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PARABOLIC EQUATIONS WITH ROUGH DATA

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Abstract. We survey recent work on local well-posedness results for parabolic equations and systems with rough initial data. The design of the function spaces is guided by tools and constructions from harmonic analysis, like maximal functions, square functions and Carleson measures. We construct solutions under virtually optimal scale invariant conditions on the initial data. Applications include BMO initial data for the harmonic map heat flow and the Ricci-DeTurck flow for initial metrics with small local oscillation. The approach is sufficiently flexible to apply to boundary value problems, quasilinear and fully nonlinear equations.

Keywords: parabolic equation; rough initial data

MSC 2010: 35K59, 53C44

1. Introduction

In this paper we survey recent work on the initial value problem for parabolic equations in a fairly broad sense. This new approach is based on basic notions in harmonic analysis like maximal function, square function, and Carleson measures. The design of the function spaces we use is modeled on maximal functions and square functions, where the version we use incorporates the regularity theory for the corresponding linear parabolic equations.

We consider it to be an appealing feature that the first local existence statement can be formulated without using function spaces, while being essentially optimal in terms of the regularity of the initial data needed, see Theorem 1.1 below.

Our proofs make only use of fairly general properties of linear equations with constant coefficients: (Gaussian) decay of the kernel and a version of the Calderon-Zygmund estimates. Moreover, the arguments are almost local in space for local in time solutions. In the flat small data situation, this idea has first been used in Koch and Tataru [20] and, closer to the core of this survey, by the authors in [19].

One of the main observations is that the methods we use are flexible enough to handle initial boundary value problems in half-spaces, parabolic systems, subelliptic parabolic equations, and higher order parabolic equations. Subelliptic parabolic equations occur in the context of the porous medium equation (see [9] and the thesis of C. Kienzler [18]) and in the context of thin films (see the thesis of D. John [17]).

It seems natural to study parabolic equations on uniform manifolds—i.e., manifolds with a metric and an atlas corresponding to balls of size one for which all the coordinate changes are uniformly in C^1 with uniform modulus of continuity. This concept of uniform manifolds has been introduced by Denzler, Koch and McCann [11] and it was recently used by Shao and Simonett [26] and Shao [25].

It is a consequence of our results—and basically this result can also be found in the papers of Whitney [35] and Kotschwar [22]—that these manifolds carry a uniform analytic metric: there is an atlas corresponding to balls of diameter 1 and a metric g such that all coordinate changes φ_{ij} satisfy bounds

$$\begin{aligned} |\partial_x^\alpha \varphi_{ij}| &\leqslant cR^{-|\alpha|}|\alpha|!, \\ |\partial_x^\alpha g^{ij}| &\leqslant cR^{-|\alpha|}|\alpha|! \end{aligned}$$

where c and R are independent of α.

Initial boundary value problems fit into the framework of uniform structures: Consider a bounded domain with a smooth boundary. Locally we can flatten the boundary, and we obtain a "uniform" structure in the spirit as discussed above.

In the following, we discuss several examples which we consider instructive and interesting.

Consider the equation

$$u_t - \sum_{i,j=1}^d \partial_i a^{ij}(t,x,u)\partial_j u = f(t,x,u,\nabla u) \tag{1.1}$$

in $\mathbb{R}^d$, where a^{ij} and f are continuous functions satisfying

$$\lambda^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^d a^{ij}(t,x,u)\xi_i\xi_j \quad \text{and} \quad |a^{ij}| \leqslant \lambda$$

for some $\lambda > 1$, uniformly for all t, x, u and ξ. The coefficients are not assumed to be symmetric.

The basic regularity assumption with respect to x and t is the requirement of *locally small oscillation:* There exists δ depending on λ, and $T > 0$ with

$$|a^{ij}(t,x,u) - a^{ij}(s,y,u)| \leqslant \delta \quad \text{for all } 0 \leqslant s,\ t \leqslant T,\ |x-y| \leqslant \sqrt{T}.$$

We assume *Lipschitz continuity* with respect to u: There exists L with

$$|a^{ij}(t,x,u) - a^{ij}(t,x,v)| \leqslant L|u-v|.$$

The nonlinearity f is assumed to be quadratic in the last component. There is a small parameter ε and we assume

$$|f(t,x,u,0)| \leqslant \varepsilon/T$$

and

$$|f(t,x,u,p) - f(t,x,v,q)| \leqslant c\big(|u-v|(\varepsilon/T + L|p|^2) + (\varepsilon/\sqrt{T} + L(|p|+|q|))|p-q|\big).$$

Higher regularity: Let $k \geqslant 1$ be a regularity index. The derivatives of a^{ij} with respect to x and u of order k are uniformly bounded:

$$T^{|\alpha|/2}|\partial_x^\alpha \partial_u^j a^{ij}| \leqslant L$$

and

$$T^{1+|\alpha|/2-|\beta|/2}|\partial_x^\alpha \partial_u^j \partial_p^\beta f| \leqslant L(1 + |T^{1/2}p|^{(2-|\beta|)_+}),$$

for $|\alpha| + j + |\beta| \leqslant k$.

