $j \in \{1, \ldots, n\}$. Thus the manifold M factorizes locally, and one may repeat the Weinstein splitting argument as long as there remains non-zero rank left. □

Theorem 14.5 (Fundamental identity ⇒ Decomposable Darboux Theorem [10])**.** *If $n \geq 3$, the fundamental identity* (4.3) *implies that π is a decomposable Darboux multi-vector field.*

Proof of Theorem 14.5. This proof essentially follows the proof of Theorem 14.4, although now one has access to the second part of Lemma 14.1 as well, so the nested integrability argument (14.6) and the Weinstein splitting procedure becomes superfluous. □

Proposition 14.6. *The determinant n-bracket* (12.2) *satisfies the fundamental identity* (4.3) *if and only if for all points $p \in M$ with $\pi_{|_p} \neq 0$, the vector fields $X_1, \ldots, X_n \in \Gamma(TM)$ are in involution at the point $p \in M$.*

Proof of the "only if" part of Proposition 14.6. One may assume the $\pi_{|_p} \neq 0$. One knows from Theorem 14.5 that the decomposable n-vector field $\pi = X_1 \wedge \ldots \wedge X_n$ can be locally written as $\pi_{|_U} = \partial_1 \wedge \ldots \wedge \partial_n$, and one knows from $\pi_{|_p} \neq 0$ that $X_1, \ldots, X_n$ are pointwise linearly independent in some neighborhood U of the point $p \in M$. Thus the following two distributions

$$\mathrm{span}_{C^\infty(U)}\{X_1, \ldots, X_n\} = \mathrm{span}_{C^\infty(U)}\{\partial_1, \ldots, \partial_n\} \tag{14.7}$$

are the same. Since the latter is in involution, so must the former be. □

15. The $n = 3$ case

For $n \geq 4$, the generalized algebraic Poisson identity (9.2) is different from the fundamental algebraic identity (5.1). However, in the $n = 3$ case, the generalized algebraic Poisson identity (9.2) is equivalent to the fundamental algebraic identity (5.1).

Remark 15.1 (Evading Algebraic Identity via Degeneracy)**.** In this paper we are particularly interested in multi-vector fields, which are not necessarily pointwise decomposable. Theorem 14.2 tells us to avoid imposing non-degenerately weighted fundamental algebraic identities (8.1). Now suppose that one is given some weighted generalized algebraic Poisson identity

$$\sum_{\sigma \in S_4} (-1)^\sigma \mu(\sigma) \{h_1, f_{\sigma(1)}, f_{\sigma(2)}\}\{f_{\sigma(3)}, f_{\sigma(4)}, h_2\} = -(h_1 \leftrightarrow h_2) \ . \tag{15.1}$$

with $\binom{4}{2} = 6$ weight functions $\mu(\sigma)$. It is easy to see that it can always be rewritten into a weighted fundamental algebraic identity (8.1) (which one would like to avoid) with three weight functions $\lambda_1, \lambda_2, \lambda_3$. The only hope to evade decomposability is that the λ_i weights might perhaps be degenerate (=zero), cf. eq. (8.2). In fact, $\lambda_i = 0$ if and only if the $\mu(\sigma)$ weights in the weighted generalized algebraic Poisson identity (15.1) satisfy

$$\forall \sigma \in S_4 : \quad \mu(\tau \circ \sigma) = -\mu(\sigma) \, . \tag{15.2}$$

Here $\tau := (4, 3, 1, 2) \in S_{2n-2=4}$ is the 180° cyclic permutation of even permutation parity $(-1)^\tau = +1$.

Theorem 15.2. *In the $n = 3$ case, an arbitrary non-degenerately weighted generalized Poisson structure*

$$\sum_{\sigma \in S_5} (-1)^\sigma \mu(\sigma) \{f_{\sigma(1)}, f_{\sigma(2)}, \{f_{\sigma(3)}, f_{\sigma(4)}, f_{\sigma(5)}\}\} = 0 \tag{15.3}$$

is always pointwise decomposable.

