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TORSIONAL ASYMMETRY IN SUSPENSION BRIDGE SYSTEMS

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Abstract. In this paper a dynamic linear model of suspension bridge center spans is formulated and three different ways of fixing the main cables are studied. The model describes vertical and torsional oscillations of the deck under the action of lateral wind. The mutual interactions of main cables, center span, and hangers are analyzed. Three variational evolutions are analyzed. The variational equations correspond to the way how the main cables are fixed. The existence, uniqueness, and continuous dependence on data are proved.

Keywords: suspension bridge; Hamilton principle; vertical oscillation; torsional oscillation; existence; uniqueness; continuous dependence on data

MSC 2010: 35L57, 35Q74

1. Introduction

The collapse of the original Tacoma suspension bridge on 7 November 1940 has been studied in many papers. A wide list of references connected with that event is possible to find, for instance, in [20]. The Tacoma bridge was opened on 1 July 1940 and since the opening day vertical oscillations appeared in lateral winds whose speed reached more than $22\,\mathrm{m\,s^{-1}}$. On 7 November 1940 the torsional oscillations appeared after the midspan cable band on one main cable loosened. The motion of the central span was primarily a one-nodded torsional oscillation with the maximum twist angle about 35° and the corresponding maximum vertical amplitude about $4.3\,\mathrm{m}$. The bridge collapsed after approximately one hour and the central span fell into the Tacoma Narrows. One can see the collapse in the clips [28] and [29]. The

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basic scheme of the central span of the original Tacoma suspension bridge, which is almost identical for all suspension bridges, is depicted in Fig. 1.

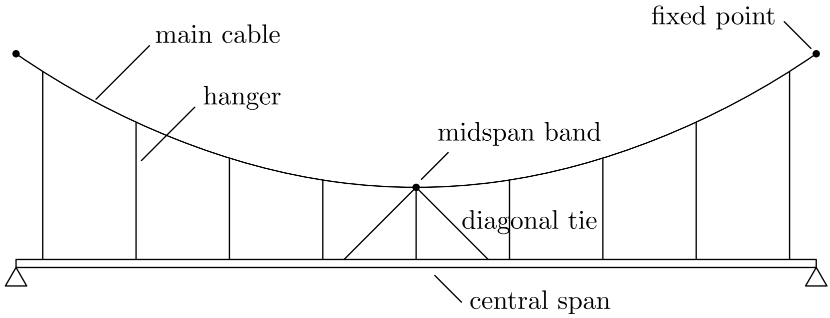

Figure 1. Specification of central span.

Here we mention some papers utilizing continuum models of the central span. The authors of the paper [19] analyzed vertical motions of the central span together with the reaction of hangers. The central span was modeled as a beam and the hangers as an elastic nonlinear continuum. The fundamental nonlinearity of the model is that the hangers strongly resist expansion, but do not resist compression. The equations formulated in the paper are nonlinear and the authors studied periodic solutions when the center span was exposed to a periodic force. The analysis showed that the equation has at least two solutions. The same equation has been studied in many papers, for instance, in [14], [2], [3], [6], [7], [4], [5], and [12], where the authors analyzed the structure of periodic solutions and proved the multiplicity of solutions. The same model was numerically studied in [10] for some concrete parameters which corresponded to the original Tacoma bridge and some other suspension bridges. A different model of the central span was presented in [26] and [8], where the main cable was modeled as a string and the central span as a beam. The hangers were studied as a nonlinear continuum with the same properties as in the previous model. The model was described by two nonlinear equations whose solution has similar properties as the solution of the equation studied in [19]. In the paper [1] the equation formulated in [19] was analyzed as a general dynamic problem with initial conditions. The authors of the paper [13] presented the model describing both the vertical and the torsional oscillations of the center span. The main cables were modeled as strings attached to the deck through a systems of hangers modeled as a continuum. The hangers resisted expansion, but did not resist compression just as the hangers in the model formulated in [19]. The authors of [13] studied a similar initial value problem as was studied in [1]. In all of the above mentioned papers the main cables were modeled as strings. In the papers [16] and [15] a different way was proposed. The main cables were modeled as a system of stiff rods connected with joints in which hangers were attached. In this model both the behavior of the main cables and the hangers is nonlinear. It seems that the loosening of the midspan cable band had a significant impact on the behavior of the original Tacoma bridge and in the end it resulted in torsional oscillations. These questions were studied, for instance, in [18] and [21].

In this paper we suppose that the equilibrium state of the bridge under gravitational forces is known. The variational equations studied in this paper were formulated in [17]. They describe deflections from the equilibrium state due to the forces induced by lateral wind. Deflections are described by two functions corresponding to vertical and torsional motions of the central span. The variational equations correspond to the way how the midspan cable bands are fixed. The equations describe the mutual reaction of the center span and the cable system as well as the reaction of the diagonal ties on the midspan cable bands. A simple analysis was carried out in [17] and some hypotheses explaining the collapse were formulated. The analysis was based on the restrictive condition that the mass of the deck is concentrated at the position of hangers. In this paper the existence, uniqueness, and continuous dependence on data for the variational equations are proved. We concentrate on vertical and torsional motions of the central span in lateral wind and neglect horizontal motions. Horizontal motions are not connected with vertical and torsional motions and can be studied independently. Moreover, the coefficients describing the action of lateral wind on horizontal motions of the central span are negligible as compared to the coefficients connected with vertical and torsional motions. These are the reasons why horizontal motions are not studied in this paper.

2. Formulation of problems and main results

In this section we fix our attention on the oscillations induced by lateral wind and concentrate on the behavior of the central span which is attached by the hangers to the deck. The analysis is based on the variational equations derived from the Hamilton principle (see [17]). Solutions to the variational equations give the deflection of the center span from the equilibrium under gravitational forces. This deflection is described by functions $u(x,t)$, $\theta(x,t)$, where $u(x,t)$ corresponds to vertical displacement and $\theta(x,t)$ corresponds to torsional deformation of the center span, where x belongs to $(-\frac{1}{2}L, \frac{1}{2}L)$ (see Fig. 2). In the derivation of the variational equations in [17] it was supposed that the equilibrium under gravitational forces was known, especially the shape of the main cable $y(x)$ and the horizontal projection H of tension forces in the main cable. The value of H is constant as follows from the theory in [22]. The formulation of the linearized models is based on the hypotheses formulated in [17]. First of all we suppose that the main cables and hangers are inextensible and flexible.

Let us recall the parameters of the deck and the cable system. These parameters are gathered in Table 1 and some of them are depicted in Fig. 2. The values of these parameters for the original Tacoma bridge can be found, for instance, in [20]. The variational equations describe the reactions of the center span and the cable system to some additional forces which are significantly smaller than the gravitational forces acting on the bridge.

D	half the width of the deck
L	the length of the central span
L_1	the sag of the main cables
M_D	the mass of the deck per unit length
I_P	the polar mass moment of inertia of the deck
M_C	the mass of the main cable per unit length
E_D	the modulus of elasticity of the deck
I_D	the moment of inertia of the deck
G_D	the shear modulus of the deck
J_D	the torsional constant of the deck
g	the gravitational acceleration

Table 1.

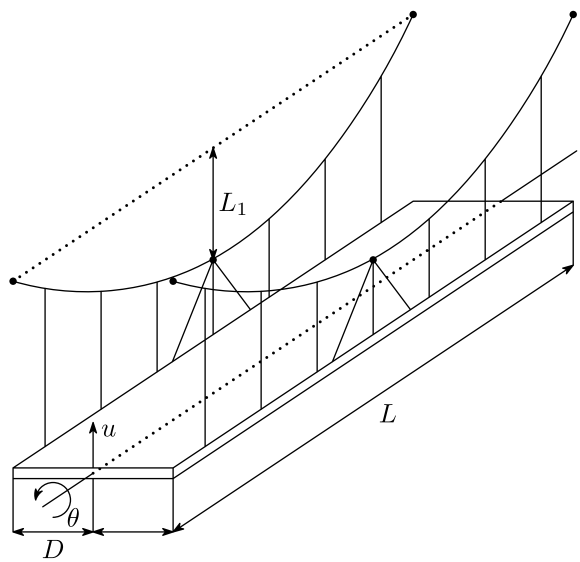

Figure 2. Perspective view of central span.