Theorem 1.1. *There exists $\delta > 0$, and for all $L > 0$ there is $\varepsilon_0 > 0$ such that, if for $T > 0$*

$$|u_0(x) - u_0(y)| \leqslant \varepsilon < \varepsilon_0 \quad \text{for } |x-y| \leqslant \sqrt{T}$$

and the assumptions above are satisfied, then there is a unique continuous solution u up to time T which satisfies

$$|(t^{1/2}\partial_x)^\alpha u(t,x)| \leqslant c_\alpha \varepsilon$$

for $|\alpha| \leqslant k$. The solution is analytic with respect to x, if a^{ij} and f are analytic with respect to x, u and Du. If a^{ij} and f are analytic with respect to all variables, then there exist c and R such that

$$|(t^{1/2}\partial_x)^\alpha (t\partial_t)^j u(t,x)| \leqslant c(|\alpha|+j)! R^{|\alpha|+j}\varepsilon.$$

Examples of equations and systems of the above type are the harmonic map heat flow, the viscous Hamilton-Jacobi equation, the Ricci-DeTurck flow and the fast diffusion equations for the relative size with respect to the Barenblatt solution. In all of these cases, continuous initial data are natural and essentially optimal, which can be seen by the examples below.

2. The fixed point formulation

We construct the solution of the parabolic equation as a fixed point using Duhamel's formula. For this we consider the abstract equation

$$u_t = Au + f[u],$$

where A is the generator of a semigroup $S(t)$. If there are function spaces X_0, X and Y such that

$$\|S(t)u_0\|_X \leqslant c\|u_0\|_{X_0} \tag{2.1}$$

$$\left\| \int_0^t S(t-s)f(s)\,\mathrm{d}s \right\|_X \leqslant c\|f\|_Y \tag{2.2}$$

$$\|f[u]-f[v]\|_Y \leqslant c(\|u\|_X + \|v\|_X + \delta)\|u-v\|_X, \tag{2.3}$$

then it is standard to deduce existence and uniqueness by the contraction mapping principle.

Alternatively, existence of the fixed point follows from the implicit function theorem, provided the maps are differentiable. The contraction property implies invertibility of the linearization. This has an important consequence: The solution depends smoothly on parameters, if the nonlinear functions are smooth, and it depends analytically, if the functions are analytic.

Possible and popular choices are

- ▷ Hölder spaces $C^\alpha(\Omega)$ and $C^{\alpha/2,\alpha}([0,T)\times\Omega)$ (see [26] for a recent contribution, discussion and references),
- ▷ the Sobolev space $X = W^{1,2,p}([0,T)\times\Omega)$, $X_0 = W^{2-2/p,p}(\Omega)$ of functions with one time and two spacial derivatives in L^2, $Y = L^p$, $p > n+2$.

To motivate our choice we take a look at fundamental objects in harmonic analysis. Consider the heat equation

$$u_t = \Delta u, \quad u(0,x) = v(x).$$

A nontangential maximal function is given by

$$Mv(x) = \sup_{t,\,|h|^2\leqslant t} |u(x+h,t)|,$$

which has the variant for $k \geqslant 0$ and $p \in [1,\infty]$

$$Mv(x) = \sup_R R^k \left(R^{-d-2} \int_{R^2/2}^{R^2} \int_{B_R(x)} |D_x^k u|^p \,\mathrm{d}x\,\mathrm{d}t \right)^{1/p}.$$

The basic property is

$$\|Mv\|_{L^p} \leqslant c\|v\|_{L^p} \quad \text{for } 1 < p \leqslant \infty \quad \text{and} \quad \|v\|_{L^p} \leqslant c\|Mv\|_{L^p} \quad \text{if } 1 < p < \infty.$$

For $p = \infty$ there is a substitute via the square function

$$\|v\|_{\mathrm{BMO}} \sim \sup_{x,R} \left(R^{-d} \int_0^{R^2} \int_{B_R(x)} |\nabla u|^2 \, \mathrm{d}y \, \mathrm{d}t \right)^{1/2}.$$

The right hand side is a Carleson measure type expression.

These tools have been used in the study of function spaces, but also for the solution of the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [4] and the study of harmonic functions in Lipschitz domains by Jerison, Kenig [16] and others.

The Carleson measure formulation of the BMO norm (or more precisely the BMO^{-1} norm) turned out to be a crucial ingredient in the study of the Navier-Stokes equations with initial data in BMO^{-1} by Koch and Tataru [20]. More recently, the authors applied these concepts to geometric problems including the harmonic map heat flow, the Ricci-DeTurck flow, and the mean curvature and Willmore flow for Lipschitz graphs, see [19].

In order to study equations of the form (1.1), we pick $p > n + 2$ and $q = p/2$. Moreover, we let $T > 0$ and define the norms

$$\|u_0\|_{X_0} = \|u_0\|_{\sup}$$

and

$$\begin{aligned} \|u\|_X = \sup_{x, t \leqslant T} |u(t,x)| &+ \sup_{x, R^2 < T} R \left(R^{-d-2} \int_{R^2/2}^{R^2} \int_{B_R(x)} |\nabla u|^p \, \mathrm{d}y \, \mathrm{d}t \right)^{1/p} \\ &+ \sup_{x, R^2 < T} \left(R^{-d} \int_0^{R^2} \int_{B_R(x)} |\nabla u|^2 \, \mathrm{d}y \, \mathrm{d}t \right)^{1/2}. \end{aligned}$$

Here the second line is similar to the L^∞ norm of a maximal function, and the last line corresponds to a Carleson measure.

