Irina Astashova

On asymptotic behavior of solutions to Emden-Fowler type higher-order differential equations

Mathematica Bohemica, Vol. 140 (2015), No. 4, 479–488

Persistent URL: http://dml.cz/dmlcz/144464

ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO EMDEN-FOWLER TYPE HIGHER-ORDER DIFFERENTIAL EQUATIONS

IRINA ASTASHOVA, Moskva

(Received October 16, 2013)

Abstract. For the equation

$$y^{(n)} + |y|^k \operatorname{sgn} y = 0, \quad k > 1, \ n = 3, 4,$$

existence of oscillatory solutions

$$y = (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad \alpha = \frac{n}{k-1}, \ x < x^*,$$

is proved, where x^* is an arbitrary point and h is a periodic non-constant function on $\mathbb{R}$. The result on existence of such solutions with a positive periodic non-constant function h on $\mathbb{R}$ is formulated for the equation

$$y^{(n)} = |y|^k \operatorname{sgn} y, \quad k > 1, \ n = 12, 13, 14.$$

Keywords: nonlinear ordinary differential equation of higher order; asymptotic behavior of solutions; oscillatory solution

MSC 2010: 34C15, 34C10

1. INTRODUCTION

During the investigation of the problem on asymptotic behavior near vertical asymptotes of positive solutions to the equation

$$y^{(n)} = p(x, y, y', \ldots, y^{(n-1)})|y|^k \operatorname{sgn} y, \quad n \geqslant 2, \ k > 1, \tag{1.1}$$

The research was supported by RFBR (grant 11-01-00989).

posed by I. T. Kiguradze [7], it was proved for sufficiently large n in [8] and for $n = 12, 13, 14$ in [1] and [6] that the equation

$$y^{(n)} = |y|^k \operatorname{sgn} y \tag{1.2}$$

with some $k > 1$ has a positive solution with non-power asymptotic behavior, namely,

$$y(x) = (x^* - x)^{-\alpha} h(\log(x^* - x)), \tag{1.3}$$

where h is a positive periodic non-constant function on $\mathbb{R}$,

$$\alpha = \frac{n}{k-1}. \tag{1.4}$$

Still, it was not clear whether solutions of that type exist if $n < 12$.

2. Preliminary results. Existence of solutions with non-power asymptotic behavior

In this section, a result on existence of positive solutions with non-power asymptotic behavior is formulated for equation (1.2) with $n = 12, 13, 14$. Precise proofs are contained in [1]. The results concerning solutions with power asymptotic behavior are contained in [2], [5], [7].

Theorem 2.1. *For $n = 12, 13, 14$ there exists $k > 1$ such that equation* (1.2) *has a solution $y(x)$ with*

$$y^{(j)}(x) = (x^* - x)^{-\alpha-j} h_j(\log(x^* - x)), \quad j = 0, 1, \ldots, n-1,$$

where α is defined by (1.4) *and h_j are periodic positive non-constant functions on $\mathbb{R}$.*

To prove this result the Hopf bifurcation theorem [9] is used:

Theorem 2.2 (Hopf). *Consider the α-parametrized dynamical system $\dot{x} = L_\alpha x + Q_\alpha(x)$ in a neighborhood of $0 \in \mathbb{R}^n$ with linear operators L_α and smooth enough functions $Q_\alpha(x) = O(|x|^2)$ as $x \to 0$. Let λ_α and $\overline{\lambda}_\alpha$ be simple complex conjugated eigenvalues of the operators L_α. Suppose $\operatorname{Re} \lambda_{\tilde{\alpha}} = \operatorname{Re} \overline{\lambda}_{\tilde{\alpha}} = 0$ for some $\tilde{\alpha}$ and the operator $L_{\tilde{\alpha}}$ has no other eigenvalues with zero real part.*

If $\operatorname{Re}(\mathrm{d}\lambda_\alpha/\mathrm{d}\alpha)(\tilde{\alpha}) \neq 0$, then there exist continuous mappings $\varepsilon \mapsto \alpha(\varepsilon) \in \mathbb{R}$, $\varepsilon \mapsto T(\varepsilon) \in \mathbb{R}$, and $\varepsilon \mapsto b(\varepsilon) \in \mathbb{R}^n$ defined in a neighborhood of 0 and such that