Let us make a few remarks about the main cables under gravitational forces. The main cables are assumed to be fixed at their end points which are immovable and ideally flexible, so the tension forces in cables are oriented in the tangential direction. If gravitation is the only force acting on the bridge and the induced forces acting on the main cables are regularly distributed along the central span, then the shape of the main cables is a parabola (see [22]). The shape of the main cables reads

$$y(x) = \frac{4L_1x^2}{L^2}, \tag{1}$$

where x belongs to $(-\frac{1}{2}L, \frac{1}{2}L)$. The horizontal projection H of the tension forces in the main cable is constant and is given by the formula (see [22])

$$H = \frac{g(M_C + \frac{1}{2}M_D)L^2}{8L_1}. \tag{2}$$

The formulas (1) and (2) approximate $y(x)$ and H in real situations.

Now we will study the variational equations which were derived in [17]. We will define a few bilinear forms connected with the formulation of our problems. Let us have a bilinear form

$$a_c(u,v) = \int_{-L/2}^{L/2} A_c \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x,$$

where A_c is a function on $(-\frac{1}{2}L, \frac{1}{2}L)$ defined by

$$A_c = H\Big(1 + \Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2\Big). \tag{3}$$

The bilinear form is connected with the potential energy of the main cable corresponding to the vertical deflection of the deck from the equilibrium under gravitational forces. The vertical deformation of the deck transfers to the main cable through the inextensible hangers. If we consider both the vertical and the torsional deflections of the deck, the potential energy of the main cables reads

$$a_c(u,u) + D^2 a_c(\theta,\theta),$$

which was derived in [17]. Let us define other two bilinear forms

$$a_{\mathrm{ver}}(u,v) = \int_{-L/2}^{L/2} E_D I_D \frac{\mathrm{d}^2u}{\mathrm{d}x^2}\frac{\mathrm{d}^2v}{\mathrm{d}x^2}\,\mathrm{d}x, \quad a_{\mathrm{tor}}(\theta,\varphi) = \int_{-L/2}^{L/2} G_D J_D \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x$$

which are connected with the bending and the torsional deformation energy of the deck. To simplify our equations for the dynamic problems, we define bilinear forms

$$m_{\mathrm{ver}}(u,v) = \int_{-L/2}^{L/2} M_{\mathrm{ver}} uv\,\mathrm{d}x, \quad m_{\mathrm{tor}}(\theta,\varphi) = \int_{-L/2}^{L/2} M_{\mathrm{tor}}\theta\varphi\,\mathrm{d}x, \tag{4}$$

where $M_{\rm ver}, M_{\rm tor}$ are functions on $(-\frac{1}{2}L, \frac{1}{2}L)$ defined by

$$
\begin{aligned}
M_{\rm ver}(x) &= 2M_C\Big(1 + \Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2\Big)^{1/2} + M_D, \\
M_{\rm tor}(x) &= 2D^2 M_C\Big(1 + \Big(\frac{\mathrm{d}y}{\mathrm{d}x}\Big)^2\Big)^{1/2} + I_P.
\end{aligned}
\tag{5}
$$

These bilinear forms correspond to the kinetic energy of the deck and the main cables.

In this paper we will analyze the aeroelastic forces induced by lateral wind. The aeroelastic forces per unit length of the deck are given (see [25], [23]) by

$$
\begin{aligned}
L_u &= H_1\dot{u} + H_2\dot{\theta} + H_3\theta, \\
M_\theta &= A_1\dot{u} + A_2\dot{\theta} + A_3\theta,
\end{aligned}
\tag{6}
$$

where L_u and M_θ are the aeroelastic vertical lift force and the torsional moment of the deck per unit length. The coefficients $H_i(x,t)$, $A_i(x,t)$ generally depend on the shape of the deck and the speed of wind, so we can say that these coefficients are functions defined on $(-\frac{1}{2}L, \frac{1}{2}L) \times (0,T)$. The coefficients are characteristic for every bridge and the values of these coefficients for the original Tacoma bridge are given, for instance, in [25]. Let us define the bilinear forms with the parameter t from $(0,T)$

$$
\begin{aligned}
f_1(\dot{u}, v; t) &= \int_{-L/2}^{L/2} H_1(x,t)\dot{u}v\,\mathrm{d}x, & g_1(\dot{u}, \varphi; t) &= \int_{-L/2}^{L/2} A_1(x,t)\dot{u}\varphi\,\mathrm{d}x, \\
f_2(\dot{\theta}, v; t) &= \int_{-L/2}^{L/2} H_2(x,t)\dot{\theta}v\,\mathrm{d}x, & g_2(\dot{\theta}, \varphi; t) &= \int_{-L/2}^{L/2} A_2(x,t)\dot{\theta}\varphi\,\mathrm{d}x, \\
f_3(\theta, v; t) &= \int_{-L/2}^{L/2} H_3(x,t)\theta v\,\mathrm{d}x, & g_3(\theta, \varphi; t) &= \int_{-L/2}^{L/2} A_3(x,t)\theta\varphi\,\mathrm{d}x
\end{aligned}
$$

which correspond to the forces given by (6).

The variational equation for the dynamic problems was derived in [17] from the Hamilton principle and reads

$$
\begin{aligned}
&m_{\rm ver}(\ddot{u}, v) + m_{\rm tor}(\ddot{\theta}, \varphi) + 2a_c(u,v) + 2D^2 a_c(\theta,\varphi) + a_{\rm ver}(u,v) + a_{\rm tor}(\theta,\varphi) \\
&\quad = f_1(\dot{u}, v; t) + f_2(\dot{\theta}, v; t) + f_3(\theta, v; t) + g_1(\dot{u}, \varphi; t) + g_2(\dot{\theta}, \varphi; t) + g_3(\theta, \varphi; t).
\end{aligned}
\tag{7}
$$

The equation holds for all sufficiently smooth functions $v(x)$, $\varphi(x)$ defined on $(-\frac{1}{2}L, \frac{1}{2}L)$. In our models we assume that the central span is hinged at its end points, so the functions u, θ satisfy the boundary conditions

$$
u(-\tfrac{1}{2}L, t) = u(\tfrac{1}{2}L, t) = \theta(-\tfrac{1}{2}L, t) = \theta(\tfrac{1}{2}L, t) = 0
\tag{8}
$$

which hold for all t from $(0,T)$. The test functions v, φ satisfy the boundary conditions

$$v(-\tfrac{1}{2}L) = v(\tfrac{1}{2}L) = \varphi(-\tfrac{1}{2}L) = \varphi(\tfrac{1}{2}L) = 0. \tag{9}$$

So far we have not considered the fact that the main cables are inextensible and fixed at the end points and fastened at the midspan cable bands. To simplify the formulation of our problems, we define three linear forms

$$h(u) = \int_{-L/2}^{L/2} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x, \quad h_r(u) = \int_{-L/2}^{0} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x, \quad h_l(u) = \int_{0}^{L/2} \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

If both the main cables are fixed at their end points, then u and θ satisfy the relations

$$h(u) = h(\theta) = 0. \tag{10}$$

If both the main cables are fixed at the midspan cable bands as well, the relations

$$h_r(u) = h_r(\theta) = h_l(u) = h_l(\theta) = 0 \tag{11}$$

hold. In the end let us study the case where both main cables are fixed at the end points and only one main cable is fixed at the midspan cable band. Then the relations

$$h_r(u - D\theta) = h_l(u - D\theta) = h(u + D\theta) = 0 \tag{12}$$

hold. These formulas were derived in [17].