Additionally, we consider nonlinearities

$$f[u] = f_0(u, \nabla u) + \partial_i F^i(u, \nabla u)$$

and decompose

$$\|f\|_Y = \|f_0\|_{Y^0} + \|F\|_{Y^1},$$

where
$$\|f_0\|_{Y^0} = \sup_{x,R^2<T} R\left(R^{-(d+2)/2}\int_{R^2/2}^{R^2}\int_{B_R(x)} |f_0|^q \,\mathrm{d}y\,\mathrm{d}t\right)^{1/q} + \sup_{x,R^2<T} R^{-d}\int_0^{R^2}\int_{B_R(x)} |f_0|\,\mathrm{d}y\,\mathrm{d}t$$
and
$$\|F\|_{Y^1} = \sup_{x,R^2<T} R\left(R^{-d-2}\int_{R^2/2}^{R^2}\int_{B_R(x)} |F|^p \,\mathrm{d}y\,\mathrm{d}t\right)^{1/p} + \sup_{x,R^2<T} \left(R^{-d}\int_0^{R^2}\int_{B_R(x)} |F|^2\,\mathrm{d}y\,\mathrm{d}t\right)^{1/2}.$$

Now we construct a function $w\colon \mathbb{R}^d \to \mathbb{R}$ such that $\|w-u_0\|_{\sup}$ and $\|\nabla w\|_{\sup}$ are small in terms of ε, and we look for u solving

$$u_t - \sum_{i,j=1}^d \partial_i a^{ij}(t,x,w)\partial_j u = f(t,x,u,\nabla u) + \sum_{i,j=1}^d \partial_i(a^{ij}(t,x,u) - a^{ij}(t,x,w))\partial_j u.$$

The estimate
$$\|u\|_X \leqslant c\|u_0\|_{X_0}$$
follows from standard kernel estimates. The estimates

$$\|f(u)-f(v)\|_{Y^0} \leqslant c(\|u\|_X + \|v\|_X + \delta)\|u-v\|_X$$

and the bound for $\|(a^{ij}(t,x,u)-a^{ij}(t,x,w))\partial_j u\|_{Y_1}$ are true by construction.

By scaling and the kernel estimates, if

$$u_t - \sum_{i,j=1}^d \partial_i a^{ij}\partial_j u = f + \sum_{i=1}^d \partial_i F^i$$

with $u(0)=0$, then
$$|u(0,1)| \leqslant c[\|f\|_{Y^0} + \|F\|_{Y^1}].$$
Energy estimates (plus kernel estimates) give

$$\left(\int_0^1\int_{B_1(0)} |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \leqslant c[\|f\|_{Y^0} + \|F\|_{Y^1}].$$

Finally, kernel estimates and Calderon-Zygmund theory imply

$$\left(\int_{1/2}^1\int_{B_1(0)} |\nabla u|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \leqslant c[\|f\|_{Y^0} + \|F\|_{Y^1}].$$

Hence, we are in the above mentioned abstract setting, in which the existence and uniqueness of a solution of (1.1) in X follows from a fixed point argument.

3. Regularity and uniqueness

We will prove regularity and existence of derivatives via the implicit function theorem. For simplicity we do that for analyticity, which goes back to Angenent (see [1], [2]). Consider

$$u_t - \Delta u = \Gamma(u)|\nabla u|^2$$

for some analytic and bounded function Γ. Define

$$u^{s,a}(t,x) = u(st, x + ta).$$

It satisfies

$$u_t - s\Delta u + a\nabla u = \Gamma(u)|\nabla u|^2,$$

which is analytic in a (for a close to zero) and s (for s close to 1). We construct the solution by the implicit function theorem. Thus $(s,a) \mapsto u^{s,a} \in X$ is analytic. The evaluation of a derivative is linear, hence for all t and x the map

$$(s,a) \to \nabla u^{s,a}(t,x)$$

is analytic. But

$$t\partial_t u = \partial_s u^{s,a}|_{s=1,a=0} \quad \text{and} \quad t\partial_j u = \partial_{a_j} u^{s,a}|_{s=1,a=0},$$

with corresponding formulas for higher derivatives.

This argument can be localized as follows: The map

$$x \to x + ta$$

is the flow map of the constant vector field $a \in \mathbb{R}^d$. This is clearly analytic with respect to a. We fix the analytic vector fields (for given a)

$$X = (1 - |x|^2)a$$

They generate a flow which is analytic with respect to x and a. The vector field vanishes at $|x| = 1$. Hence also

$$X_+ = (1 - |x|^2)_+ a$$

generates a C^∞ flow which is analytic with respect to the parameter $a \in \mathbb{R}^d$. This argument shows that analyticity with respect to x is a local problem—in contrast to analyticity in time: The fundamental solution is smooth but not analytic at $t = 0$ and $x \in \mathbb{R}^d$, $x \neq 0$.

We need slightly more for Theorem 1.1. The properties on the nonlinearity are too weak for a direct implementation of this argument. In order to overcome this difficulty, we first obtain bounds for the derivatives, and then we implement this argument on the second half of the time interval.