$\alpha(0)=\tilde{\alpha}$, $T(0)=2\pi/\operatorname{Im}\lambda_{\tilde{\alpha}}$, $b(0)=0$, $b(\varepsilon)\neq 0$ *for* $\varepsilon\neq 0$, *and the solutions to the problems*

$$\dot{x}=L_{\alpha(\varepsilon)}x+Q_{\alpha(\varepsilon)}(x),\quad x(0)=b(\varepsilon)$$

are $T(\varepsilon)$*-periodic and non-constant.*

To use this theorem, equation (1.2) is transformed (see [5] or [2], Chapter I, (5.1)) by using the substitution

$$x^*-x=\mathrm{e}^{-t},\quad y=(C+v)\mathrm{e}^{\alpha t}, \tag{2.1}$$

where α is defined by (1.4) and

$$C=(\alpha(\alpha+1)\dots(\alpha+n-1))^{1/(k-1)}. \tag{2.2}$$

Suppose V is the vector with coordinates $V_j=v^{(j)}$, $j=0,\dots,n-1$. Then equation (1.2) can be written as

$$\frac{\mathrm{d}V}{\mathrm{d}t}=AV+F(V), \tag{2.3}$$

where A is a constant $n\times n$ matrix

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0\\ 0 & 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 0 & 1 & \dots & 0\\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot\\ 0 & 0 & 0 & 0 & \dots & 1\\ -\tilde{a}_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-1}\end{pmatrix}$$

with

$$\begin{aligned}-\tilde{a}_0&=a_0-kc^{k-1}p_0=a_0-k\alpha(\alpha+1)\dots(\alpha+n-1)\\ &=a_0-(\alpha+1)\dots(\alpha+n-1)(\alpha+n)\end{aligned}$$

and eigenvalues satisfying the equation

$$\begin{aligned}0=\ &\det(A-\lambda E)=(-1)^{n+1}(-\tilde{a}_0-a_1\lambda-\ldots-a_{n-1}\lambda^{n-1}-\lambda^n)\\ &=(-1)^{n+1}\big((\alpha+1)(\alpha+2)\dots(\alpha+n)-(\lambda+\alpha)\dots(\lambda+\alpha+n-1)\big),\end{aligned}$$

which is equivalent to

$$\prod_{j=0}^{n-1}(\lambda+\alpha+j)=\prod_{j=0}^{n-1}(1+\alpha+j). \tag{2.4}$$

The mappings $F\colon\ \mathbb{R}^n \to \mathbb{R}^n$ and $G\colon\ \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ satisfy the following estimates as $t \to \infty$:

$$\begin{cases} \|F(V)\| &= O(\|V\|^2), \\ \|F'_V(V)\| &= O(\|V\|). \end{cases} \tag{2.5}$$

F is a vector function with $n-1$ zero components: $F(V) = (0,\ldots,0,F_{n-1}(V))$ and

$$F_{n-1}(V) = ((C+V_0)^k - C^k - kC^{k-1}V_0) = O({V_0}^2), \quad V_0 \to 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}V}F_{n-1}(V) = O(|V_0|), \quad V_0 \to 0.$$

If equation (2.4) has a pair of pure imaginary roots, we have to check the other conditions of the Hopf bifurcation theorem and then apply it.

Lemma 2.1. *For any integer $n > 11$ there exist $\alpha > 0$ and $q > 0$ such that*

$$\prod_{j=0}^{n-1}(q\mathrm{i}+\alpha+j) = \prod_{j=0}^{n-1}(1+\alpha+j) \quad \textit{with } \mathrm{i}^2 = -1. \tag{2.6}$$

Lemma 2.2. *For any $\alpha > 0$ and any integer $n > 1$ all roots $\lambda \in \mathbb{C}$ to equation* (2.4) *are simple.*

Lemma 2.3. *If $12 \leqslant n \leqslant 14$, $\alpha > 0$, and $q > 0$ satisfy the polynomial equation*

$$\prod_{j=0}^{n-1}((\alpha+j)^2+q^2) = \prod_{j=0}^{n-1}(\alpha+j+1)^2,$$

then $2\alpha + 4 < q^2 < 3\alpha + 5$.