Let $u(x,t)$, $\theta(x,t)$ be functions defined on $(-\frac{1}{2}L, \frac{1}{2}L) \times (0,T)$. To simplify our notation, the symbols $u(t)$, $\theta(t)$ denote the functions whose values are the functions defined by $u(t)(x) = u(x,t)$ and $\theta(t)(x) = \theta(x,t)$. Let us consider the embeddings

$$H^2(-\tfrac{1}{2}L, \tfrac{1}{2}L) \subset L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \quad H^1(-\tfrac{1}{2}L, \tfrac{1}{2}L) \subset L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \tag{13}$$

then we can define spaces where we are looking for solutions to our problems. Let us suppose that

$$\begin{aligned} u(t) &\in L^2(0,T; H^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)), \quad \dot{u}(t) \in L^2(0,T; L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)),\\ \theta(t) &\in L^2(0,T; H^1(-\tfrac{1}{2}L, \tfrac{1}{2}L)), \quad \dot{\theta}(t) \in L^2(0,T; L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)), \end{aligned}$$

where the over dots represent the generalized time derivative and the embeddings (13) are applied. The test functions v and φ in (7) belong respectively to $H^2(-\frac{1}{2}L, \frac{1}{2}L)$ and $H^1(-\frac{1}{2}L, \frac{1}{2}L)$. Moreover, $u(t)$ and $\theta(t)$ satisfy the initial conditions

$$\begin{aligned} u(0) &= u_0 \in H^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \quad \dot{u}(0) = u_1 \in L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \\ \theta(0) &= \theta_0 \in H^1(-\tfrac{1}{2}L, \tfrac{1}{2}L), \quad \dot{\theta}(0) = \theta_1 \in L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L). \end{aligned} \tag{14}$$

If we talk about solutions to the variational equation (7), we have in mind that this equation holds in the generalized sense with respect to t, which means that the equality

$$\int_0^T M(t)\ddot{\xi}(t) + A(t)\xi(t)\,\mathrm{d}t = \int_0^T F(t)\xi(t)\,\mathrm{d}t$$

holds for all $\xi(t) \in C_0^\infty(0,T)$ which is the space of smooth functions on $(0,T)$ with compact support, where

$$\begin{aligned} M(t) &= m_{\mathrm{ver}}(u(t), v) + m_{\mathrm{tor}}(\theta(t), \varphi), \\ A(t) &= 2a_c(u(t), v) + 2D^2 a_c(\theta(t), \varphi) + a_{\mathrm{ver}}(u(t), v) + a_{\mathrm{tor}}(\theta(t), \varphi), \\ F(t) &= f_1(\dot{u}(t), v; t) + f_2(\dot{\theta}(t), v; t) + f_3(\theta(t), v; t) \\ &\quad + g_1(\dot{u}(t), \varphi; t) + g_2(\dot{\theta}(t), \varphi; t) + g_3(\theta(t), \varphi; t). \end{aligned}$$

Now we will formulate three dynamic problems connected with the way how the main cables are fixed, which puts some restrictions on solutions and initial conditions. The first dynamic problem $\mathcal{D}_1$ describes oscillations of the center span if the main cables are fixed at the end points. The second dynamic problem $\mathcal{D}_2$ describes oscillations of the center span if the main cables are fixed at the end points and the midspan bands. The third dynamic problem $\mathcal{D}_3$ describes oscillations of the center span if the main cables are fixed at the end points, one midspan band holds and the other loosens. Let us define spaces W_1, W_2, W_3, which are subspaces of $H^2(-\frac{1}{2}L, \frac{1}{2}L) \times H^1(-\frac{1}{2}L, \frac{1}{2}L)$, as follows:

$$\begin{aligned} W_1 &= \{(v, \varphi)\colon\ h(v) = h(\varphi) = 0\}, \\ W_1 &= \{(v, \varphi)\colon\ h_r(v) = h_r(\varphi) = h_l(v) = h_l(\varphi) = 0\}, \\ W_1 &= \{(v, \varphi)\colon\ h_r(v - D\varphi) = h_l(v - D\varphi) = h(v + D\varphi) = 0\}. \end{aligned}$$

The subspaces V_1, V_2, V_3 are the closures of W_1, W_2, W_3 in $L^2(-\frac{1}{2}L, \frac{1}{2}L) \times L^2(-\frac{1}{2}L, \frac{1}{2}L)$.

Functions $u(t)$, $\theta(t)$ are a solution to $\mathcal{D}_1$ if they satisfy the relations

$$h(u(t)) = h(\theta(t)) = 0$$

for all t, the boundary conditions (8), and the variational equation (7). The variational equation (7) holds for all v, φ which satisfy the relations

$$h(v) = h(\varphi) = 0$$

and the boundary conditions (9). The initial conditions (14) are compatible with $\mathcal{D}_1$, which means that $(u_0, \theta_0) \in W_1$ and $(u_1, \theta_1) \in V_1$. Moreover, u_0, θ_0 satisfy the boundary conditions (9).

The functions $u(t)$, $\theta(t)$ are a solution to the dynamic problem $\mathcal{D}_2$ if they satisfy the relations

$$h_r(u(t)) = h_r(\theta(t)) = h_l(u(t)) = h_l(\theta(t)) = 0$$

for all t, the boundary conditions (8), and the variational equation (7). The variational equation (7) holds for all v, φ which satisfy the relations

$$h_r(v) = h_r(\varphi) = h_l(v) = h_l(\varphi) = 0$$

and the boundary conditions (9). The initial conditions (14) are compatible with $\mathcal{D}_2$, which means that $(u_0, \theta_0) \in W_2$ and $(u_1, \theta_1) \in V_2$. Moreover, u_0, θ_0 satisfy the boundary conditions (9).

The functions $u(t)$, $\theta(t)$ are a solution to the third dynamic problem $\mathcal{D}_3$ if they satisfy the relations

$$h_r(u(t) - D\theta(t)) = h_l(u(t) - D\theta(t)) = h(u(t) + D\theta(t)) = 0$$

for all t, the boundary conditions (8), and the variational equation (7). The variational equation (7) holds for all v, φ which satisfy the relations

$$h_r(v - D\varphi) = h_l(v - D\varphi) = h(v + D\varphi) = 0$$

and the boundary conditions (9). The initial conditions (14) are compatible with $\mathcal{D}_3$, which means that $(u_0, \theta_0) \in W_3$ and $(u_1, \theta_1) \in V_3$. Moreover, u_0, θ_0 satisfy the boundary conditions (9).

Now we are going to formulate two main theorems whose proofs will be given in Section 4. The coefficients $M_C, M_D, I_P, I_D, J_D, E_D, G_D, H$ are positive.

Theorem 2.1. *Let y belong to $C^1([-\frac{1}{2}L, \frac{1}{2}L])$ and let the initial conditions be compatible with $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$. Let H_1, H_2, A_1, A_2 belong to $C^1([-\frac{1}{2}L, \frac{1}{2}L] \times [0,T])$ and H_3, A_3 belong to $C([-\frac{1}{2}L, \frac{1}{2}L] \times [0,T])$. Then the problems $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ are uniquely solvable.*

Let us study the continuous dependence on the forces acting on the deck and the initial conditions. In the rest of this paper the superscripts n are connected with the continuous dependence on the data while the subscripts 0, 1, 2, 3 describe the concrete components of forces or initial conditions, which one can see in the text.

Let $H_1^n, H_2^n, A_1^n, A_2^n$, $n = 0, 1, 2, \ldots$ belong to $C^1([-\frac{1}{2}L, \frac{1}{2}L] \times [0, T])$ and H_3^n, A_3^n belong to $C([-\frac{1}{2}L, \frac{1}{2}L] \times [0, T])$. Moreover, assume that

$$(15)\quad H_1^n, H_2^n, H_3^n, A_1^n, A_2^n, A_3^n \to H_1^0, H_2^0, H_3^0, A_1^0, A_2^0, A_3^0 \quad \text{in } C([-\tfrac{1}{2}L, \tfrac{1}{2}L] \times [0, T])$$

as $n \to \infty$. Let u_0^n and θ_0^n, $n = 0, 1, 2, \ldots$ belong to $H^2(-\frac{1}{2}L, \frac{1}{2}L)$ and $H^1(-\frac{1}{2}L, \frac{1}{2}L)$, satisfy the boundary conditions (9) and u_1^n, θ_1^n, $i = 0, 1, 2, \ldots$ belong to $L^2(-\frac{1}{2}L, \frac{1}{2}L)$. Moreover, assume that

$$(16)\qquad \begin{aligned} u_0^n &\to u_0^0 \quad \text{in } H^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \quad & \theta_0^n &\to \theta_0^0 \quad \text{in } H^1(-\tfrac{1}{2}L, \tfrac{1}{2}L),\\ u_1^n &\to u_1^0 \quad \text{in } L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L), \quad & \theta_1^n &\to \theta_1^0 \quad \text{in } L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L) \end{aligned}$$

as $n \to \infty$. Let us study the sequence of solutions u^n, θ^n which correspond to the forces and the initial conditions. Then the following theorem holds.