Let us further comment on the uniqueness result claimed in Theorem 1.1. The fixed point map gives a unique fixed point in X, but Theorem 1.1 claims uniqueness for weak solutions satisfying

$$t^{1/2}\|\nabla u\|_{L^\infty} \leqslant c_1\varepsilon,$$

which does not imply the Carleson measure bound. Let u be a solution as in the theorem. For $t > 0$ we can solve the initial value problem for the initial data $u(t)$. It is unique, and hence the shifted solution is uniformly bounded in X. The limit $t \to 0$ shows that the solution is in X and hence unique.

The above framework allows to deal with rougher initial data. It is obvious that we may allow small perturbations of the initial data in L^∞. We may also allow small BMO perturbations, if we require that all the structure assumptions hold uniformly in u. Here we take a caloric extension w of the initial data, and make the ansatz $u = w + v$. Then we apply a fixed point argument in order to find a function v such that u is a solution of our problem.

4. Modifications and generalizations

4.1. Uniform manifolds and initial boundary value problems. Parabolic equations have an infinite speed of propagation but heat kernels have Gaussian decay. Therefore we only need local in time estimates if we want to construct local solutions. The simplest version is for uniformly small local oscillations as in the above theorem. This result can be extended to uniform manifolds: We only need uniform local coordinate maps. The uniqueness argument is elementary but delicate.

The estimates mentioned at the end of Section 2 required Calderon-Zygmund type estimates and pointwise bounds of the heat kernel. Both are available for boundary value problems in a half-space.

Now consider a parabolic equation in a bounded domain with smooth boundary. Locally we can flatten the boundary. We take the half-space problem as model, and consider the bounded domain with smooth boundary as a uniform manifold.

This allows to deal with Dirichlet boundary conditions and conormal boundary conditions

$$\sum_{i,j=1}^{d} \nu_i a^{ij}(t,x,u)\partial_j u = \sum_i g_i(t,x,u)\partial_i u + f(t,x,u),$$

where we assume (with ν denoting the exterior normal vector)

$$\sum_{i=1}^{n} g_i(t,x,u)\nu_i(x) = 0,$$

which expresses that (g_i) has values in the tangent space of the boundary.

4.2. Systems. The same arguments apply to systems of equations

$$u_t^k - \sum_{i,j=1}^{d} \sum_l \partial_i a_{kl}^{ij}(t,x,u)\partial_j u^l = f^k(t,x,u,\nabla u),$$

as soon as the Calderon-Zygmund estimates and the Gaussian estimates are available.

A sufficient condition is that

$$\sum_{i,j,k,l} a_{kl}^{ij} A_i^k A_j^l \geqslant \lambda^{-1}|A|^2$$

holds uniformly. This implies that

$$\int \sum_{i,j,k,l} \tilde{a}_{kl}^{ij}(x)\partial_i\varphi^k\partial_j\varphi^l \,\mathrm{d}x \geqslant \lambda^{-1}\||\nabla\varphi|\|_{L^2}^2, \tag{4.1}$$

where $\tilde{a}_{kl}^{ij}(x) = a_{kl}^{ij}(t,x,v(t,x))$ and $v(t,x)$ is the mean of the initial data on a ball of radius $\sqrt{t}$ and $0 < t \leqslant T$.

The positivity condition (4.1) implies rank-1 positivity,

$$\sum_{i,j,k,l} a_{kl}^{ij}\eta^k\eta^l\xi_i\xi_j \geqslant \lambda^{-1}|\xi|^2|\eta|^2.$$

On the other hand, for uniformly continuous coefficients, rank-1 positivity implies

$$\int \sum_{i,j,k,l} \tilde{a}_{kl}^{ij}(x)\partial_i\varphi^k\partial_j\varphi^l \,\mathrm{d}x \geqslant (2\lambda)^{-1}\|\nabla\varphi\|_{L^2}^2 - C\|\varphi\|_{L^2}^2, \tag{4.2}$$

but for discontinuous coefficients no good algebraic characterization of the coefficients satisfying (4.2) seems to be available.

In any case, (4.1) for $a^{ij}_{kl}(t,x) = a^{ij}_{kl}(t,x,w(x))$, where w is the heat extension of the initial data, is enough to handle small BMO perturbations of uniformly continuous initial data. The situation is different for small L^∞ perturbations. Here we deal with small perturbations of uniformly continuous coefficients and rank-1 positivity suffices.

There is an important and natural weaker notion of parabolicity. Assume that the coefficients a^{ij}_{kl} have values in a compact set K of tensors. We call the equation parabolic if in this compact set K, for all $\xi \in \mathbb{R}^d \setminus \{0\}$, the matrix

$$A_{kl}(\xi) = \sum_{i,j=1}^{d} a^{ij}_{kl}\xi_i\xi_j$$

has its spectrum in the open left complex half-plane.

The situation of boundary value problems is considerably more complex. Again, positivity is sufficient, and for Dirichlet boundary conditions this is again equivalent to rank-1 positivity. In general, no good characterization of positivity is known, but there are many important sufficient conditions, see Simpson and Spector [29].

Again, parabolicity in the sense of Solonnikov [30] is sufficient for an analogue of Theorem 1.1 for initial boundary value problems for systems.