The condition $\mathrm{Re}(\mathrm{d}\lambda_\alpha/\mathrm{d}\alpha)(\tilde{\alpha}) \neq 0$ needed for the Hopf theorem, expressed explicitly by means of the implicit function theorem, reads

$$\left[\sum_{j=0}^{n-1}\frac{\alpha+j}{q^2+(\alpha+j)^2}\right]^2 + \left[\sum_{j=0}^{n-1}\frac{q}{q^2+(\alpha+j)^2}\right]^2$$
$$\neq \sum_{j=0}^{n-1}\frac{\alpha+j}{q^2+(\alpha+j)^2}\sum_{j=0}^{n-1}\frac{1}{1+\alpha+j}.$$

Lemma 2.4. *If* $12 \leqslant n \leqslant 14$, $\alpha > 0$ *and* $0 < q^2 < 3\alpha + 5$, *then*

$$\left[\sum_{j=0}^{n-1} \frac{\alpha+j}{q^2+(\alpha+j)^2}\right]^2 + \left[\sum_{j=0}^{n-1} \frac{q}{q^2+(\alpha+j)^2}\right]^2 > \sum_{j=0}^{n-1} \frac{\alpha+j}{q^2+(\alpha+j)^2} \sum_{j=0}^{n-1} \frac{1}{1+\alpha+j}. \tag{2.7}$$

To apply the Hopf bifurcation theorem we need to check that equation (2.4) cannot have more than a single pair of imaginary conjugated roots. It can be easily obtained by considering equation (2.6).

Now the Hopf bifurcation theorem and the lemmas formulated provide, for $n = 12, 13, 14$, existence of a family $\alpha_\varepsilon > 0$ such that equation (2.4) with $\alpha = \alpha_0$ has imaginary roots $\lambda = \pm q\mathrm{i}$ and for sufficiently small ε system (2.3) with $\alpha = \alpha_\varepsilon$ has a periodic solution $V_\varepsilon(t)$ with period $T_\varepsilon \to T = 2\pi/q$ as $\varepsilon \to 0$. In particular, the coordinate $V_{\varepsilon,0}(t) = v(t)$ of the vector $V_\varepsilon(t)$ is also a periodic function with the same period. Then, taking into account (2.1), we obtain

$$y(x) = \bigl(C + v(-\ln(x^* - x))\bigr)(x^* - x)^{-\alpha}.$$

Putting $h(s) = C + v(-s)$, which is a non-constant continuous periodic and positive for sufficiently small ε function we obtain the required equality

$$y(x) = (x^* - x)^{-\alpha} h(\ln(x^* - x)).$$

In a similar way we obtain the related expressions for $y^{(j)}(x)$, $j = 0, 1, \ldots, n-1$.

3. Main results

Theorem 3.1. *For* $n = 4$ *and any real* $k > 1$ *there exists an oscillatory periodic function* $h\colon \mathbb{R} \to \mathbb{R}$ *such that the functions* $x \mapsto |x - x_*|^{-\alpha} h(\log|x - x_*|)$ *defined on* $(x_*; \infty)$ *and* $(-\infty; x_*)$ *with* $\alpha = n/(k-1)$ *and arbitrary* x_* *are solutions to the equation*

$$y^{(n)}(x) + |y(x)|^k \operatorname{sgn} y(x) = 0. \tag{3.1}$$

Proof. Any solution to equation (3.1) with positive initial data can change the signs of its derivatives of order 0, 1, 2, 3 according to the following cyclic scheme only:

$$\begin{array}{ccccccc} \boxed{++++} & \to & \boxed{+++-} & \to & \boxed{++--} & \to & \boxed{+---} \\ \uparrow & & & & & & \downarrow \\ \boxed{-+++} & \leftarrow & \boxed{--++} & \leftarrow & \boxed{---+} & \leftarrow & \boxed{----} \end{array}$$

Moreover, all these sign changes must occur infinitely many times. In other words, non-extensible solutions cannot keep ultimately any of the sign combinations from the above scheme.

Thus, to any non-trivial initial data $(0, v_1, v_2, v_3)$ with $v_j \geqslant 0$, $j = 1, 2, 3$, at a point $x_0 \in \mathbb{R}$ we can assign the point $x_1 > x_0$ where the solution with these initial data vanishes the next time. This correspondence is continuous according to the implicit function theorem and due to the fact that the solutions considered vanish at x_1 with non-zero first order derivative.

Passing from the point x_1 to the values of derivatives at this point we obtain another continuous mapping $F\colon \{0\} \times \mathbb{R}^3_+ \setminus \{0\} \to \{0\} \times \mathbb{R}^3_- \setminus \{0\}$.