Theorem 2.2. *Let y belong to $C^1([-\frac{1}{2}L, \frac{1}{2}L])$. Let H_1^n, H_2^n, A_1^n, A_2^n belong to $C^1([-\frac{1}{2}L, \frac{1}{2}L] \times [0, T])$, H_3^n, A_3^n belong to $C([-\frac{1}{2}L, \frac{1}{2}L] \times [0, T])$ and satisfy* (15)*. Let the initial conditions u_0^n, θ_0^n, u_1^n, θ_1^n be compatible with $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ and satisfy* (16)*. Then the solutions u^n, θ^n satisfy the limits*

$$\begin{aligned} u^n &\to u^0 \quad \text{in } L^2(0, T; H^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)), \quad & \dot{u}^n &\to \dot{u}^0 \quad \text{in } L^2(0, T; L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)),\\ \theta^n &\to \theta^0 \quad \text{in } L^2(0, T; H^1(-\tfrac{1}{2}L, \tfrac{1}{2}L)), \quad & \dot{\theta}^n &\to \dot{\theta}^0 \quad \text{in } L^2(0, T; L^2(-\tfrac{1}{2}L, \tfrac{1}{2}L)) \end{aligned}$$

as $n \to \infty$.

Let us close this section with a few remarks. From the relations (10)–(11) it follows that the variational equation (7) for the problems $\mathcal{D}_1, \mathcal{D}_2$ can be rewritten into the two variational equations

$$(17)\qquad \begin{aligned} &m_{\text{ver}}(\ddot{u}, v) + 2a_c(u, v) + a_{\text{ver}}(u, v) = f_1(\dot{u}, v; t) + f_2(\dot{\theta}, v; t) + f_3(\theta, v; t),\\ &m_{\text{tor}}(\ddot{\theta}, \varphi) + 2D^2 a_c(\theta, \varphi) + a_{\text{tor}}(\theta, \varphi) = g_1(\dot{u}, \varphi; t) + g_2(\dot{\theta}, \varphi; t) + g_3(\theta, \varphi; t). \end{aligned}$$

The problem $\mathcal{D}_3$ cannot admit such a reformulation, because the relations (12) cannot be rewritten in an equivalent form so that the new relations would contain either u or θ. Moreover, it is not possible to rewrite the variational equations (7) and (17) into partial differential equations, because the test functions v, φ satisfy the restrictions (10)–(12).

3. Some auxiliary abstract results

In this section we prove some auxiliary assertions which we apply in the proof of the uniqueness, existence, and continuous dependence on data for the three problems formulated in Section 2. Let us recall some facts whose proofs we can find, for instance, in [27], [9], [11]. Let V be a real Banach space, then $L^2(0,T;V)$ is the space of all measurable functions from the real interval $(0,T)$ to V which satisfy

$$\int_0^T \|u(t)\|_V^2 \,\mathrm{d}t < \infty.$$

Let $u \in L^2(0,T;V)$ have the generalized derivative $\dot{u} \in L^2(0,T;V)$. Then $u \in C([0,T];V)$ and the inequality

$$\|u\|_{C([0,T];V)} \leqslant C(\|u\|_{L^2(0,T;V)} + \|\dot{u}\|_{L^2(0,T;V)}) \tag{18}$$

holds, where the constant C is independent of u. If $u \in L^2(0,T;V)$ then the functions $w(t)$ and $v(t)$ defined by

$$w(t) = \int_0^t u(s)\,\mathrm{d}s, \quad v(t) = \int_t^T u(s)\,\mathrm{d}s \tag{19}$$

belong to $C([0,T];V)$ and their generalized derivatives are $u(t)$ and $-u(t)$, respectively. Let V, H be separable Hilbert spaces with the embedding $V \subset H$ which is continuous, i.e.

$$\|u\|_H \leqslant C\|u\|_V$$

for all $u \in V$. Moreover, V is dense in H. Let W_V and W_H be the Banach space of continuous bilinear forms on V and H with the norms

$$\|a(\cdot,\cdot)\|_{W_V} = \sup_{u,v\neq 0;\, u,v\in V} \frac{|a(u,v)|}{\|u\|_V\|v\|_V}, \quad \|m(\cdot,\cdot)\|_{W_H} = \sup_{u,v\neq 0;\, u,v\in H} \frac{|m(u,v)|}{\|u\|_H\|v\|_H}.$$

In this section for brevity we set

$$|u| = \|u\|_H, \quad \|v\| = \|v\|_V.$$

Let $m(\cdot,\cdot)$, $a(\cdot,\cdot)$ be continuous symmetric bilinear forms on H and V which satisfy the inequalities

$$\alpha|u|^2 \leqslant m(u,u), \quad \alpha\|v\|^2 \leqslant a(v,v), \tag{20}$$

where α is a positive constant and the inequalities hold for all $u \in H$ and $v \in V$. In the sequel we frequently use the triple

$$V \subset H \subset V^*,$$

where the embedding $H \subset V^*$ is given by

$$\langle u, v\rangle_V = m(u, v).$$

In the last formula $u \in H$ and $v \in V$. Moreover, we say that V is embedded in V^* via $m(\cdot, \cdot)$.

Let $b(\cdot, \cdot; t), c(\cdot, \cdot; t)$ be a continuous bilinear form on H with the parameter $t \in [0, T]$ and

$$b(\cdot, \cdot; t) \in C^1([0, T]; W_H), \quad c(\cdot, \cdot; t) \in C([0, T]; W_H). \tag{21}$$

We study the initial value problem

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m(u(t), v) + a(u(t), v) = b(\dot{u}(t), v; t) + c(u(t), v; t), \tag{22}$$

$$u(0) = u_0 \in V, \quad \dot{u}(0) = u_1 \in H, \tag{23}$$

$$u \in L^2(0, T; V), \quad \dot{u} \in L^2(0, T; H). \tag{24}$$

The equation (22) is valid for all $v \in V$ in the generalized sense, which means that the equality

$$\int_0^T m(u(t), v)\ddot{\varphi}(t) + (a(u(t), v) - b(\dot{u}(t), v; t) - c(u(t), v; t))\varphi(t)\,\mathrm{d}t = 0$$

holds for all $\varphi \in C_0^\infty(0, T)$ which is the space of smooth functions on $(0, T)$ with compact support. Moreover, the relations (20) and (21) hold.

Lemma 3.1. *Let the equation* (22) *be satisfied, then there exists* $\ddot{u} \in L^2(0, T; V^*)$, *where* V *is embedded in* V^* *via* $m(\cdot, \cdot)$, *such that the equality*

$$\int_0^T \langle \ddot{u}(t), v\rangle_V \varphi(t)\,\mathrm{d}t = \int_0^T m(u(t), v)\ddot{\varphi}(t)\,\mathrm{d}t \tag{25}$$

holds for all $v \in V$ *and* $\varphi \in C_0^\infty(0, T)$. *Moreover, there exists* $N \subset (0, T)$ *of measure zero such that the equality*

$$\langle \ddot{u}(t), v\rangle_V + a(u(t), v) = b(\dot{u}(t), v; t) + c(u(t), v; t) \tag{26}$$

holds for all $v \in V$ *and* $t \in (0, T) \setminus N$.