4.3. More derivatives. Consider the parabolic equation

$$u_t - \sum_{i,j=1}^{d} a^{ij}(t,x,u,\nabla u)\partial^2_{ij}u = f(t,x,u,\nabla u)$$

in $\mathbb{R}^d$ and let $T > 0$ be given. We assume boundedness with a parameter ε

$$\|a^{ij}\|_{\sup} \leqslant \lambda, \quad \|f\|_{\sup} \leqslant \varepsilon T^{-1/2},$$

parabolicity

$$\sum_{i,j=1}^{d} a^{ij}(t,x,u,p)\xi_i\xi_j \geqslant |\xi|^2/\lambda,$$

and Lipschitz continuity

$$|a^{ij}(t,x,u,p) - a^{ij}(t,x,v,q)| \leqslant L(|p-q| + T^{-1/2}|u-v|),$$
$$|f(t,x,u,p) - f(t,x,v,q)| \leqslant \varepsilon(T^{-1/2}|p-q| + T^{-1}|u-v|).$$

We assume again locally small oscillation: There exists δ depending only on λ with

$$\sup_{|x-y|\leqslant\sqrt{T}, 0\leqslant t,s\leqslant T} |a^{ij}(t,x,u,p) - a^{ij}(s,y,v,q)| \leqslant \delta$$

and regularity with $k \geqslant 1$,

$$T^{|\alpha|/2+l/2}|\partial_x^\alpha \partial_u^l \partial_p^\beta a^{ij}| \leqslant c \tag{4.3}$$

and, with $k \geqslant 1$,

$$T^{1/2+|\alpha|/2+l/2}|\partial_x^\alpha \partial_u^l \partial_p^\beta f| \leqslant c \tag{4.4}$$

for $|\alpha| \leqslant k$.

Theorem 4.1. *There exists $\delta > 0$ such that for all $L > 0$ there is $\varepsilon_0 > 0$ such that if $T > 0$,*

$$|\nabla u_0(x) - \nabla u_0(y)| \leqslant \varepsilon \leqslant \varepsilon_0 \quad \text{for } |x-y| \leqslant \sqrt{T}$$

and if the assumptions above are satisfied, then there is a unique continuous solution u up to time T, which satisfies

$$t^{-1/2}|(t^{1/2}\partial_x)^\alpha u(t,x)| \leqslant c_\alpha \varepsilon$$

for $1 \leqslant |\alpha| \leqslant 1+k$. The solution is analytic with respect to x, if a^{ij} and f are analytic. If a^{ij} and f are analytic with respect to all variables, then there exist c and R such that

$$t^{-1/2}|(t^{1/2}\partial_x)^\alpha (t\partial_t)^j u(t,x)| \leqslant c(|\alpha|+j)!R^{-(|\alpha|+j)}\varepsilon$$

for $|\alpha| + j \geqslant 1$, where c and R are independent of x, t, j and α.

The mean curvature flow in arbitrary codimension provides an example of this structure. Here, bounded first derivatives seem to be appropriate if one wants to deal with the flow for graphs, see e.g. [33], [34]. Note that Šverák [31] has constructed Lipschitz continuous singular solutions to the stationary problem. These solutions indicate either that solutions to the parabolic equation become nonunique, or that the smallness condition for the initial data is needed for solutions in the function space X.

Similarly we deal with the fully nonlinear equation

$$u_t - F(t,x,u,\nabla u,\nabla^2 u) = 0$$

with initial data in $C^{1,1}$. We assume Lipschitz continuity

$$|F(t,x,u,p,A) - F(t,x,u,p,B)| \leqslant \lambda |A-B|$$

and ellipticity

$$F(t,x,u,p,A+B) - F(t,x,u,p,A) \geqslant \lambda^{-1}\lambda_{\min}(B),$$

where A is symmetric and B positive definite with $\lambda_{\min}$ denoting the smallest eigenvalue. The condition of locally small oscillation takes the form:

$$\begin{aligned}(4.5)\qquad \sup_{|s-t|\leqslant T, |x-y|\leqslant\sqrt{T}} &|F(t,x,.,.,A+H) - F(t,x,.,.,A) \\ &- (F(s,y,.,.,A+H) - F(s,y,.,.,A))| \leqslant \delta|H|.\end{aligned}$$

The Lipschitz condition involving a small parameter ε is

$$|F(t,x,u,p,A) - F(t,x,v,q,A)| \leqslant \frac{\varepsilon}{T}|u-v| + \frac{\varepsilon}{\sqrt{T}}|p-q|$$

and

$$|F(t,x,u,p,0)| \leqslant \varepsilon.$$

Let $k \geqslant 1$. The higher regularity condition is

$$|(T^{1/2}\partial_x)^\alpha (T\partial_u)^l (T\partial_p)^\beta (\partial_A)^\gamma F| \leqslant L$$

for $l + |\alpha| + |\beta| + |\gamma| \leqslant k$.

