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BASIC EQUATIONS OF G-ALMOST GEODESIC MAPPINGS OF THE SECOND TYPE, WHICH HAVE THE PROPERTY OF RECIPROCITY

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Abstract. We study G-almost geodesic mappings of the second type $\underset{\theta}{\pi}2(e)$, $\theta = 1, 2$ between non-symmetric affine connection spaces. These mappings are a generalization of the second type almost geodesic mappings defined by N. S. Sinyukov (1979). We investigate a special type of these mappings in this paper. We also consider e-structures that generate mappings of type $\underset{\theta}{\pi}2(e)$, $\theta = 1, 2$. For a mapping $\underset{\theta}{\pi}2(e, F)$, $\theta = 1, 2$, we determine the basic equations which generate them.

Keywords: non-symmetric affine connection; almost geodesic mapping; G-almost geodesic mapping; property of reciprocity; almost geodesic mapping of the second type

MSC 2010: 53B05, 53B20, 53C15

1. Introduction

Let us consider two N-dimensional differentiable manifolds GA_N and $G\bar{A}_N$ and a differentiable mapping $f\colon GA_N \to G\bar{A}_N$. We can consider these manifolds together with this *mapping system of local coordinates.* Namely, if $f\colon M \in GA_N \to \overline{M} \in G\bar{A}_N$ and if $(\mathcal{U}, \varphi)$ is the local chart around the point M then $\varphi(M) = x = (x^1, \ldots, x^N) \in E^N$. In this case, we define mapping $\overline{\varphi} = \varphi \circ f^{-1}$ for the coordinate mapping in $G\bar{A}_N$, and then

$$\overline{\varphi}(\overline{M}) = (\varphi \circ f^{-1})(f(M)) = \varphi(M) = x = (x^1, \ldots, x^N) \in E^N.$$

The points M and $\overline{M} = f(M)$ have the same local coordinates in this case. If the connection coefficients $L^i_{jk}(x)$ and $\bar{L}^i_{jk}(x)$ of the affine connections introduced

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in GA_N and $G\bar{A}_N$, respectively, are non-symmetric in lower indices then GA_N and $G\bar{A}_N$ are *general affine connection spaces.*

One says that the reciprocal mapping $f\colon GA_N \to G\bar{A}_N$ is *geodesic*, [17], [16] if geodesics of the space GA_N pass to geodesics of the space $G\bar{A}_N$. Generalizing the concept of a geodesic mapping between Riemannian spaces and symmetric affine connection ones, Sinyukov [18] introduced the following notions:

A curve $l\colon x^h = x^h(t)$ is called an *almost geodesic line* if its tangential vector $\lambda^h(t) = \mathrm{d}x^h/\mathrm{d}t \neq 0$ satisfies the equation

$$\bar{\lambda}^h_{(2)} = \bar{a}(t)\lambda^h + \bar{b}(t)\bar{\lambda}^h_{(1)},$$

where $\bar{\lambda}^h_{(1)} = \lambda^h_{||p}\lambda^p$, $\bar{\lambda}^h_{(2)} = \bar{\lambda}^h_{(1)||p}\lambda^p$. Here $\bar{a}(t)$ and $\bar{b}(t)$ are functions of a parameter t and $||$ denotes the covariant derivative with regard to the connection in $\bar{A}_N$.

Definition 1.1. A mapping f of a symmetric affine connection space A_N onto a space $\bar{A}_N$ is called an *almost geodesic mapping* if any geodesic line of the space A_N is mapped into an almost geodesic line of the space $\bar{A}_N$.

A lot of research papers and monographs [1]–[23] have been dedicated to the theory of geodesic mappings of Riemannian spaces, affine connected ones and their generalizations. Sinyukov [18] and Mikeš [1], [2], [12], [13], [23] gave some other significant contributions to the study of almost geodesic mappings of affine connected spaces and singled out three types π_1, π_2, π_3 of almost geodesic mappings between affine connected spaces without torsion.

In a general affine connection space GA_N, with non-symmetric affine connection L, one can define four kinds of a covariant derivative [15], [14]. Let us denote a covariant derivative of a kind θ ($\theta = 1, \ldots, 4$) with regard to affine connections of GA_N and $G\bar{A}_N$ by $\underset{\theta}{|}$ and $\underset{\theta}{||}$, respectively.

For example, a tensor a^i_j in GA_N satisfies

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{\alpha m}a^\alpha_j - L^\alpha_{jm}a^i_\alpha \quad \text{and} \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + L^i_{m\alpha}a^\alpha_j - L^\alpha_{mj}a^i_\alpha.$$

Thus, in the case of a space with a non-symmetric affine connection we can define two kinds of almost geodesic lines and two kinds of almost geodesic mappings [20]–[19].