P r o o f. Let us consider the expression

$$\int_0^T b(\dot{u}(t), v(t); t) + c(u(t), v(t); t) - a(u(t), v(t)) \, \mathrm{d}t,$$

where $v(t) \in L^2(0,T;V)$. This expression is a linear continuous functional on $L^2(0,T;V)$, which yields the existence of $w(t) \in L^2(0,T;V^*)$ such that the equality

$$\int_0^T \langle w(t), v(t)\rangle_V \, \mathrm{d}t = \int_0^T b(\dot{u}(t), v(t); t) + c(u(t), v(t); t) - a(u(t), v(t)) \, \mathrm{d}t$$

holds for all $v(t) \in L^2(0,T;V)$. Comparing the last equation with (22), we obtain that the equality

$$\int_0^T \langle w(t), v\rangle_V \varphi(t) \, \mathrm{d}t = \int_0^T m(u(t), v) \ddot{\varphi}(t) \, \mathrm{d}t$$

holds for all $v \in V$ and $\varphi \in C_0^\infty(0,T)$, which yields $w(t) = \ddot{u}(t)$, and the equation

$$\int_0^T (\langle \ddot{u}(t), v\rangle_V + a(u(t), v) - b(\dot{u}(t), v; t) - c(u(t), v; t))\varphi(t) \, \mathrm{d}t = 0$$

holds for all $v \in V$ and $\varphi \in C_0^\infty(0,T)$. Let v_n be a dense sequence in V, then the last equation yields that there exists a set N of measure zero such that (26) holds for all v_n and $t \in (0,T) \setminus N$. Thus (26) holds for all $v \in V$ and $t \in (0,T) \setminus N$. $\square$

The last lemma and (18) implies that $u \in C([0,T];H)$ and $\dot{u} \in C([0,T];V^*)$, thus the initial conditions (23) make sense.

Lemma 3.2 (Uniqueness). *A solution to* (22)–(24) *is unique.*

P r o o f. It suffices to show that the only solution with $u_0 = u_1 = 0$ is $u = 0$. Let us define

$$v(t) = \int_t^s u(\tau) \, \mathrm{d}\tau$$

on the interval $(0,s)$ where $t \leqslant s \leqslant T$ and $v(t) = 0$ on the interval (s,T). Then from (19) it follows that $v \in C([0,T];V)$ and $\dot{v}(t) = -u(t)$ on $(0,s)$ in the generalized sense. Lemma 3.1 yields the equation

$$\int_0^s \langle \ddot{u}, v\rangle_V + a(u,v) - b(\dot{u}, v; t) - c(u, v; t) \, \mathrm{d}t = 0. \tag{27}$$

Assume that $u \in C^2([0,T];V)$ is an arbitrary function. Since $m(\cdot,\cdot)$ is symmetric and $v(s)=0$, the relations

$$\begin{aligned}
\int_0^s \langle \ddot{u}, v\rangle_V \,\mathrm{d}t &= \int_0^s m(\ddot{u}, v)\,\mathrm{d}t = \int_0^s m(\dot{u}, u)\,\mathrm{d}t - \langle \dot{u}(0), v(0)\rangle_V \\
&= \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} m(u(t), u(t))\,\mathrm{d}t - \langle \dot{u}(0), v(0)\rangle_V \\
&= \frac{1}{2} m(u(s), u(s)) - \frac{1}{2} m(u(0), u(0)) - \langle \dot{u}(0), v(0)\rangle_V
\end{aligned}$$

hold. Denoting

$$b_t(u,v;t) = \frac{\partial}{\partial t} b(u,v;t),$$

we have the relation

$$\int_0^s b(\dot{u}, v; t)\,\mathrm{d}t = \int_0^s b(u,u;t) - b_t(u,v;t)\,\mathrm{d}t - b(u(0), v(0); 0).$$

Since $a(\cdot,\cdot)$ is symmetric, the relations

$$\begin{aligned}
\int_0^s a(u(t), v(t))\,\mathrm{d}t &= -\int_0^s a(\dot{v}(t), v(t))\,\mathrm{d}t \\
&= -\frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} a(v(t), v(t))\,\mathrm{d}t = \frac{1}{2} a(v(0), v(0))
\end{aligned}$$

hold. Let $u \in L^2(0,T;V)$, $\dot{u} \in L^2(0,T;H)$, $\ddot{u} \in L^2(0,T;V^*)$, then there exists a sequence $u_n \in C^2([0,T];V)$ (see [24], [9], mollifier technique) such that $u_n, \dot{u}_n, \ddot{u}_n$ converge to $u, \dot{u}, \ddot{u}$ in the spaces $L^2(0,T;V), L^2(0,T;H), L^2(0,T;V^*)$. From (18) and (19) it follows that $u_n, \dot{u}_n$ converge to $u, \dot{u}$ in the spaces $C([0,T];H), C([0,T];V^*)$ and v_n converges to v in the space $C([0,T];V)$, where

$$v_n(t) = \int_t^s u_n(\tau)\,\mathrm{d}\tau$$

on the interval $(0,s)$ and $v_n(t) = 0$ on the interval (s,T). These facts and the relations above yield the equations

$$\begin{aligned}
\int_0^s \langle \ddot{u}, v\rangle_V \,\mathrm{d}t &= \frac{1}{2} m(u(s), u(s)), \\
\int_0^s a(u,v)\,\mathrm{d}t &= \frac{1}{2} a(v(0), v(0)), \\
\int_0^s b(\dot{u}, v; t)\,\mathrm{d}t &= \int_0^s b(u,u;t) - b_t(u,v;t)\,\mathrm{d}t,
\end{aligned}$$

where u is the solution. We have applied $u(0) = \dot{u}(0) = 0$. The last relations and (27) imply the equation

$$\frac{1}{2}m(u(s), u(s)) + \frac{1}{2}a(v(0), v(0)) = \int_0^s b(u, u; t) - b_t(u, v; t) + c(u, v; t)\,\mathrm{d}t$$

which gives the inequality

$$|u(s)|^2 + \|v(0)\|^2 \leqslant C\biggl(\int_0^s |u(t)|^2 + \|v(t)\|^2\,\mathrm{d}t\biggr).$$

Let us define

$$w(t) = \int_0^t u(\tau)\,\mathrm{d}\tau, \quad 0 \leqslant t \leqslant s,$$

then $v(t) = w(s) - w(t)$ and the last inequality can be rewritten into

$$|u(s)|^2 + \|w(s)\|^2 < C\biggl(\int_0^s |u(t)|^2 + \|w(s) - w(t)\|^2\,\mathrm{d}t\biggr).$$

Let us consider the inequality

$$\|w(t) - w(s)\|^2 \leqslant 2\|w(t)\|^2 + 2\|w(s)\|^2,$$

then the last inequality yields

$$|u(s)|^2 + \|w(s)\|^2 \leqslant C\biggl(\int_0^s |u(t)|^2 + \|w(t)\|^2\,\mathrm{d}t + s\|w(s)\|^2\biggr).$$

If s satisfies the relation $Cs \leqslant \frac{1}{2}$, then on the interval $(0, S)$, where $CS \leqslant \frac{1}{2}$, the inequality

$$|u(s)|^2 + \|w(s)\|^2 \leqslant C\int_0^s |u(t)|^2 + \|w(t)\|^2\,\mathrm{d}t$$

holds. Then Gronwall's inequality yields $u(t) = w(t) = 0$ on the interval $(0, S)$. Applying the same argument for the intervals $(S, 2S), (2S, 3S), \ldots$, we have the desired result. $\square$

Lemma 3.3 (Existence)**.** *There exists a solution to* (22)–(24) *and the inequality*

$$\|u\|_{L^2(0,T;V)} + \|\dot{u}\|_{L^2(0,T;H)} \leqslant C(\|u_0\| + |u_1|) \tag{28}$$

holds.