Theorem 4.2. *There exists $\varepsilon_0 > 0$ such that if*

$$|D^2u_0(x) - D^2u_0(y)| \leqslant \varepsilon \leqslant \varepsilon_0 \quad \textit{for } |x-y| \leqslant \sqrt{T},$$

then there is a unique continuous solution u up to time T, which satisfies

$$t^{-1}|(t^{1/2}\partial_x)^\alpha u(t,x)| \leqslant c_\alpha \varepsilon$$

for $2 \leqslant |\alpha| \leqslant 2+k$. The solution is analytic with respect to x, if a^{ij} and F are analytic. If a^{ij} and F are analytic with respect to all variables, then there exist c and R such that

$$t^{-1}|(t^{1/2}\partial_x)^\alpha (t\partial_t)^j u(t,x)| \leqslant c(|\alpha|+j)!R^{-(|\alpha|+j)}\varepsilon,$$

where c and R are independent of x, t, j and α.

It is not clear whether the smallness condition is needed. Note, however, that Nadirashvili and Vlăduţ [24] and Nadirashvili, Tkachev and Vlăduţ [23] have constructed singular solutions in $C^{1,1}$. So again, either the parabolic flow is nonunique for large $C^{1,1}$ initial data, or the smallness assumption is needed.

5. Applications

5.1. Navier-Stokes equations. Consider the Navier-Stokes equations

$$u_t - \Delta u + u\nabla u + \nabla p = 0$$
$$\nabla u = 0$$

with divergence-free initial data u_0.

Let v be the caloric extension, i.e., the solution to the heat equation of the initial data u_0. The Carleson measure characterization of the BMO norm is

$$\|u_0\|_{\mathrm{BMO}} \sim \sup_{R,x} \left(R^{-d} \int_0^{R^2} \int_{B_R(x)} |\nabla v(t,y)|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{1/2},$$

which we use to define the local BMO^{-1} norm by

$$\|u_0\|_{\mathrm{BMO}_T^{-1}} \sim \sup_{R^2 \leqslant T, x} \left(R^{-d} \int_0^{R^2} \int_{B_R(x)} |v(t,y)|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{1/2}.$$

We also let

$$\|u\|_{X_T} = \|t^{1/2} |u(t,x)|\|_{\sup} + \sup_{x, R \leqslant \sqrt{T}} \left(R^{-d} \int_0^{R^2} \int_{B_R(x)} |u|^2 \,\mathrm{d}y \,\mathrm{d}t \right)^{1/2}.$$

Theorem 5.1. *There exists $\varepsilon > 0$ depending only on the space dimension d such that given u_0 with $\|u_0\|_{\mathrm{BMO}_T^{-1}} < \varepsilon$ there exists a unique solution $u \in X_T$ up to time T with*

$$\|u\|_{X_T} \leqslant c\|u_0\|_{\mathrm{BMO}^{-1}}.$$

The solution is a classical solution for $T > 0$. It assumes the initial data in the weak sense. See Koch and Tataru [20] for more details.

5.2. Hamilton-Jacobi equations and harmonic map heat flow. Consider

$$u_t - \sum_{i,j=1}^{d} \partial_i a^{ij}(x,u) \partial_j u = \sum_{i,j=1}^{d} f^{ij}(u) \partial_i u \partial_j u$$

on a bounded domain Ω with smooth boundary and homogeneous Dirichlet initial data, where the coefficients a^{ij} are bounded, uniformly elliptic, with uniformly bounded derivatives. Also f is supposed to be bounded with uniformly bounded derivatives. The harmonic map heat flow is a particular example, for which the coefficients a^{ij} are independent of u. In this form the type of the equations does not change when we change dependent and independent variables.

Theorem 5.2. *There exists ε such that the following is true: Let $\varphi_0 \in C(\overline{\Omega})$ satisfy $\varphi_0 = 0$ at the boundary. Then there exists $T > 0$ such that whenever*

$$\|u_0 - \varphi_0\|_{\mathrm{BMO}} \leqslant \varepsilon,$$

there is a unique smooth solution up to time T.

Here we use the heat extension with Dirichlet boundary conditions to define the space BMO. It is remarkable that the initial data is not required to satisfy the boundary condition.

Let us consider an example on $B_1(0) \subset \mathbb{R}^2$: We want to solve the equation

$$u_t - \Delta u = |\nabla u|^2$$

with initial data

$$u_0(x) = \ln(1 - \ln|x|),$$

which is in BMO.

Our results yield the existence of a unique smooth solution which assumes the initial data in a weak sense. It is remarkable that the constant map $u(t,x) = u_0$ is also a weak solution.

The harmonic map heat flow on $(0,T) \times \mathbb{R}^d$ has been considered previously by the authors [19], and with small BMO initial data by Wang [32]. We extend these results to uniform manifolds.

5.3. Ricci-DeTurck flow. The Ricci flow

$$\partial_t g = -2\,\mathrm{Ric}(g) \quad \text{in } M^n \times (0,T) \quad \text{and} \quad g(0,\cdot) = g_0, \tag{5.1}$$

is the most natural parabolic deformation of a metric on a Riemannian manifold. Due to the invariance under coordinate changes it is not parabolic. DeTurck [13] introduced a condition fixing the coordinates: He considered a Ricci flow coupled with the harmonic map heat flow with respect to a background metric. In local coordinates the Ricci-DeTurck flow can be written as

$$\begin{aligned}(\partial_t - \nabla_a g^{ab}\nabla_b) g_{ij} = &-\nabla_a g^{ab}\nabla_b g_{ij} - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\ &+ \frac{1}{2} g^{ab} g^{pq} \times (\nabla_i g_{pa}\nabla_j g_{qb} + 2\nabla_a g_{jp}\nabla_q g_{ib} - 2\nabla_a g_{jp}\nabla_b g_{iq} \\ &- 2\nabla_j g_{pa}\nabla_b g_{iq} - 2\nabla_i g_{pa}\nabla_b g_{jq}),\end{aligned}$$

where we use a fixed background metric h. This is a particular instance of Theorem 1.1, where we require that the initial metric lies in a compact convex set of positive definite matrices.