Indirect proof of Theorem 15.2. We cannot allow any non-degenerately weighted fundamental algebraic identities (8.1), cf. Theorem 14.2. The weighted generalized Poisson identity (15.3) has $\binom{5}{2} = 10$ weight functions $\mu(\sigma)$. As a shorthand let us from now on write $\mu(\sigma)$ as $\mu_{\sigma(1),\sigma(2)}$. The weighted generalized Poisson identity (15.3) implies $2n - 1 = 5$ associated weighted generalized algebraic Poisson identities of the form (15.1), cf. Remark 10.3. Because of Remark 15.1, one must demand

$$\begin{aligned}
&(15.4) && k=1: && \mu_{23}=-\mu_{45}, && \mu_{24}=-\mu_{35}, && \mu_{25}=-\mu_{34},\\
&(15.5) && k=2: && \mu_{13}=-\mu_{45}, && \mu_{14}=-\mu_{35}, && \mu_{15}=-\mu_{34},\\
&(15.6) && k=3: && \mu_{12}=-\mu_{45}, && \mu_{14}=-\mu_{25}, && \mu_{15}=-\mu_{24},\\
&(15.7) && k=4: && \mu_{12}=-\mu_{35}, && \mu_{13}=-\mu_{25}, && \mu_{15}=-\mu_{23},\\
&(15.8) && k=5: && \mu_{12}=-\mu_{34}, && \mu_{13}=-\mu_{24}, && \mu_{14}=-\mu_{23}.
\end{aligned}$$

It is not hard to check that this implies that the all coefficient $\mu(\sigma)=0$ must vanish. This contradicts the non-degeneracy (10.2). In other words, there is no identity (15.3) to start with. □

Appendix A. Pre-multi-symplectic manifolds

Let M be a d-dimensional manifold, let $n\geq 1$ be an integer, and let

$$Z^n(M) := \{\omega \in \Gamma(\bigwedge^n T^*M) \mid d\omega = 0\} \tag{A.1}$$

denote the set of closed n-forms

$$\omega = \frac{1}{n!}\omega_{i_1\ldots i_n} dx^{i_1}\wedge\ldots\wedge dx^{i_n} \ \in\ \Gamma(\bigwedge^n T^*M)\,, \qquad d\omega = 0\,, \tag{A.2}$$

on M.

Definition A.1. A closed n-form ω is called a **pre-multi-symplectic n-form**, and the pair $(M;\omega)$ is called an **n-pre-multi-symplectic manifold**.

The **flat map** $\flat\colon \Gamma(TM) \to \Gamma(\bigwedge^{n-1}T^*M)$ takes a vector field $X = X^j\partial_j \in \Gamma(TM)$ into a differential $n-1$ form

$$\frac{1}{(n-1)!}\flat(X)_{i_1\ldots i_{n-1}} dx^{i_1}\wedge\ldots\wedge dx^{i_{n-1}} = \flat(X) = i_X\omega \in \Gamma(\bigwedge^{n-1}T^*M) \tag{A.3}$$

with components $\flat(X)_{i_1\ldots i_{n-1}} = X^j\omega_{ji_1\ldots i_{n-1}}$.

Definition A.2. The **rank** of an n-form $\omega_{|_p} \in \bigwedge^n T^*M$ in a point $p\in M$ is the dimension d of the manifold M minus the dimension the kernel of the flat map, $\mathrm{rank}(\omega_{|_p}) := d - \dim(\ker(\flat_{|_p}))$.

Recall by Poincaré Lemma, there locally exists a **pre-multi-symplectic potential $(n-1)$-form** $\vartheta \in \Gamma(\bigwedge^{n-1}T^*M_{|U})$, so that $\omega_{|U} = d\vartheta$.

Definition A.3. A **Darboux coordinate system** $(x^1,\ldots,x^{nr},y^{nr+1},\ldots,y^d)$ in a local neighborhood U, where $r\in\{0,1,2,\ldots,[d/n]\}$, satisfies

$$\omega_{|_U} = \sum_{m=1}^{r} dx^{(m-1)n+1}\wedge\ldots\wedge dx^{mn} \tag{A.4}$$

in the whole neighborhood U.¶

The rank, $\mathrm{rank}(\omega_{|_U}) = nr$, of the n-multi vector field π is then a multiplum of the order n, corresponding to that **canonical coordinates** $(x^1,\ldots,x^{nr})$ come in n-tuples. The y-coordinate functions $y^{nr+1},\ldots,y^d$ are called local **Casimir functions** in U.

Definition A.4. An n-form ω has a **conformal vector field** X with **conformal weight function** $\lambda \in C^\infty(M)$ if

$$\mathcal{L}_X\omega = \lambda\omega\,. \tag{A.6}$$

Remark A.5. It follows from the proof of Poincaré Lemma that if a pre-multi-symplectic n-form ω has Darboux coordinates in some neighborhood U, then there exists a local conformal vector field $X \in \Gamma(TM_{|U})$ for $\omega_{|U}$ with conformal weight $\lambda = 1$, which can be made to vanish $X_{|p} = 0$ in any point $p \in U$.