Proof. Let w_k be a sequence of linearly independent elements of V such that the linear span of this sequence is dense in V, so it is dense in H as well. Let us consider

$$u_m(t) = \sum_{k=1}^{m} d_m^k(t) w_k, \tag{29}$$

where $d_m^k(t)$, $k = 1, \ldots, m$, are real functions from $C^2([0,T])$ such that

$$u_m(0) \to u_0 \text{ in } V, \quad \dot{u}_m(0) \to u_1 \text{ in } H \tag{30}$$

as $m \to \infty$. Moreover, these functions are solutions to the system of ordinary differential equations

$$m(\ddot{u}_m(t), w_k) + a(u_m(t), w_k) = b(\dot{u}_m(t), w_k; t) + c(u_m(t), w_k; t),$$

where $k = 1, \ldots, m$, $t \in [0, T]$. The last equations yield

$$m(\ddot{u}_m, \dot{u}_m) + a(u_m, \dot{u}_m) = b(\dot{u}_m, \dot{u}_m; t) + c(u_m, \dot{u}_m; t). \tag{31}$$

From the symmetry of $m(\cdot,\cdot)$, $a(\cdot,\cdot)$ we obtain

$$m(\ddot{u}_m, \dot{u}_m) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} m(\dot{u}_m, \dot{u}_m), \quad a(u_m, \dot{u}_m) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} a(u_m, u_m).$$

The last formulas and (31) give

$$\begin{aligned}\frac{1}{2} m(\dot{u}_m(s), \dot{u}_m(s)) + \frac{1}{2} a(u_m(s), u_m(s)) &= \frac{1}{2} m(\dot{u}_m(0), \dot{u}_m(0)) + \frac{1}{2} a(u_m(0), u_m(0)) \\ &\quad + \int_0^s b(\dot{u}_m(t), \dot{u}_m(t); t) + c(u_m(t), \dot{u}_m(t); t)\,\mathrm{d}t.\end{aligned}$$

The last equality, (30), and Gronwall's inequality give the inequality

$$|\dot{u}_m(s)| + \|u_m(s)\| \leqslant C(\|u_0\| + |u_1|), \tag{32}$$

where $s \in [0,T]$ and the constant C is independent of m. The inequality (32) shows that u_m, $\dot{u}_m$, $m = 1, 2, \ldots$, are bounded in $L^2(0,T;V)$, $L^2(0,T;H)$. Thus there exist subsequences denoted by u_m, $\dot{u}_m$ again such that

$$u_m \rightharpoonup u \quad \text{in } L^2(0,T;V), \quad \dot{u} \rightharpoonup \dot{u} \quad \text{in } L^2(0,T;H)$$

as $m \to \infty$. The symbol $\rightharpoonup$ denotes the weak convergence. The last limits yield that u is a solution to (22) and the relations (24) are fulfilled. Moreover, (32) gives the inequality (28). It remains to prove the initial conditions (23). Let us consider $v(t) = w_k\varphi(t)$, where w_k is an arbitrary element from the sequence defined at the beginning of the proof and $\varphi(t)$ is a smooth function which satisfies $\varphi(T) = \dot{\varphi}(T) = 0$. If we follow the ideas in the proof of Lemma 3.2, then (26) yields the equality

$$\begin{aligned}\int_0^T m(u(t), w_k)\ddot{\varphi}(t) + (a(u(t), w_k) - b(\dot{u}(t), w_k; t) - c(u(t), w_k; t))\varphi(t)\,\mathrm{d}t \\ = m(u(0), w_k)\dot{\varphi}(0) - \langle \dot{u}(0), w_k\rangle_V \varphi(0).\end{aligned}$$

Moreover, for all $m \geqslant k$ the equalities

$$\begin{aligned}\int_0^T m(u_m(t), w_k)\ddot{\varphi}(t) + (a(u_m(t), w_k) - b(\dot{u}_m(t), w_k; t) - c(u_m(t), w_k; t))\varphi(t)\,\mathrm{d}t \\ = m(u_m(0), w_k)\dot{\varphi}(0) - m(\dot{u}_m(0), w_k)\varphi(0)\end{aligned}$$

hold. Then the last two equalities and (30) yield

$$m(u(0), w_k)\dot{\varphi}(0) - \langle \dot{u}(0), w_k\rangle_V \varphi(0) = m(u_0, w_k)\dot{\varphi}(0) - m(u_1, w_k)\varphi(0).$$

Since $\dot{\varphi}(0)$, $\varphi(0)$, and k are arbitrary, the initial conditions (23) are satisfied. $\square$

Let $b^n(\cdot,\cdot;t)$, $c^n(\cdot,\cdot;t)$, $n = 0, 1, \ldots$, belong to $C^1([0,T];W_H)$, $C([0,T];W_H)$ and

$$b^n(\cdot,\cdot;t) \to b^0(\cdot,\cdot;t), \quad c^n(\cdot,\cdot;t) \to c^0(\cdot,\cdot;t) \quad \text{in } C([0,T];W_H) \tag{33}$$

as $n \to \infty$. Let u_0^n, u_1^n, $n = 0, 1, \ldots$, belong to V and H and

$$u_0^n \to u_0^0 \quad \text{in } V, \quad u_1^n \to u_1^0 \quad \text{in } H \tag{34}$$

as $n \to \infty$. We study the sequence of the initial value problems

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m(u^n(t), v) + a(u^n(t), v) = b^n(\dot{u}^n(t), v; t) + c^n(u^n(t), v; t), \tag{35}$$

$$u^n(0) = u_0^n, \quad \dot{u}^n(0) = u_1^n, \tag{36}$$

$$u^n \in L^2(0,T;V), \quad \dot{u}^n \in L^2(0,T;H), \tag{37}$$

where (35) holds for all $v \in V$ and these equations are satisfied in the generalized sense.

Lemma 3.4 (Continuous dependence)**.** *Let $m(\cdot,\cdot)$, $a(\cdot,\cdot)$ be continuous symmetric bilinear forms on H and V satisfying* (20)*. Let u_0^n, u_1^n, $n=0,1,\ldots$, satisfy the limits* (34) *and $b^n(\cdot,\cdot;t)$, $c^n(\cdot,\cdot;t)$ from $C^1([0,T];W_H), C([0,T];W_H)$ satisfy the limits* (33)*. If u^n are solutions to* (35)–(37)*, then*

$$u^n \to u^0 \quad \text{in } L^2(0,T;V), \quad \dot{u}^n \to \dot{u}^0 \quad \text{in } L^2(0,T;H)$$

as $n\to\infty$.

P r o o f. Let w_k, $k=1,2,\ldots$, be the same sequence as in the proof of Lemma 3.3 and $u_m^n(t)$ are the approximations of solutions to the n-th problem (35)–(37), then we have

$$m(\ddot{u}_m^n(t),w_k)+a(u_m^n(t),w_k)=b^n(\dot{u}_m^n(t),w_k;t)+c^n(u_m^n(t),w_k;t), \tag{38}$$

where $k=1,\ldots,m$ and $t\in[0,T]$. If we follow the proof of Lemma 3.4 and consider the limits (34), we have the inequality

$$|\dot{u}_m^n(t)|+\|u_m^n(t)\|\leqslant C(\|u_0\|+|u_1|), \tag{39}$$

where $t\in[0,T]$ and C is a constant independent of n and m. The equations (38) yield the equations

$$\begin{aligned} m(\ddot{u}_m^n(t)-\ddot{u}_m^0(t),\dot{u}_m^n(t)-\dot{u}_m^0(t))+a(u_m^n(t)-u_m^0(t),\dot{u}_m^n(t)-\dot{u}_m^0(t))&\\ =A_1(t)+A_2(t)+A_3(t),&\end{aligned} \tag{40}$$

where

$$\begin{aligned} A_1(t)&=b^0(\dot{u}_m^n(t)-\dot{u}_m^0(t),\dot{u}_m^n(t)-\dot{u}_m^0(t))+c^0(u_m^n(t)-u_m^0(t),\dot{u}_m^n(t)-\dot{u}_m^0(t)),\\ A_2(t)&=b^n(\dot{u}_m^n(t),\dot{u}_m^n(t)-\dot{u}_m^0(t))-b^0(\dot{u}_m^n(t),\dot{u}_m^n(t)-\dot{u}_m^0(t)),\\ A_3(t)&=c^n(u_m^n(t),\dot{u}_m^n(t)-\dot{u}_m^0(t))-c^0(u_m^n(t),\dot{u}_m^n(t)-\dot{u}_m^0(t)).\end{aligned}$$