Now consider the compact set

$$E = \left\{Y = (Y_0, Y_1, Y_2, Y_3) \in \mathbb{R}^4\colon\ |Y_0|^{1/4} + |Y_1|^{1/(k+3)} + |Y_2|^{1/(2k+2)} + |Y_3|^{1/(3k+1)} = 1\right\}$$

homeomorphic to the 3D sphere.

Note that for any point $Y = (Y_0, Y_1, Y_2, Y_3) \in \mathbb{R}^4 \setminus \{0\}$ there exists a unique constant $B > 0$ such that $(B^\alpha Y_0, B^{\alpha+1} Y_1, B^{\alpha+2} Y_2, B^{\alpha+3} Y_3) \in E$. The related mapping $\Psi\colon \mathbb{R}^4 \setminus \{0\} \to E$ defined as

$$\Psi(Y_0, Y_1, Y_2, Y_3) = (B^\alpha Y_0, B^{\alpha+1} Y_1, B^{\alpha+2} Y_2, B^{\alpha+3} Y_3)$$

with the appropriate $B > 0$ is surely continuous.

Next, consider the subset

$$E_0 = \{Y = (0, Y_1, Y_2, Y_3) \in E\colon\ Y_1 \geqslant 0,\ Y_2 \geqslant 0,\ Y_3 \geqslant 0\} \subset E$$

and the composition $-\Psi \circ F|_{E_0}$ continuously mapping E_0 into itself. Since E_0 is homeomorphic to the 2D disk, we can apply Brouwer's fixed-point theorem. This yields the existence of a solution $y(x)$ to equation (3.1) on a segment $[x_0; x_1]$ satisfying the conditions $y(x_0) = y(x_1) = 0$ and, with some constant $B > 0$,

$$-B^{\alpha+1} y'(x_1) = y'(x_0), \quad -B^{\alpha+2} y''(x_1) = y''(x_0), \quad -B^{\alpha+3} y'''(x_1) = y'''(x_0).$$

Note that for any solution $y_1(x)$ to equation (3.1) the function $y_2(x)$ defined as $\pm b^\alpha y_1(bx + C)$ with arbitrary constants $b > 0$ and C is also a solution to this equation and $y_2^{(j)}(x) = \pm b^{\alpha+j} y_1^{(j)}(bx + C)$ for all $j = 0, \dots, n$.

Hence, the solution $y(x)$ obtained before can be smoothly extended to the segment $[x_1; x_2]$ with $x_2 = x_1 + B(x_1 - x_0)$ by the formula

$$y(x) = -B^{-\alpha} y\Big(\frac{x - x_1}{B} + x_0\Big), \quad x \in [x_1; x_2], \quad \frac{x - x_1}{B} + x_0 \in [x_0; x_1].$$

The extended solution satisfies $y(x_1) = y(x_2) = 0$ and

$$-B^{\alpha+1} y'(x_2) = y'(x_1), \quad -B^{\alpha+2} y''(x_2) = y''(x_1), \quad -B^{\alpha+3} y'''(x_2) = y'''(x_1).$$

So, the process can be repeated infinitely many times with the same constant $B > 0$. Similarly, the solution can be extended to the left. The domain of the extended solution is the union of the segments $[x_j; x_{j+1}]$, $j \in \mathbb{Z}$, and for all j we have

$$y(x_j) = 0,$$

$$x_{j+1} - x_j = B(x_j - x_{j-1}),$$
$$-B^{\alpha+i} y^{(i)}(x_{j+1}) = y^{(i)}(x_j), \quad i = 1, 2, 3,$$

and

$$y(x) = -B^{-\alpha} y\Big(\frac{x - x_j}{B} + x_{j-1}\Big), \quad x \in [x_j; x_{j+1}]. \tag{3.2}$$

In order to investigate whether B is greater or less than 1, denote by x_j''', x_j'', x_j' the points in $(x_{j-1}; x_j)$ where the related derivatives of the solution vanish. Now the following inequalities will be proved:

$$|y(x_j')| < |y(x_{j+1}''')| < |y(x_{j+1}'')| < |y(x_{j+1}')|, \tag{3.3}$$
$$|y'(x_j'')| < |y'(x_j)| < |y'(x_{j+1}''')| < |y'(x_{j+1}'')|, \tag{3.4}$$
$$|y''(x_j''')| < |y''(x_j')| < |y''(x_j)| < |y''(x_{j+1}''')|, \tag{3.5}$$
$$|y'''(x_j)| < |y'''(x_{j+1}'')| < |y'''(x_{j+1}')| < |y'''(x_{j+1})|. \tag{3.6}$$