Definition 1.2. A curve $l\colon x^h = x^h(t)$ on $G\bar{A}_N$ is called [20]–[19] a *G-almost geodesic line of the first kind* if its tangent vector $\lambda^h(t) = \mathrm{d}x^h/\mathrm{d}t \neq 0$ satisfies the equation

$$\underset{1}{\bar{\lambda}}{}^h{}_{(2)} = \underset{1}{\bar{a}}(t)\lambda^h + \underset{1}{\bar{b}}(t)\underset{1}{\bar{\lambda}}{}^h{}_{(1)},$$

where $\underset{1}{\bar{\lambda}}{}^h{}_{(1)} = \lambda^h{}_{\underset{1}{||}\alpha}\lambda^\alpha$, $\underset{1}{\bar{\lambda}}{}^h{}_{(2)} = \underset{1}{\bar{\lambda}}{}^h{}_{(1)\underset{1}{||}\alpha}\lambda^\alpha$ and $\underset{1}{\bar{a}}(t)$ and $\underset{1}{\bar{b}}(t)$ are functions of a parameter t.

Definition 1.3. A curve $l\colon x^h = x^h(t)$ is called a *G-almost geodesic line of the second kind* if its tangential vector $\lambda^h(t) = \mathrm{d}x^h/\mathrm{d}t \neq 0$ satisfies the equation

$$\underset{2}{\bar\lambda}{}^h{}_{(2)} = \underset{2}{\bar a}(t)\lambda^h + \underset{2}{\bar b}(t)\underset{2}{\bar\lambda}{}^h{}_{(1)},$$

where $\underset{2}{\bar\lambda}{}^h{}_{(1)} = \lambda^h_{\underset{2}{||}\alpha}\lambda^\alpha$, $\underset{2}{\bar\lambda}{}^h{}_{(2)} = \underset{2}{\bar\lambda}{}^h{}_{(1)\underset{2}{||}\alpha}\lambda^\alpha$, $\underset{2}{\bar a}(t)$ and $\underset{2}{\bar b}(t)$ are functions of a parameter t.

Definition 1.4. A mapping f of the space GA_N onto a space $G\bar A_N$ is called a *G-almost geodesic mapping of the first kind* if any geodesic line of the space GA_N turns into an almost geodesic line of the first kind of the space $G\bar A_N$.

Definition 1.5. A mapping f is called a *G-almost geodesic mapping of the second kind* if any geodesic line of the space GA_N turns into almost geodesic line of the second kind of the space $G\bar A_N$.

We can put

$$P^h_{ij}(x) = \bar L^h_{ij}(x) - L^h_{ij}(x),$$

where $L^h_{ij}(x)$, $\bar L^h_{ij}(x)$ are connection coefficients of the spaces GA_N and $G\bar A_N$, $N > 2$, together with the mapping f system of local coordinates, and P^h_{ij} is a deformation tensor. From [20], it follows that the succeeding results hold:

Theorem 1.1. *A mapping f of the space GA_N onto $G\bar A_N$ is a G-almost geodesic mapping of the first kind if and only if the deformation tensor P^h_{ij} satisfies the conditions*

$$(P^h_{\alpha\beta\underset{1}{|}\gamma} + P^h_{\delta\alpha}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = \underset{1}{b}P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \underset{1}{a}\lambda^h \tag{1.1}$$

identically, where $\underset{1}{a}$ and $\underset{1}{b}$ are functions.

Theorem 1.2. *A mapping f of the space GA_N onto $G\bar A_N$ is a G-almost geodesic mapping of the second kind if and only if the deformation tensor P^h_{ij} satisfies the conditions*

$$(P^h_{\alpha\beta\underset{2}{|}\gamma} + P^h_{\alpha\delta}P^\delta_{\beta\gamma})\lambda^\alpha\lambda^\beta\lambda^\gamma = \underset{2}{b}P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta + \underset{2}{a}\lambda^h \tag{1.2}$$

identically, where $\underset{2}{a}$ and $\underset{2}{b}$ are functions.

We are going to present basic equations of G-almost geodesic mappings of the type $\underset{\theta}{\pi}_2(e)$, $\theta = 1, 2$, between non-symmetric affine connection spaces $\mathbb{GA}_N$ and $\mathbb{G}\bar{\mathbb{A}}_N$ in this paper.