By Whitney's result [35] we may approximate a uniform C^1 Riemannian manifold by a uniform C^k Riemannian manifold. Altogether, using Theorem 1.1 we arrive at the following:

Theorem 5.3. *Let (M, g_0) be a uniform C^1 manifold with a uniformly continuous metric g_0. Choose an atlas which makes M a uniform C^3 manifold with h a C^2 background metric with uniformly bounded second derivatives. Then there exist $\varepsilon > 0$ (independent of g_0), $T > 0$ and a continuous solution g of the Ricci-DeTurck flow on $(0, T) \times M$ with $g(0, \cdot) = g_0$ which satisfies*

$$t^{1/2} \|\nabla(g(t) - h)\|_{L^\infty} \leqslant \varepsilon.$$

Moreover, the solution is unique among all other solutions satisfying the same bound for the gradient.

We note that there are several interesting existence results for the Ricci flow under various curvature assumptions using more geometric arguments by Cabezas-Rivas and Wilking [5] and Simon [27], [28].

Uniqueness results were previously obtained under some curvature bounds by Chen and Zhu [7], Chen [6] and Kotschwar [21].

5.4. Asymptotics for fast diffusion. Consider the fast diffusion equation

$$u_t = \frac{1}{m} \Delta u^m$$

with $(d-2)_+/d < m < 1$. Let

$$\beta = (2 - (1-m)d)^{-1} \tag{5.2}$$

and

$$u_B = (B + |x|^2)^{-1/(1-m)}.$$

Then

$$u(t, x) = t^{-\beta d} \Big(B + \frac{|x|}{t^\beta}^2 \Big)^{-1/(1-m)}$$

is the Barenblatt solution.

Conformal coordinates lead to the equation

$$v_t = \frac{1}{m}(B + |x|^2)\Delta v^m + \frac{2}{1-m} x \nabla (v - 2v^m) + \Big(d + 2\frac{B + |x|^2}{1-m} |x|^2 \Big)(v - v^m).$$

This equation is uniformly parabolic on the cigar manifold given by the Riemannian metric

$$\delta_{ij}(B + |x|^2)^{-1},$$

provided the relative size $v = u/u_B$ is bounded from below and above. The cigar manifold is a uniform manifold in the sense of Section 4.1. It has been shown by Vazquez that under weak assumptions on the initial data, $v \to 0$ uniformly in x as $t \to \infty$. It is remarkable that the spectrum and the eigenfunctions of the linearization can be computed explicitly, see Denzler and McCann [12].

Using the formulation on the above manifold but not the approach discussed here, Denzler, McCann and the first author [11] derived precise information on the large time asymptotics from the information on the linearized operator. Due to the fact that the cigar is noncompact, there are important issues about the continuous spectrum for which we refer the reader to [11].

5.5. Perturbed traveling wave solutions to the porous medium equation. The porous medium equation

$$\varrho_t = \Delta \varrho^m$$

with $m > 1$ is an idealized model for the propagation of gas in a porous medium. It has special solutions: The Barenblatt solution

$$\varrho(t,x) = t^{-\beta d}\Big(B - \frac{|x|^2}{t^\beta}\Big)_+^{1/(m-1)},$$

which has compact support in x for fixed t. Here β is defined by (5.2).

A second explicit solution is given by the traveling wave solution

$$\varrho(t,x)^{m-1} = (t + x_n)_+.$$

The quantity

$$v = \frac{m}{m-1}\varrho^{m-1}$$

corresponds to the physical pressure. It satisfies formally

$$v_t - (m-1)v\Delta v = |\nabla v|^2.$$

Theorem 5.4 (Kienzler [18]). *Suppose that the nonnegative function* $\varrho_0\colon \mathbb{R}^d \to \mathbb{R}$ *satisfies*

$$\Big|\nabla\Big(\frac{m}{m-1}\varrho_0^{m-1}\Big) - e_n\Big| < \delta$$

on the set of positivity. Then the unique solution to the porous medium equation satisfies

$$\Big|\nabla\Big(\frac{m}{m-1}\varrho^{m-1}\Big) - e_n\Big| < C\delta \quad \textit{and} \quad t^{k+|\alpha|-1}|\partial_t^k\partial_x^\alpha\varrho^{m-1}| \leqslant c_{k+|\alpha|}\delta,$$

where ϱ *is positive whenever* $1 \leqslant |\alpha| \leqslant 2$.

Existence and uniqueness of solutions to nonnegative initial data is well understood with the final contribution of Dahlberg and Kenig [8]. The regularity of solutions is more difficult. There are local regular solutions to regular initial data satisfying a suitable nondegeneracy condition (see Daskalopoulos and Hamilton [9] and Daskalopoulos, Hamilton and Lee [10]).

The Aronson-Graveleau solutions [3] describe the self-similar filling of a hole by gas. It is a consequence that at the time of the filling the pressure does not remain Lipschitz continuous.