Indeed,

$$\frac{1}{k+1}(|y(x_j')|^{k+1} - |y(x_{j+1}''')|^{k+1}) = -\int_{x_j'}^{x_{j+1}'''} y'(x)|y(x)|^{k-1} y(x)\,\mathrm{d}x$$
$$= \int_{x_j'}^{x_{j+1}'''} y'(x) y^{\mathrm{IV}}(x)\,\mathrm{d}x = y'(x) y'''(x)\Big|_{x_j'}^{x_{j+1}'''} - \int_{x_j'}^{x_{j+1}'''} y''(x) y'''(x)\,\mathrm{d}x < 0$$

since $y'(x'_j) = y'''(x'''_{j+1}) = 0$ and $y''(x)y'''(x) > 0$ on $[x'_j; x'''_{j+1}]$. This yields the first of inequalities (3.3), while the others follow from $y(x)y'(x) > 0$ on (x'''_{j+1}, x'_{j+1}).

Similarly, for the first inequality of (3.4),

$$\begin{aligned} y'(x''_j)^2 - y'(x_j)^2 &= -2 \int_{x''_j}^{x_j} y'(x)y''(x)\,\mathrm{d}x \\ &= -2y(x)y''(x)\Big|_{x''_j}^{x_j} + 2 \int_{x''_j}^{x_j} y(x)y'''(x)\,\mathrm{d}x < 0 \end{aligned}$$

since $y(x_j) = y''(x''_j) = 0$ and $y(x)y'''(x) < 0$ on $[x''_j; x_j]$. The others follow from $y'(x)y''(x) > 0$ on (x_j, x''_{j+1}).

Inequalities (3.5) and (3.6) can be proved just in the same way. The inequalities proved show that for the solution considered the sequence

$$|y(x_j)|^{1/4} + |y'(x_j)|^{1/(k+3)} + |y''(x_j)|^{1/(2k+2)} + |y'''(x_j)|^{1/(3k+1)}$$

is strictly increasing, whence $B < 1$ and the solution $y(x)$ is defined in non-extensible way on the semi-axis $(-\infty; x_*)$ with $x_* = x_0 + (x_1 - x_0)/(1 - B) = x_{j-1} + (x_j - x_{j-1})/(1 - B)$, $j \in \mathbb{Z}$.

Since equation (3.1) is invariant under substitutions $x \mapsto -x$ and $x \mapsto x + a$ with any $a \in \mathbb{R}$, similar solutions can be defined on the semi-axes $(-\infty; x_*)$ and $(x_*; \infty)$ with arbitrary x_*.

Now the function $h(t)$ can be defined as $h(t) = \mathrm{e}^{\alpha t} y\left(x_* - \mathrm{e}^t\right)$. Its periodicity is proved by straightforward calculations. Indeed, if $x_* - \mathrm{e}^t \in [x_j; x_{j+1}]$ for some $j \in \mathbb{Z}$, then

$$h(t - \log B) = \mathrm{e}^{\alpha t} B^{-\alpha} y\Big(x_* - \frac{\mathrm{e}^t}{B}\Big)$$

and, according to (3.2),

$$h(t) = \mathrm{e}^{\alpha t} y(x_* - \mathrm{e}^t) = -\mathrm{e}^{\alpha t} B^{-\alpha}\; y\Big(\frac{x_* - \mathrm{e}^t - x_j}{B} + x_{j-1}\Big).$$

The expression in the last parentheses is equal to

$$\frac{x_{j-1} + \dfrac{x_j - x_{j-1}}{1 - B} - \mathrm{e}^t - x_j}{B} + x_{j-1} = -\frac{x_{j-1}}{1 - B} + \frac{x_j}{1 - B} - \frac{\mathrm{e}^t}{B} + x_{j-1} = x_* - \frac{\mathrm{e}^t}{B}.$$

So, we have $h(t - \log B) = -h(t)$ for all $t \in \mathbb{R}$ and the function $h(t)$ is periodical with period $-2\log B$. □

It was proved [2]–[4], [7] that equation (1.1) with $n > 2$, under some conditions on the function $p(x, y_0, \ldots, y_{n-1})$, has oscillatory solutions. The following theorem was also proved [2]–[4]:

Theorem 3.2. *For $n = 3$ there exists a constant $B \in (0, 1)$ such that any oscillatory solution $y(x)$ of* (1.2) *satisfies the conditions*

(1) $(x_{i+1} - x_i)/(x_i - x_{i-1}) = B^{-1}, \quad i = 2, 3, \ldots,$
(2) $y(x'_{i+1})/y(x'_i) = -B^{\alpha}, \quad i = 1, 2, 3, \ldots,$
(3) $y'(x_{i+1})/y'(x_i) = -B^{\alpha+1}, \quad i = 1, 2, 3, \ldots,$
(4) $|y(x'_i)| = M(x'_i - x_*)^{-\alpha}, \quad i = 1, 2, 3, \ldots,$

for certain $M > 0$ and x_*, where $x_1 < x_2 < \ldots < x_i < \ldots$ and $x'_1 < x'_2 < \ldots < x'_i < \ldots$ are the sequences satisfying $y(x_j) = 0$, $y'(x'_j) = 0$, $y(x) \neq 0$ if $x \in (x_i, x_{i+1})$, $y'(x) \neq 0$ if $x \in (x'_i, x'_{i+1})$.

Now, with help of this theorem, the method of proving Theorem 3.1 can be applied to obtain

Theorem 3.3. *For $n = 3$ and any real $k > 1$ there exists an oscillatory periodic function $h\colon \mathbb{R} \to \mathbb{R}$ such that the functions $x \mapsto |x - x_*|^{-\alpha} h(\log |x - x_*|)$ with $\alpha = n/(k-1)$ and arbitrary x_* are solutions respectively to* (3.1) *if defined on $(-\infty; x_*)$ and to* (1.2) *if defined on $(x_*; \infty)$.*

References

[1] *I. V. Astashova*: On power and non-power asymptotic behavior of positive solutions to Emden-Fowler type higher-order equations. Adv. Difference Equ. *2013* (2013), Article No. 2013:220, 15 pages.
[2] *I. V. Astashova*: Qualitative properties of solutions to quasilinear ordinary differential equations. Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition (I. V. Astashova, ed.). UNITY-DANA, 2012, pp. 22–290. (In Russian.)
[3] *I. V. Astashova*: Application of dynamical systems to the study of asymptotic properties of solutions to nonlinear higher-order differential equations. J. Math. Sci., New York *126* (2005), 1361–1391; translated from Sovrem. Mat. Prilozh. *8* (2003), 3–33. (In Russian.)
[4] *I. V. Astashova*: On asymptotic behavior of oscillatory solutions of some nonlinear differential equations of the third and forth order. Reports of extended session of a seminar of the I. N. Vekua Institute of Applied Mathematics 3. Tbilisi, 1988, pp. 9–12. (In Russian.)
[5] *I. V. Astashova*: Asymptotic behavior of solutions of certain nonlinear differential equations. Reports of the extended sessions of a seminar of the I. N. Vekua Institute of Applied Mathematics I (I. T. Kiguradze, ed.). Tbilis. Gos. Univ., Tbilisi, 1985, pp. 9–11. (In Russian.)
[6] *I. V. Astashova, S. A. Vyun*: On positive solutions with non-power asymptotic behavior to Emden-Fowler type twelfth order differential equation. Differ. Equ. *48* (2012), 1568–1569. (In Russian.)
[7] *I. T. Kiguradze, T. A. Chanturia*: Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Mathematics and Its Applications (Soviet Series) 89, Kluwer Academic Publishers, Dordrecht, 1993, translated from the Russian original, Nauka, Moskva, 1985.

[8] *V. A. Kozlov*: On Kneser solutions of higher order nonlinear ordinary differential equations. Ark. Mat. *37* (1999), 305–322.
[9] *J. E. Marsden, M. McCracken*: The Hopf Bifurcation and Its Applications. With contributions by P. Chernoff et al. Applied Mathematical Sciences 19, Springer, New York, 1976.

Author's addresses: *Irina Astashova*, Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Department of Differential Equations, GSP-1, 1 Leninskie Gory, Main Building, 119 991 Moscow, Russia; and Moscow State University of Economics, Statistics and Informatics, Department of Higher Mathematics, Nezhinskaya 7, 119 501 Moscow, Russia, e-mail: ast@diffiety.ac.ru.