2. G-ALMOST GEODESIC MAPPINGS OF THE SECOND TYPE

Sinyukov (see [18]) introduced almost geodesic mapping of the second type π_2 for affine connection spaces without torsion with the condition

$$b = \frac{b_{\gamma\delta}\lambda^\gamma\lambda^\delta}{\sigma_\alpha\lambda^\alpha},$$

where $\sigma_\alpha\lambda^\alpha \neq 0$ and $b_{\gamma\delta}$ is a twice covariant tensor.

Analogously, a G-almost geodesic mapping of the first kind of a non-symmetric affine connection space is an almost geodesic mapping of the second type $\underset{1}{\pi}_2$ if the function $\underset{1}{b}$ satisfies the condition

$$\underset{1}{b} = \frac{\underset{1}{b}_{\gamma\delta}\lambda^\gamma\lambda^\delta}{\sigma_\alpha\lambda^\alpha},$$

where $\sigma_\alpha\lambda^\alpha \neq 0$ and $\underset{1}{b}_{\gamma\delta}$ is a twice covariant tensor.

Let

$$P^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = 2\sigma_\alpha\lambda^\alpha F^h_\beta\lambda^\beta + 2\psi_\alpha\lambda^\alpha\lambda^h.$$

Then

$$(P^h_{\alpha\beta} - 2\sigma_\alpha F^h_\beta - 2\psi_\alpha\delta^h_\beta)\lambda^\alpha\lambda^\beta \equiv 0,$$

wherefrom

$$P^h_{\underline{ij}} = \psi_i\delta^h_j + \psi_j\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i.$$

Here, ψ_i and σ_i are vectors, F^i_j is a tensor, $\underline{ij}$ denotes a symmetrization with division, $\underset{\vee}{ij}$ denotes an anti-symmetrization with division and δ^h_i is the Kronecker symbol. We can put $P^h_{\underset{\vee}{ij}} = \xi^h_{ij}$.

Then

$$P^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i + \xi^h_{ij}. \tag{2.1}$$

In the equation (2.1), magnitudes ψ_i, σ_i are covariant vectors, F^h_i is a tensor and ξ^h_{ij} is an anti-symmetric tensor.

After substituting the equation (2.1) in the equation (1.1), we conclude that

$$F^h_{i\underset{1}{|}j} + F^h_{j\underset{1}{|}i} + F^h_\delta F^\delta_i\sigma_j + F^h_\delta F^\delta_j\sigma_i + \xi^h_{\delta i}F^\delta_j + \xi^h_{\delta j}F^\delta_i = \mu_i F^h_j + \nu_j F^h_i + \nu_i\delta^h_j + \nu_j\delta^h_i, \tag{2.2}$$

where μ_i and ν_i are covariant vectors.

Conditions (2.1) and (2.2) are the *basic equations* of the mapping $\underset{1}{\pi}_2$.

A G-*almost geodesic mapping of the second kind* is a G-almost geodesic mapping of the second type $\underset{2}{\pi}2$ if it satisfies the following condition for the function $\underset{2}{b}$ in (1.2):

$$\underset{2}{b} = \frac{\underset{2}{b}_{\gamma\delta}\lambda^\gamma\lambda^\delta}{\sigma_\alpha\lambda^\alpha},$$

where $\sigma_\alpha\lambda^\alpha \neq 0$ and $\underset{2}{b}_{\gamma\delta}$ is a twice covariant tensor.

Using the method from the previous case, we get

$$\begin{aligned}(2.3)\qquad F^h_{i\underset{2}{|}j} + F^h_{j\underset{2}{|}i} + F^h_\delta F^\delta_i\sigma_j + F^h\delta F^\delta_j\sigma_i + \xi^h_{i\delta}F^\delta_j + \xi^h_{j\delta}F^\delta_i \\ = \mu_i F^h_j + \mu_j F^h_i + \nu_i\delta^h_j + \nu_j\delta^h_i,\end{aligned}$$

where μ_i, ν_i are covariant vectors.

Conditions (2.1) and (2.3) are the *basic equations* of G-almost geodesic mappings of the type $\underset{2}{\pi}2$.

Remark 2.1. If $\sigma_i \equiv 0$ in the equation (2.1) then almost geodesic mappings are reduced to the geodesic ones. On the other hand, if $\psi_i \equiv 0$, then this mapping is a canonical almost geodesic one (see [21]). In the case $\sigma_i \equiv 0$ and $\psi_i \equiv 0$, we have a trivial almost geodesic mapping. We are working with nontrivial almost geodesic mappings only in the sequel.

