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ON SOME L^p-ESTIMATES FOR SOLUTIONS OF ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. In this review article we present an overview on some a priori estimates in L^p, $p > 1$, recently obtained in the framework of the study of a certain kind of Dirichlet problem in unbounded domains. More precisely, we consider a linear uniformly elliptic second order differential operator in divergence form with bounded leading coeffcients and with lower order terms coefficients belonging to certain Morrey type spaces. Under suitable assumptions on the data, we first show two L^p-bounds, $p > 2$, for the solution of the associated Dirichlet problem, obtained in correspondence with two different sign assumptions. Then, putting together the above mentioned bounds and using a duality argument, we extend the estimate also to the case $1 < p < 2$, for each sign assumption, and for a data in $L^p \cap L^2$.

Keywords: elliptic equation; discontinuous coefficient; a priori bound

MSC 2010: 35J25, 35B45, 35R05

1. Introduction

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geqslant 2$, and consider the elliptic second order linear differential operator in variational form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}\frac{\partial}{\partial x_i} + d_j\Big) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c \tag{1.1}$$

and the associated Dirichlet problem

$$\begin{cases} u \in \mathring{W}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega). \end{cases} \tag{1.2}$$

The first authors to approach this problem, in the framework of unbounded domains, were G. Bottaro and M. Marina who proved, in [4], the solvability of (1.2) for $n \geqslant 3$, under the hypotheses

$$(\mathrm{h}_1) \qquad \begin{cases} a_{ij} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n, \\ \exists\, \nu > 0\colon \sum\limits_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqslant \nu|\xi|^2 \quad \text{a.e. in } \Omega,\ \xi \in \mathbb{R}^n, \end{cases}$$

$$(\mathrm{h}_2') \qquad \begin{cases} b_i, d_i \in L^n(\Omega), \quad i = 1,\ldots,n, \\ c \in L^{n/2}(\Omega) + L^\infty(\Omega), \end{cases}$$

$$(\mathrm{h}_3) \qquad c - \sum_{i=1}^{n} (d_i)_{x_i} \geqslant \mu, \quad \mu \in \mathbb{R}_+.$$

The assumption (h_2') has been weakened in the later paper [20], where the authors consider coefficients b_i, d_i and c verifying (h_2') only locally and also extend the dimension of the space to $n \geqslant 2$. A further generalization has been obtained in [21], for $n \geqslant 3$, since the b_i, d_i and c are taken in certain spaces of Morrey type with lower summabilities.

In the works [4], [20], [21] the bound

$$(1.3) \qquad \|u\|_{W^{1,2}(\Omega)} \leqslant C\|f\|_{W^{-1,2}(\Omega)}$$

is also shown.

Our aim, in this review article, is to put in evidence the main novelties and difficulties that arose in our study (cf. [14], [17], [18]) concerning the achievement of an a priori bound of the type

$$(1.4) \qquad \|u\|_{L^p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}$$

for the solution of problem (1.2), in the same settings as in [20] and [21].

Let us therefore begin by introducing the Morrey type spaces involved. For $q \in [1,\infty[$ and $\lambda \in [0,n[$, the space of Morrey type $M^{q,\lambda}(\Omega)$ is the set of all functions g in $L^q_{\mathrm{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega)} = \sup_{\substack{\tau\in]0,1] \\ x\in\Omega}} \tau^{-\lambda/q}\|g\|_{L^q(\Omega(x,\tau))} < \infty,$$

endowed with the norm just defined, where $\Omega(x,\tau)$ is the intersection of Ω and the open ball of center x and radius τ. Moreover, $M^{q,\lambda}_\circ(\Omega)$ is the closure of $C^\infty_\circ(\Omega)$ in $M^{q,\lambda}(\Omega)$. We refer to [7] and [21] for the main properties of these spaces.

For reader's convenience, we recall that if $g \in M^{q,\lambda}(\Omega)$, with $q > 2$ and $\lambda = 0$ if $n = 2$, and $q \in]2,n]$ and $\lambda = n-q$ if $n > 2$, then the multiplication operator

$$(1.5) \qquad u \in \mathring{W}^{1,2}(\Omega) \longrightarrow gu \in L^2(\Omega)$$

is bounded (see [8] for details).