The equations (40) can be rewritten into

$$\begin{aligned} &\frac{1}{2}m(\dot{u}_m^n(s)-\dot{u}_m^0(s),\dot{u}_m^n(s)-\dot{u}_m^0(s))+\frac{1}{2}a(u_m^n(s)-u_m^0(s),u_m^n(s)-u_m^0(s))\\ &\quad=\frac{1}{2}m(\dot{u}_m^n(0)-\dot{u}_m^0(0),\dot{u}_m^n(0)-\dot{u}_m^0(0))+\frac{1}{2}a(u_m^n(0)-u_m^0(0),u_m^n(0)-u_m^0(0))\\ &\qquad+\int_0^s A_1(t)+A_2(t)+A_3(t)\,\mathrm{d}t,\end{aligned}$$

where $s \in [0,T]$. Then the inequality (39), Gronwall's inequality, and the shapes of $A_1(t)$, $A_2(t)$, $A_3(t)$ imply the inequality

$$\begin{aligned}|\dot u_m^n(s) - \dot u_m^0(s)|^2 + \|u_m^n(s) - u_m^0(s)\|^2 \\ \leqslant C(\|u_0^n - u_0^0\|^2 + |u_1^n - u_1^0|^2 + \|b^n(\cdot,\cdot;t) - b^0(\cdot,\cdot;t)\|_{C([0,T];W_H)} \\ + \|c^n(\cdot,\cdot;t) - c^0(\cdot,\cdot;t)\|_{C([0,T];W_H)}),\end{aligned}$$

where C is independent of n and m, from which the assertion of this lemma follows. □

Let $d_i(\cdot)$, $i = 1,\dots,k$, be a linear continuous functional on V. Let us define the subspace $\widehat V \subset V$ as follows:

$$\widehat V = \{u\colon\ d_i(u) = 0,\ i = 1,\dots,k\}.$$

The subspace $\widehat H \subset H$ is the closure of $\widehat V$ in H. We say that the initial conditions $u_0 \in V$, $u_1 \in H$ are compatible with $d_i(\cdot)$, $i = 1,\dots,k$, if $u_0 \in \widehat V$ and $u_1 \in \widehat H$.

Let u_0^n, u_1^n, $n = 0,1,\dots$, belong to V and H and let

$$u_0^n \to u_0^0 \ \text{ in } V, \quad u_1^n \to u_1^0 \ \text{ in } H \tag{41}$$

as $n \to \infty$. We study the problems

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} m(u^n(t),v) + a(u^n(t)v) = b^n(\dot u^n(t),v) + c^n(u^n(t),v), \tag{42}$$

$$u^n(0) = u_0^n \in V, \quad \dot u^n(0) = u_1^n \in H, \tag{43}$$

$$u^n \in L^2(0,T;V), \quad \dot u^n \in L^2(0,T;H), \tag{44}$$

$$d_i(u^n(t)) = 0, \quad i = 1,\dots,k, \tag{45}$$

where u_0^n, u_1^n are compatible with $d_i(\cdot)$, $i = 1,\dots,k$, the equations (42) are fulfilled in the generalized sense for all v satisfying $d_i(v) = 0$, $i = 1,\dots,k$, and the relations (45) are satisfied almost everywhere on $(0,T)$.

Theorem 3.1. *Let $m(\cdot,\cdot), a(\cdot,\cdot)$ be symmetric continuous bilinear forms on H and V satisfying* (20). *Let $b^n(\cdot,\cdot;t)$, $c^n(\cdot,\cdot;t)$ belong to $C^1([0,T];W_H)$ and $C([0,T];W_H)$ and satisfy the limits* (33). *Let the initial conditions u_0^n, u_1^n satisfy the limits* (41) *and be compatible with $d_i(\cdot)$, $i = 1,\dots,k$. Then the problems* (42)–(45) *are uniquely solvable and*

$$u^n \to u^0 \quad \text{in } L^2(0,T;V), \quad \dot u^n \to \dot u^0 \quad \text{in } L^2(0,T;H)$$

as $n \to \infty$.

P r o o f. The subspaces $\widehat{V}$ and $\widehat{H}$ were defined above. Let $\widehat{H}^{\perp}$ be the orthogonal complement of $\widehat{H}$ in H. Let $u \in L^2(0,T;V)$ and $\dot{u} \in L^2(0,T;H)$, then for all $v \in H$ and $\varphi \in C_0^{\infty}(0,T)$ the formula

$$\int_0^T (\dot{u},v)_H \varphi \,\mathrm{d}t = \int_0^T (u,v)_H \dot{\varphi} \,\mathrm{d}t$$

holds. Moreover, $u(t) \in \widehat{V}$ for almost all $t \in (0,T)$ and thus for every $w \in \widehat{H}^{\perp}$ the equality

$$\int_0^T (\dot{u},w)_H \varphi \,\mathrm{d}t = 0$$

holds. This yields that $\dot{u}(t) \in \widehat{H}$ for almost every $t \in (0,T)$. Applying Lemmas 3.2–3.4 and substituting in these lemmas the spaces V, H for the spaces $\widehat{V}$, $\widehat{H}$, we have the assertion of this theorem. $\square$

4. Proofs of main theorems

In this section we apply the abstract results from Section 3 to prove the theorems formulated in Section 2, which includes the existence, uniqueness, and continuous dependence on data for the problems $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$. To apply the abstract results from Section 3, let us define some auxiliary spaces, bilinear forms and linear functionals. The Hilbert space V is the subspace of $H^2(-\frac{1}{2}L,\frac{1}{2}L) \times H^1(-\frac{1}{2}L,\frac{1}{2}L)$, where (u,θ) belongs to V if the functions u, θ satisfy the boundary conditions (9). The Hilbert space H is the space $L^2(-\frac{1}{2}L,\frac{1}{2}L) \times L^2(-\frac{1}{2}L,\frac{1}{2}L)$. The spaces V, H are equipped with scalar products

$$\begin{aligned}
((u,\theta),(v,\varphi))_V &= (u,v)_{H^2(-L/2,L/2)} + (\theta,\varphi)_{H^1(-L/2,L/2)},\\
((u,\theta),(v,\varphi))_H &= (u,v)_{L^2(-L/2,L/2)} + (\theta,\varphi)_{L^2(-L/2,L/2)}.
\end{aligned}$$

Let us define a bilinear form on V

$$a((u,\theta),(v,\varphi)) = 2a_c(u,v) + 2D^2 a_c(\theta,\varphi) + a_{\mathrm{ver}}(u,v) + a_{\mathrm{tor}}(\theta,\varphi)$$

and a bilinear form on H

$$m((u,\theta),(v,\varphi)) = m_{\mathrm{ver}}(u,v) + m_{\mathrm{tor}}(\theta,\varphi).$$

Let us define other bilinear forms on H with the parameter t

$$\begin{aligned}
b((\dot{u},\dot{\theta}),(v,\varphi);t) &= f_1(\dot{u},v;t) + f_2(\dot{\theta},v;t) + g_1(\dot{u},\varphi;t) + g_2(\dot{\theta},\varphi;t),\\
c((u,\theta),(v,\varphi);t) &= f_3(\theta,v;t) + g_3(\theta,\varphi;t),
\end{aligned}$$

where $f_i(\cdot,\cdot,t)$ and $g_i(\cdot,\cdot,t)$ are defined in Section 2 and correspond to the functions $H_i(x,t)$ and $A_i(x,t)$, $i=1,2,3$.

Lemma 4.1. *Let y belong to $C^1([-\frac{1}{2}L,\frac{1}{2}L])$, H_1, H_2, A_1, A_2 belong to $C^1([-\frac{1}{2}L,\frac{1}{2}L]\times[0,T])$ and H_3, A_3 belong to $C([-\frac{1}{2}L,\frac{1}{2}L]\times[0,T])$. Then $b(\cdot,\cdot;t)$ and $c(\cdot,\cdot;t)$ belong to $C^1([0,T];W_H)$ and $C([0,T];W_H)$. The space W_H is defined in Section 3. Moreover, there exists a positive number α such that the inequalities*

$$\alpha\|u\|_H^2 \leqslant m(u,u), \quad \alpha\|v\|_V^2 \leqslant a(v,v) \tag{46}$$

hold for all $u\in H$ and $v\in V$.