3. The property of reciprocity of G-almost geodesic mappings of the second type

A mapping $f\colon GA_N \to G\bar{A}_N$ of the type $\underset{1}{\pi}2$ has the *property of reciprocity*, if its inverse mapping $f^{-1}\colon G\bar{A}_N \to GA_N$ (provided it exists) is of the $\underset{1}{\pi}2$ type, and f^{-1} corresponds to the same tensor F^h_i, see also [21]. Since the inverse mapping $f^{-1}\colon G\bar{A}_N \to GA_N$ satisfies

$$\overline{P}^h_{ij} = -P^h_{ij},$$

we can put the following in the equation (2.1):

$$\overline{\psi}_i = -\psi_i, \quad \overline{\sigma}_i = -\sigma_i, \quad \overline{F}^h_i = F^h_i, \quad \bar{\xi}^h_{ij} = -\xi^h_{ij}.$$

A mapping $f\colon GA_N \to G\bar{A}_N$ of the type $\underset{1}{\pi}2$ has the property of reciprocity if and only if the tensor F^h_i of the space $G\bar{A}_N$ satisfies the equation of the form (2.2), i.e.,

$$(3.1)\qquad F^h_{(i\underset{1}{||}j)} - F^h_\alpha F^\alpha_{(i}\sigma_{j)} - \xi^h_{\alpha(i}F^\alpha_{j)} = \bar{\mu}_{(i}F^h_{j)} + \bar{\nu}_{(i}\delta^h_{j)},$$

where (ij) is a symmetrization without division with respect to i and j, and $\underset{1}{||}$ is a covariant derivative of the first kind in $G\bar{A}_N$. Inserting a covariant derivative of the first kind in GA_N into the equation (3.1) we get

$$F^h_\alpha F^\alpha_{(i}\sigma_{j)} + \xi^h_{\alpha(i}F^\alpha_{j)} = \bar{\bar{\mu}}_{(i}F^h_{j)} + \bar{\bar{\nu}}_{(i}\delta^h_{j)},$$

where vectors $\bar{\bar{\mu}}_i$, $\bar{\bar{\nu}}_i$ are expressed by μ_i, ν_i, $\bar{\mu}_i$, $\bar{\nu}_i$, ψ_i, σ_i, F^h_i. Since $\sigma \neq 0$, we get

$$F^h_\alpha F^\alpha_i = p\delta^h_i + qF^h_i, \tag{3.2}$$

where p and q are functions.

Based on the facts given above, we have:

Theorem 3.1. *The relation* (3.2) *expresses the necessary and sufficient condition for a mapping* $\underset{1}{\pi}2\colon GA_N \to G\bar{A}_N$ *to have the property of reciprocity.*

The equations (2.1) and (2.2) are invariant under the mapping $\underset{1}{\pi}2$ of a tensor

$$\widetilde{F}^h_i = rF^h_i + s\delta^h_i, \quad r \neq 0.$$

Then we have

$$\widetilde{F}^h_\alpha \widetilde{F}^\alpha_i = \tilde{p}\delta^h_i + \tilde{q}\widetilde{F}^h_i,$$

where

$$\tilde{p} = r^2p - s^2 - srq, \quad \tilde{q} = 2s + rq.$$

Here we can select invariants r and s such that

$$\tilde{q} \equiv 0, \quad \tilde{p} = \tilde{e}\ (= \pm 1, 0).$$

In this case, we have

$$\widetilde{F}^h_\alpha \widetilde{F}^\alpha_i = \tilde{e}\delta^h_i.$$

Based on the facts given above, we can put

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \quad e = \pm 1, 0. \tag{3.3}$$

Substituting the equation (3.3) into the condition (2.2), we get

$$F^h_{(i\underset{1}{|}j)} + \xi^h_{\alpha(i}F^\alpha_{j)} = \mu_{(i}F^h_{j)} + \nu_{(i}\delta^h_{j)}. \tag{3.4}$$

Hence, a G-almost geodesic mapping $f\colon\ GA_N \to G\bar{A}_N$ of the type $\underset{1}{\pi}2$ which has the property of reciprocity is determined by the equations (2.1), (3.3) and (3.4) (see [21]). This mapping is denoted by $\underset{1}{\pi}2(e)$.

In the case of the G-almost geodesic mapping $f\colon\ GA_N \to G\bar{A}_N$ of the type $\underset{2}{\pi}2$ which has the property of reciprocity, it is determined by the equations

$$
\begin{aligned}
(3.5)\qquad P^h_{ij} &= \psi_i\delta^h_j + \psi_j\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i + \xi^h_{ij},\\
F^h_{(i\underset{2}{|}j)} - \xi^h_{\alpha(i}F^\alpha_{j)} &= \mu_{(i}F^h_{j)} + \nu_{(i}\delta^h_{j)},\\
F^h_\alpha F^\alpha_i &= e\delta^h_i, \quad e = \pm 1, 0.
\end{aligned}
$$

This mapping is denoted by $\underset{2}{\pi}2(e)$.