In our analysis we assume that the leading coefficients satisfy the hypothesis (h_1). For the lower order terms coefficients we suppose that

$$(\mathrm{h}_2)\qquad \begin{cases} b_i, d_i \in M_o^{2t,\lambda}(\Omega), \quad i = 1, \dots, n, \\ c \in M^{t,\lambda}(\Omega), \\ \text{with } t > 1 \text{ and } \lambda = 0 \quad \text{if } n = 2, \\ \text{with } t \in]1, n/2] \text{ and } \lambda = n - 2t \quad \text{if } n > 2. \end{cases}$$

Furthermore, the sign assumptions (h_3) or

$$(\mathrm{h}_4)\qquad c - \sum_{i=1}^{n} (b_i)_{x_i} \geqslant \mu, \quad \mu \in \mathbb{R}_+,$$

are satisfied.

Our first a priori bounds are proved in [17], [18], considering a sufficiently regular set Ω and supposing that (h_1)–(h_3) or (h_1), (h_2) and (h_4) hold, respectively. In these articles we show that if $f \in L^2(\Omega) \cap L^\infty(\Omega)$, then estimate (1.4) holds for any bounded solution u of (1.2) and for every $p > 2$.

Successively, in [14], we improve the results of [17], [18] showing that given a bounded data $f \in L^2(\Omega)$, a bounded solution u corresponds to it. This allows us to prove, by means of an approximation argument, that if f belongs to $L^2(\Omega) \cap L^p(\Omega)$, $p > 2$, then the solution is in $L^p(\Omega)$ too and verifies (1.4). The main result, namely estimate (1.4) for $p > 1$, for each sign hypothesis and for $f \in L^2(\Omega) \cap L^p(\Omega)$, is finally achieved, by means of a duality argument, putting together the two preliminary L^p-estimates, $p > 2$, obtained under different sign assumptions, and adding the further hypothesis that the a_{ij} are also symmetric.

As evidenced in [15], [16], estimate (1.4) for $p > 1$ finds a natural field of application in the study of certain weighted and no-weighted non variational problems with leading coefficients satisfying hypotheses of Miranda's type (see [13]).

Always in the framework of unbounded domains, we refer to [10], [11] for the study of some different variational problems and to [9] where quasilinear elliptic equations with quadratic growth are considered. A very general case involving principal coefficients having vanishing mean oscillation (VMO) can be found in [5], and in [2] and [3] in a weighted contest.

2. Test functions

This section is devoted to some specific functions that, chosen as test in the variational formulation of our problem, allow us to show the rather technical Lemma 3.1 which is the core of the proof of our L^p-a priori bound, $p > 2$.

We start with some notation. Let $G\colon\ t \in \mathbb{R} \to G(t)$ be a uniformly Lipschitz real function and suppose that

$$G_{|_{[-k,k]}} = 0, \quad k \in \mathbb{R}_+ \tag{2.1}$$

and that its derivative G' has a finite number of discontinuity points.

A known result proved by G. Stampacchia in the case of bounded domains (see Lemma 1.1 in [19]) ensures that given a function u in $\mathring{W}^{1,2}$ also the composition $G \circ u$ is in $\mathring{W}^{1,2}$.

Successively, in [4], G. Bottaro and M. Marina observed that the proof of this result holds true also for an unbounded open subset Ω of $\mathbb{R}^n$, $n \geqslant 2$. More precisely,

$$u \in \mathring{W}^{1,2}(\Omega) \Rightarrow G(u) = G \circ u \in \mathring{W}^{1,2}(\Omega). \tag{2.2}$$

Later on, we showed the following further generalization of (2.2), always in the case of unbounded domains. The complex proof of Lemma 2.1 can be found in [17].

Lemma 2.1. *Let G be a uniformly Lipschitz function as in* (2.1) *and such that its derivative G' has a finite number of discontinuity points. If Ω has the uniform C^1-regularity property* (*see* [1]), *then for every $u \in \mathring{W}^{1,2}(\Omega) \cap L^\infty(\Omega)$ one has*

$$|u|^{p-2}G(u) \in \mathring{W}^{1,2}(\Omega), \quad p \in \left]2, \infty\right[. \tag{2.3}$$

Now, let $h \in \mathbb{R}_+ \cup \{\infty\}$ and $k \in \mathbb{R}$, with $0 \leqslant k \leqslant h$. For each $t \in \mathbb{R}$ we set

$$G_{kh}(t) = \begin{cases} t-k, & \text{if } t > k, \\ 0, & \text{if } -k \leqslant t \leqslant k, \\ t+k, & \text{if } t < -k, \end{cases} \quad \text{if } h = \infty,$$

$$G_{kh}(t) = G_{k\infty}(t) - G_{h\infty}(t), \quad \text{if } h \in \mathbb{R}_+.$$

In [21] the authors prove the following result:

Lemma 2.2. *Let $g \in M_o^{q,\lambda}(\Omega)$, $u \in \mathring{W}^{1,2}(\Omega)$ and $\varepsilon \in \mathbb{R}_+$. Then there exist $r \in \mathbb{N}$ and $k_1, \ldots, k_r \in \mathbb{R}$, with $0 = k_r < k_{r-1} < \ldots < k_1 < k_0 = \infty$, such that, setting*

$$u_s = G_{k_s k_{s-1}}(u), \quad s = 1, \ldots, r, \tag{2.4}$$

one has $u_1, \ldots, u_r \in \mathring{W}^{1,2}(\Omega)$ and

$$\|g\chi_{\operatorname{supp}(u_s)_x}\|_{M^{q,\lambda}(\Omega)} \leqslant \varepsilon, \quad s = 1, \ldots, r, \tag{2.5}$$

$$|u_s| \leqslant |u|, \quad s = 1, \ldots, r, \tag{2.6}$$

$$u_1 + \ldots + u_r = u, \tag{2.7}$$

$$r \leqslant c, \tag{2.8}$$

with $c = c(\varepsilon, q, \|g\|_{M^{q,\lambda}(\Omega)})$ a positive constant.

We are now in a position to show (see also [17], [18]) the lemma allowing us to take the products $|u|^{p-2}u_s$ and $|u_s|^{p-2}u_s$ as test functions in the variational formulation of our problem (according to hypotheses (h_3) and (h_4), respectively). We explicitly observe that Lemma 2.1 is the main tool in the following proof.

Lemma 2.3. *If Ω has the uniform C^1-regularity property, then for every $u \in \mathring{W}^{1,2}(\Omega) \cap L^\infty(\Omega)$ and for any $p \in]2, \infty[$ one has*

$$|u|^{p-2}u_s \in \mathring{W}^{1,2}(\Omega), \tag{2.9}$$

$$|u_s|^{p-2}u_s \in \mathring{W}^{1,2}(\Omega), \tag{2.10}$$

where u_s, for $s = 1, \ldots, r$, are the functions of Lemma 2.2.

Proof. To prove (2.9) observe that if $r = 1$, then $u_1 = G_{0\infty}(u) = u$, therefore, by Lemma 3.2 in [6], one has $|u|^{p-2}u \in \mathring{W}^{1,2}(\Omega)$. If $r > 1$ and $s < r$, then $u_s = G_{k_s k_{s-1}}(u)$, therefore $|u|^{p-2}u_s = |u|^{p-2}G(u)$ for the choice $k = k_s$ in (2.1). This entails that $|u|^{p-2}u_s \in \mathring{W}^{1,2}(\Omega)$, by means of Lemma 2.1. In view of these considerations and (2.7) being true, we also get $|u|^{p-2}u_r = |u|^{p-2}u - \sum_{s=1}^{r-1} |u|^{p-2}u_s \in \mathring{W}^{1,2}(\Omega)$. Lemma 3.2 in [6] also yields (2.10). □

3. Main results

Let us associate with the operator L in (1.1) the bilinear form

$$a(u,v)=\int_\Omega\Big(\sum_{i,j=1}^n(a_{ij}u_{x_i}+d_ju)v_{x_j}+\Big(\sum_{i=1}^n b_iu_{x_i}+cu\Big)v\Big)\,\mathrm{d}x, \tag{3.1}$$

$u,v\in \mathring{W}^{1,2}(\Omega)$, and observe that, in view of the boundedness of the multiplication operator (1.5), the form a is continuous on $\mathring{W}^{1,2}(\Omega)\times\mathring{W}^{1,2}(\Omega)$ and so the operator $L\colon \mathring{W}^{1,2}(\Omega)\to W^{-1,2}(\Omega)$ is continuous as well.

Now, let u_s be the functions of Lemma 2.2 obtained in correspondence with a given $u\in\mathring{W}^{1,2}(\Omega)\cap L^\infty(\Omega)$, with $g=\sum\limits_{i=1}^n|b_i-d_i|$ and with a positive real number ε specified in the proofs of Lemmas 4.1 of [17] and [18]. One has:

Lemma 3.1. *Let a be the bilinear form defined in* (3.1)*. If Ω has the uniform C^1-regularity property and* (h_1) *and* (h_2) *hold, then under hypothesis* (h_3) *there exists a constant $C_1\in\mathbb{R}_+$ such that*

$$\int_\Omega|u|^{p-2}((u_s)_x^2+u_s^2)\,\mathrm{d}x\leqslant C_1\sum_{h=1}^s a(u,|u|^{p-2}u_h),\quad s=1,\dots,r,\ p\in\,]2,\infty[, \tag{3.2}$$

while under hypothesis (h_4) *there exists a constant $C_2\in\mathbb{R}_+$ such that*

$$\int_\Omega|u_s|^{p-2}((u_s)_x^2+u_s^2)\,\mathrm{d}x\leqslant C_2\sum_{h=s}^r a(u,|u_h|^{p-2}u_h),\quad s=1,\dots,r,\ p\in\,]2,\infty[. \tag{3.3}$$