Definition A.6. An n-form $\omega_{|_p} \in \bigwedge^n T^*M$ is called **non-degenerate** in a point $p \in M$ if the flat map $\flat_{|_p} : T_pM \to \bigwedge^{n-1}T^*_pM$ is injective.

The rank of a non-degenerate n-form is just the dimension d of the manifold.

Definition A.7. An n-form $\omega \in \Gamma(\bigwedge^n T^*M)$ is called **invertible** if there exists an n-multi-vector field $\pi \in \Gamma(\bigwedge^n TM)$ such that $\sharp\circ\flat\colon TM \to TM$ is a pointwise invertible map, i.e., the map $J_{|_p} := \sharp_{|_p} \circ \flat_{|_p} : T_pM \to T_pM$ is a bijection for all $p \in M$.

An invertible n-form is always non-degenerate.

Definition A.8. An **n-multi-symplectic**‖ **manifold** $(M;\omega)$ is a d-dimensional manifold M with an invertible closed n-form $\omega \in \Gamma(\bigwedge^n T^*M)$.

¶Pandit and Gangal considered the $n = 3$ case in Ref. [18] and Ref. [17]. Beware that definitions vary from author to author. In de Donder-Weyl theory (also known as covariant Hamiltonian field theory), a **Darboux coordinate system** in a neighborhood $U \subseteq M$ means that an n-pre-multi-symplectic manifold M of dimension d is locally isomorphic to a $(n-1)$-multi-cotangent bundle $U \cong \bigwedge^{n-1}T^*Q_{|V}$; where Q is an k-dimensional position manifold; where $V \subseteq Q$ is a neighborhood with position coordinates $(q^1,\ldots,q^k)$; where the fibers in $(n-1)$-multi-cotangent bundle $\bigwedge^{n-1}T^*Q_{|V}$ have momentum coordinates $p_{\mu_1\cdots\mu_{n-1}}$ with $1 \leq \mu_1 < \cdots < \mu_{n-1} \leq k$; and where the pre-multi-symplectic n-form is locally given as

$$\omega_{|_U} = \frac{1}{(n-1)!}\sum_{\mu_1,\ldots,\mu_{n-1}=1}^{k} dp_{\mu_1\cdots\mu_{n-1}} \wedge dq^{\mu_1} \wedge \ldots \wedge dq^{\mu_{n-1}}\ , \tag{A.5}$$

see e.g., Ref. [12] and Ref. [5]. In particular, the dimensions must in this case satisfy $d := \dim(M) = k + \binom{k}{n-1}$.

‖Beware that definitions may vary from author to author. For instance, relative to our conventions, Ref. [3] shifts the order n and calls a manifold with a non-degenerate closed n-form for an $(n-1)$**-plectic manifold**. As another example, Ref. [2] calls a manifold equipped with a certain kind of Lie-algebra-valued symplectic 2-form for a k**-symplectic manifold**.

We next salvage what we can from Moser's local trick for the $n=2$ case [14] when we consider general order $n\geq 2$. Sadly, it isn't much, mainly because the flat map $\flat\colon TM\to\bigwedge^{n-1}T^*M$ is *never* surjective for $n\geq 3$ and $d\geq 4$.

Theorem A.9 (n-th order version of Moser's local trick)**.** *Let there be given two non-degenerate pre-multi-symplectic n-forms $\omega_0,\omega_1\in\Gamma(\bigwedge^n T^*M)$ such that*

1. *their corresponding flat maps $\flat_{|\omega_0},\flat_{|\omega_1}$ have pointwise the same image,*

$$\Delta := \mathrm{Im}(\flat_{|\omega_0}) = \mathrm{Im}(\flat_{|\omega_1}) \subseteq \bigwedge\nolimits^{n-1}T^*M\,; \tag{A.7}$$

2. *they agree $\omega_{0|_p}=\omega_{1|_p}$ in a point $p\in M$;*
3. *they have conformal vector fields $Y_0,Y_1\in\Gamma(TM)$ with conformal weights $\lambda=1$;*
4. *and the conformal vector fields Y_0,Y_1 vanish in the point $p\in M$,*

$$Y_{0|p} = 0 = Y_{1|p}\,. \tag{A.8}$$

Then there exists two neighborhoods U_0 and U_1 of $p\in M$, and a diffeomorphism $\Psi\colon U_0\to U_1$, with the point $p\in M$ as a fixed point $\Psi(p)=p$, such that the pullback $\Psi^*\omega_1=\omega_0$ in the neighborhood U_0.