Proof. The definition of $f_1(\cdot,\cdot;t)$ yields the inequality

$$\begin{aligned}&|f_1(u,v\colon\, t_1)-f_1(u,v\colon\, t_2)|\\ &\leqslant \|H_1(t_1,\cdot)-H_1(t_2,\cdot)\|_{C([-L/2,L/2])}\|u\|_{L^2(-L/2,L/2)}\|v\|_{L^2(-L/2,L/2)}\end{aligned}$$

which holds for all u,v from $L^2(-\frac{1}{2}L,\frac{1}{2}L)$. We can prove similar inequalities for all $f_i(\cdot,\cdot;t)$, $g_i(\cdot,\cdot;t)$, $i=1,2,3$. If we consider the definitions of $b(\cdot,\cdot;t)$, $c(\cdot,\cdot;t)$, and W_H, then we see that $b(\cdot,\cdot;t)$ and $c(\cdot,\cdot;t)$ belong to $C([0,T];W_H)$.

The definition of $f_1(\cdot,\cdot;t)$ yields the equality

$$\frac{\partial}{\partial t}f_1(u,v;t)=\int_{-L/2}^{L/2}\frac{\partial}{\partial t}H_1(x,t)uv\,\mathrm{d}x$$

which holds for all u,v from $L^2(-\frac{1}{2}L,\frac{1}{2}L)$. The last equality yields the inequality

$$\begin{aligned}&\Big|\frac{\partial}{\partial t}f_1(u,v;t_1)-\frac{\partial}{\partial t}f_1(u,v;t_2)\Big|\\ &\leqslant \Big\|\frac{\partial}{\partial t}H_1(t_1,\cdot)-\frac{\partial}{\partial t}H_1(t_2,\cdot)\Big\|_{C([-L/2,L/2])}\|u\|_{L^2(-L/2,L/2)}\|v\|_{L^2(-L/2,L/2)}\end{aligned}$$

which holds for all u,v from $L^2(-\frac{1}{2}L,\frac{1}{2}L)$. We can prove similar inequalities for all $f_i(\cdot,\cdot;t)$, $g_i(\cdot,\cdot;t)$, $i=1,2$, which yields that $b(\cdot,\cdot;t)$ belongs to $C^1([0,T];W_H)$. The first inequality in (46) is obvious. To prove the other inequality in (46), it is necessary to find a positive number β such that the inequalities

$$\begin{aligned}\beta\|u\|^2_{H^2(-L/2,L/2)} &\leqslant \Big\|\frac{\mathrm{d}^2u}{\mathrm{d}x^2}\Big\|^2_{L^2(-L/2,L/2)},\\ \beta\|v\|^2_{H^1(-L/2,L/2)} &\leqslant \Big\|\frac{\mathrm{d}v}{\mathrm{d}x}\Big\|^2_{L^2(-L/2,L/2)}\end{aligned} \tag{47}$$

hold for all u and v from $H^2(-\frac{1}{2}L,\frac{1}{2}L)$ and $H^1(-\frac{1}{2}L,\frac{1}{2}L)$ which satisfy the boundary conditions (9). Let us prove the first inequality by contradiction. Then there exist sequences β_n and $u_n \in H^2(-\frac{1}{2}L,\frac{1}{2}L)$ such that $\beta_n \to 0$ as $n \to \infty$, $\|u_n\|^2_{H^2(-L/2,L/2)} = 1$ and the inequalities

$$\beta_n \|u_n\|^2_{H^2(-L/2,L/2)} > \left\| \frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} \right\|^2_{L^2(-L/2,L/2)}$$

hold. This yields

$$\frac{\mathrm{d}^2 u_n}{\mathrm{d}x^2} \to 0 \quad \text{in } L^2(-\tfrac{1}{2}L,\tfrac{1}{2}L)$$

as $n \to \infty$. From Rellich's theorem it follows that there exists a convergent subsequence u_n in $H^2(-\frac{1}{2}L,\frac{1}{2}L)$ whose limit is w. The norm of w in $H^2(-\frac{1}{2}L,\frac{1}{2}L)$ is 1, w satisfies the boundary conditions (9), and the second derivative of w vanishes, which yields that w is a linear polynomial. Since w satisfies the boundary conditions (9), it vanishes, which is a contradiction. The remaining inequality in (47) can be proved in a similar way. $\square$

Lemma 4.2. *Let the limits*

$$H_1^n, H_2^n, H_3^n, A_1^n, A_2^n, A_3^n \to H_1^0, H_2^0, H_3^0, A_1^0, A_2^0, A_3^0 \quad \text{in } C([-\tfrac{1}{2}L,\tfrac{1}{2}L]\times[0,T])$$

hold as $n \to \infty$. Then we have

$$\begin{aligned} b^n(\cdot,\cdot;t) &\to b^0(\cdot,\cdot;t) \quad \text{in } C([0,T];W_H),\\ c^n(\cdot,\cdot;t) &\to c^0(\cdot,\cdot;t) \quad \text{in } C([0,T];W_H), \end{aligned}$$

as $n \to \infty$.

Proof. Lemma 4.1 implies that $b^n(\cdot,\cdot;t)$ and $c^n(\cdot,\cdot;t)$ belong to $C([0,T];W_H)$. The definitions yield the inequality

$$\begin{aligned} \sup_{t\in[0,T]} |c^n(u,v;t) - c^0(u,v;t)| \leqslant C(&\|H_3^n - H_3^0\|_{C([-L/2,L/2]\times[0,T])} \\ &+ \|A_3^n - A_3^0\|_{C([-L/2,L/2]\times[0,T])})\|u\|_H\|v\|_H \end{aligned}$$

which holds for all u, v from H, where C is a constant independent of u, v, H_3^n, A_3^n. A similar inequality holds for $b^n(\cdot,\cdot;t)$. $\square$

Let us study the questions connected with the way the main cables are fixed and deal with the problems $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ separately. Let us start with $\mathcal{D}_1$, and define linear functionals on V

$$d_1((u,\theta)) = h(u), \quad d_2((u,\theta)) = h(\theta). \tag{48}$$

Let us continue with $\mathcal{D}_2$ and define linear functionals

$$d_1((u,\theta)) = h_l(u),\ \ d_2((u,\theta)) = h_r(u),\ \ d_3((u,\theta)) = h_l(\theta),\ \ d_4((u,\theta)) = h_r(\theta). \tag{49}$$

Let us finish with $\mathcal{D}_3$ and define the functionals

$$d_1((u,\theta)) = h_l(u - D\theta), \quad d_2((u,\theta)) = h_r(u - D\theta), \quad d_3((u,\theta)) = h(u + D\theta). \tag{50}$$

If we consider that the linear forms $d_i(\cdot)$ in in Theorem 3.1 are defined by (48), (49), (50), which correspond to $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, then Theorem 2.1 immediately follows from Lemma 4.1 and Theorem 3.1 and Theorem 2.2 immediately follows from Lemmas 4.1–4.2 and Theorem 3.1.

5. Conclusion

The original Tacoma bridge exhibited relatively small vertical oscillations since the time it was opened. The bridge was stable with respect to torsional oscillations until one midspan cable band loosened. This led to torsional oscillations which lasted for approximately one hour and then the deck broke. The problems formulated in this paper describe motions of the center span and main cables under time dependent forces created by lateral winds. The problems describe deflections of the center span from the steady state equilibrium under the gravitational forces acting on the center span and main cables. The evolution variational equations were formulated and analyzed. These equations describe the behavior of the center span and the main cables in three different situations: both main cables have the fastened midspan cable bands, only one cable has the fastened midspan cable band, and the main cables have no fastened midspan cable bands. The problems were analyzed and the existence, uniqueness and continuous dependence on data were proved.

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