Describing the graph is equivalent to describing the function. We describe the graph of p as a graph of a function v with

$$x_n = p, \quad y_n = w.$$

It is defined on the half-plane $x_n > 0$. The traveling wave solution becomes

$$y_n - t \quad \text{and} \quad v = w - (y_n - t)$$

satisfies with

$$\sigma = \frac{m-2}{m-1} > -1 \tag{5.3}$$

$$\frac{1}{m-1}v_t - \Big(x_n^{-\sigma}\sum_{j=1}^{d-1}\partial_j(x_n^{1+\sigma}\partial_j v)\Big) - x_n^{-\sigma}\partial_n\Big(x_n^{1+\sigma}\frac{\partial_n v - \sum_{j=1}^{d-1}(\partial_j v)^2}{1+\partial_n v}\Big) = 0 \tag{5.4}$$

in the upper half-plane $x_n \geqslant 0$. The result in transformed coordinates reads as follows:

Theorem 5.5 (Kienzler [18]). *There exists $\delta > 0$ such that the following is true. Suppose that*

$$v_0 \colon H \to \mathbb{R}$$

satisfies

$$|v_0(x) - v_0(y)| \leqslant \varepsilon |x - y|.$$

Then there is a unique solution which satisfies

$$|t^{j+|\alpha|-1} \partial_t^j \partial_x^\alpha v| \leqslant c\varepsilon$$

whenever $1 \leqslant |\alpha| \leqslant 2$.

For the proof we observe that the second order part of the operator

$$x_n^{-\sigma} \nabla (x_n^{1+\sigma} \nabla u)$$

is the second order part of the Laplace-Beltrami operator on the upper half-plane with the Riemannian metric

$$\langle u, v \rangle_x = x_n^{-1} uv.$$

This is half way between Euclidean space and the Poincaré half-plane.

On an abstract level the steps are the same as on $\mathbb{R}^d$.

(1) The intrinsic geometry defines balls and space-time cylinders. On $L^2(x_n^\sigma)$ we obtain a self-adjoint semigroup.
(2) Energy arguments give L^2 estimates with Gaussian weights, the Davies-Gaffney estimates for the analogue of the heat equation

$$(m-1)v_t - x_n \Delta v - (1+\sigma) v_n = 0.$$

(3) Local regularity gives pointwise bounds of derivatives for solutions to the homogeneous equation in cylinders.
(4) Both together imply Gaussian estimates for the fundamental solution and its derivatives in the intrinsic geometry.
(5) The Gaussian estimates and the energy estimates are good enough for the Calderón-Zygmund theory on spaces of homogeneous type.

See [18] for a complete proof.

5.6. Flat solutions to the thin film equation. Nonnegative solutions to the thin film equation

$$h_t + \nabla (h \nabla \Delta h) = 0$$

supposedly describe the dynamics of thin films. While existence of weak solutions is reasonably well understood, there are only few instances where uniqueness or higher regularity is known. This is a question with relevance for modeling: The equation has solutions with zero contact angle and nonzero contact angle, and hence at least the contact angle is needed for a complete description. Here we study existence and uniqueness of a class of solutions with zero contact angle. The only previous uniqueness results with a moving contact line are in this setting in one space dimension by Giacomelli, Knüpfer and Otto [15] and by Giacomelli, Gnann, Knüpfer and Otto [14].

There is a trivial stationary solution

$$h = ((x_n)_+)^2$$

and we want to study solutions in a neighborhood of h.

Theorem 5.6 (D. John)**.** *Suppose that*

$$|\nabla \sqrt{h_0} - e_n| \leqslant \delta.$$

Then there exists a unique solution h which satisfies

$$|\nabla \sqrt{h} - e_n| \leqslant c\delta$$

and, for $1 \leqslant |\alpha| \leqslant 2$,

$$t^{2k+|\alpha|-1} |\partial_t^k \partial_x^\alpha \sqrt{h}| \leqslant c(k,\alpha) \|\nabla \sqrt{h_0} - e_n\|_{\sup}.$$

This formulation is slightly different from what is proven by D. John in his thesis [17], but his proof gives also the simpler statement above. Again we transform the problem to a degenerate quasilinear problem on the upper half-plane.

Let $\tilde{h} = h^{1/2}$ and note that it solves the equation

$$\begin{aligned} \partial_t \tilde{h} + \tilde{h}^2 \Delta^2 \tilde{h} + 6h \nabla \tilde{h} \nabla \Delta \tilde{h} + \tilde{h}(\Delta \tilde{h})^2 \\ + 2h|\Delta' \tilde{h}|^2 + 2|\nabla \tilde{h}|^2 \Delta \tilde{h} + 4\partial_i \tilde{h} \partial_j \tilde{h} \partial_{ij}^2 \tilde{h} = 0. \end{aligned}$$

Letting

$$w = y_n, \quad x_n = \tilde{h},$$

we obtain with $u = w - x_n$

$$u_t + L_0 u = f_0[u] + x_n f_1[u] + x_n^2 f_2[u],$$

where

$$L_0 = x_n^{-1} \Delta x_n^3 \Delta - 4\Delta_{R^{n-1}}.$$

The abstract procedure is the same as for the porous medium equation, but filling in the details is demanding.

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