Proof of Theorem A.9. One may define two pre-multi-symplectic potential $n-1$ forms
(A.9)

$$\vartheta_i := \flat_{|\omega_i}(Y_i) = i_{Y_i}\omega_i\,,\quad \omega_i = \mathcal{L}_{Y_i}\omega_i = [d,i_{Y_i}]\omega_i = d\vartheta_i\,,\quad \vartheta_{i|p}=0,,\quad i\in\{0,1\}\,.$$

Next define convex linear combinations

$$\omega_t := t\omega_0+(1-t)\omega_1\,,\quad \vartheta_t := t\vartheta_0+(1-t)\vartheta_1\,,\quad \omega_t = d\vartheta_t\,,\quad t\in\mathbb{R}\,. \tag{A.10}$$

Since $\omega_{t|_p}$ in the point $p\in M$ is independent of $t\in\mathbb{R}$, one may assume** (by perhaps restricting to a local neighborhood U of the point $p\in M$) that $\flat_{|\omega_t}\colon TM_{|_U}\to\Delta_{|_U}\subseteq\bigwedge^{n-1}T^*M_{|_U}$ is a pointwise injective map for all $t\in\mathbb{R}$.

Now a vector field X_t is uniquely specified via

$$\flat_{|\omega_t}(X_t) = \flat_{|\omega_0}(Y_0)-\flat_{|\omega_1}(Y_1)\in\Delta_{|_U}\,,\qquad t\in\mathbb{R}\,. \tag{A.11}$$

The corresponding flow equation is

$$\frac{d\Psi_t(q)}{dt} = X_{t|_{\Psi_t(q)}}\,,\qquad \Psi_{t=0}(q)=q\,,\qquad q\in U\,. \tag{A.12}$$

Notice that $\vartheta_{0|_p}=0=\vartheta_{1|_p}$, so that $X_{t|_p}=0$, and hence the constant solution $\Psi_t(p)=p$, $t\in\mathbb{R}$, is the unique solution in the point $p\in M$. The ODE (A.12) has

**For instance, put $r(t):=\sqrt{t^2+(1-t)^2}>0$ and define angle $\varphi(t)\in]-\frac{\pi}{4},\frac{3\pi}{4}[$ via $r(t)\exp(i\varphi(t)) = t+i(1-t)$, where $t\in\mathbb{R}$. By continuity, it must be possible to cover the line $\{p\}\times\mathbb{R}\subseteq M\times\mathbb{R}$ with open box neighborhoods $U_{(k)}\times]t'_{(k)},t''_{(k)}[$ in which the map $r(t)^{-1}\flat_{|\omega_t}=\cos(\varphi(t))\flat_{|\omega_0}+\sin(\varphi(t))\flat_{|\omega_1}$ is pointwise injective. Since $\exp(i\varphi(t))$ belongs to a compact set in $\mathbb{C}$, there exists a finite subcover that does the job. Pick the set $U\subseteq M$ as a corresponding finite intersection, which must be open and include the point $p\in M$.

for each $q \in U$ a unique solution for $t \in [0,1]$ (by perhaps shrinking U further). It remains to check that $\Psi := \Psi_{t=1}$ is the sought-for diffeomorphism. One calculates

$$\begin{aligned} \frac{d}{dt}\left(\Psi_t^*\omega_t\right) = \Psi_t^*\Big(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\Big) &= \Psi_t^*\big([d, i_{X_t}]\omega_t + \omega_1 - \omega_0\big) \\ &= \Psi_t^* d\big(i_{X_t}\omega_t + \vartheta_1 - \vartheta_0\big) \\ &= \Psi_t^* d\big(\flat_{|\omega_t}(X_t) + \flat_{|\omega_1}(Y_1) - \flat_{|\omega_0}(Y_0)\big) \overset{(A.11)}{=} 0\,. \end{aligned} \tag{A.13}$$

So the n-form

$$\Psi_t^*\omega_t = \begin{cases} \Psi_0^*\omega_0 = \omega_0 \\ \Psi_1^*\omega_1 = \Psi^*\omega_1 \end{cases} \tag{A.14}$$

does not depend on the parameter $t \in [0,1]$. □

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