To get the claimed L^p-bound, $p>1$, a further assumption on the leading coefficients is required:

$$a_{ij}=a_{ji},\quad i,j=1,\dots,n. \tag{h_0}$$

Theorem 3.1. *Assume that hypotheses* (h_0)–(h_3) *or* (h_0)–(h_2) *and* (h_4) *are satisfied. If the set Ω has the uniform C^1-regularity property and the data $f\in L^2(\Omega)\cap L^p(\Omega)$ for some $p\in\,]1,\infty[$, then the solution u of problem* (1.2) *is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)}\leqslant C\|f\|_{L^p(\Omega)}, \tag{3.4}$$

with $C=C(n,t,p,\nu,\mu,\|b_i-d_i\|_{M^{2t,\lambda}(\Omega)})$.

P r o o f. *Step 1.* The first step consists in showing that if $f \in L^2(\Omega) \cap L^\infty(\Omega)$, then any bounded solution u of (1.2) is in L^p for every $p > 2$, and estimate (3.4) holds. As shown in [17], [18], this can be done exploiting Lemma 3.1.

Step 2. One proves some regularity results following a technique introduced by C. Miranda in [12]. More precisely, one shows that if $u \in \mathring{W}^{1,2}(\Omega)$ is the solution of (1.2) with $f \in L^2(\Omega) \cap L^\infty(\Omega)$, then, the data f being more regular, one also has $u \in L^\infty(\Omega)$, see [14] for the details. Thus, by Step 1 one obtains that therefore $u \in L^p(\Omega)$ for every $p > 2$, and satisfies (3.4).

Step 3. As proved in [14], one considers $f \in L^2(\Omega) \cap L^p(\Omega)$ for some $p > 2$, and then obtains, by means of some approximation arguments, that the solution u of problem (1.2) is in $L^p(\Omega)$ and satisfies the bound (3.4).

Step 4. It remains to show (3.4) for $1 < p \leqslant 2$. For $p = 2$ the result is already known under hypotheses (h_0)–(h_3), see [20], [21]. Under hypotheses (h_0)–(h_2) and (h_4) the proof of estimate (1.3), together with the solvability of problem (1.2), is given in [18]. Thus, let us assume that $1 < p < 2$. We suppose that (h_0)–(h_3) hold true; a similar argument, with suitable modifications, can be used for the other set of hypotheses (we refer to [14] where all the details can be found).

Let us define the bilinear form

$$a^*(w, v) = a(v, w), \quad w, v \in \mathring{W}^{1,2}(\Omega).$$

By (h_0) one has

$$a^*(w, v) = \int_\Omega \Bigg(\sum_{i,j=1}^n (a_{ij} w_{x_i} + b_j w) v_{x_j} + \Bigg(\sum_{i=1}^n d_i w_{x_i} + cw \Bigg) v \Bigg) \,\mathrm{d}x. \tag{3.5}$$

Now consider the problem

$$\begin{cases} w \in \mathring{W}^{1,2}(\Omega), \\ a^*(w, v) = \int_\Omega gv \,\mathrm{d}x, \quad g \in L^2(\Omega) \cap L^{p'}(\Omega), \end{cases} \tag{3.6}$$

where, since $1 < p < 2$, one gets $p' = p/(p-1) > 2$.

As mentioned above, the solution w of (3.6) exists and is unique. Furthermore, by Step 3 (for the second set of hypotheses) one also has

$$\|w\|_{L^{p'}(\Omega)} \leqslant C \|g\|_{L^{p'}(\Omega)}. \tag{3.7}$$

Hence, if we denote by u the solution of problem (1.2) with $f \in L^2(\Omega) \cap L^p(\Omega)$ which exists and is unique in view of the results proved in [20] and [21], we obtain

$$\int_\Omega gu \,\mathrm{d}x = a^*(w,u) = a(u,w) = \int_\Omega fw \,\mathrm{d}x \leqslant \|f\|_{L^p(\Omega)} \|w\|_{L^{p'}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}. \tag{3.8}$$

Finally, taking $g = |u|^{p-1} \operatorname{sign} u$ in (3.8), we get the claimed result. □

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