4. On e-structures that determine G-almost geodesic mappings of type $\pi_2(e)$ of first and second kinds

Definition 4.1. A tensor F^h_i which satisfies the conditions (3.3) and (3.4) is called an *e-structure* which determines a G-almost geodesic mapping $f\colon\ GA_N \to G\bar{A}_N$ of the type $\underset{1}{\pi}2(e)$.

Theorem 4.1. *An e-structure F^h_i determines a G-almost geodesic mapping $f\colon$ $GA_N \to G\bar{A}_N$ of the type $\underset{1}{\pi}2(e)$, $e = \pm 1$, if and only if it satisfies the conditions*

$$(4.1)\qquad F^h_{(i\underset{1}{|}j)} + \xi^h_{\alpha(i}F^\alpha_{j)} = \mu_{(i}F^h_{j)} - \mu_\alpha F^\alpha_{(i}\delta^h_{j)},$$

$$(4.2)\qquad F^h_\alpha F^\alpha_i = e\delta^h_i.$$

P r o o f. Based on the covariant derivative of the first kind of the condition (4.2) in the direction x^j, we get

$$(4.3)\qquad F^h_{\alpha\underset{1}{|}j}F^\alpha_i + F^\alpha_{i\underset{1}{|}j}F^h_\alpha = 0.$$

After the symmetrization of the equation (4.3) with respect to the indices i and j, we have

$$(4.4)\qquad F^h_{\alpha\underset{1}{|}j}F^\alpha_i + F^h_{\alpha\underset{1}{|}i}F^\alpha_j + F^\alpha_{(i\underset{1}{|}j)}F^h_\alpha = 0.$$

Based on the equations (3.4) and (4.4), we conclude that

$$F^h_{\alpha\underset{1}{|}i}F^\alpha_j + F^h_{\alpha\underset{1}{|}j}F^\alpha_i + e\delta^h_{(i}\mu_{j)} + F^h_{(i}\nu_{j)} + F^h_\alpha F^\beta_{(i}\xi^\alpha_{j)\beta} = 0.$$

Composing the previous relation with F^j_k, one obtains

$$(4.5)\qquad eF^h_{k\underset{1}{|}i} + F^h_{\alpha\underset{1}{|}\beta}F^\alpha_i F^\beta_k + e\delta^h_i \mu_\alpha F^\alpha_k + e\mu_i F^h_k + F^h_i \nu_\alpha F^\alpha_k + e\delta^h_k \nu_i$$
$$+ F^h_i \nu_\alpha F^\alpha_k + e\delta^h_k \nu_i + F^h_\alpha F^\beta_i F^\gamma_k \xi^\alpha_{\gamma\beta} + eF^h_\alpha \xi^\alpha_{ik} = 0.$$

After symmetrizing of the equation (4.5) by indices i and k, we infer

$$(4.6)\qquad eF^h_{(i\underset{1}{|}k)} + F^h_{(\alpha\underset{1}{|}\beta)}F^\alpha_i F^\beta_k + e\delta^h_{(i}F^\alpha_{k)}\mu_\alpha + e\mu_{(i}F^h_{k)} + \nu_\alpha F^h_{(i}F^\alpha_{k)} + e\delta^h_{(i}\nu_{k)} = 0.$$

From the equation (3.4), we have

$$(4.7)\qquad eF^h_{(i\underset{1}{|}k)} + F^h_{(\alpha\underset{1}{|}\beta)}F^\alpha_i F^\beta_k = F^h_i(\nu_\beta F^\beta_k + e\mu_k) + F^h_k(\nu_\alpha F^\alpha_i + e\mu_i)$$
$$+ e\delta^h_i(\mu_\beta F^\beta_k + \nu_k) + e\delta^h_k(\mu_\alpha F^\alpha_i + \nu_i).$$

Now, from the equations (4.6) and (4.7) we obtain

$$F^h_i(F^\alpha_k \nu_\alpha + e\mu_k) + F^h_k(F^\alpha_i \nu_\alpha + e\mu_i) + e\delta^h_i(F^\alpha_k \mu_k + \nu_k) + e\delta^h_k(F^\alpha_i \mu_\alpha + \nu_i) = 0.$$

By examining the last equality, we conclude that

$$(4.8)\qquad F^\alpha_i \mu_\alpha + \nu_i = 0, \quad \text{i.e.} \quad \nu_i = -F^\alpha_i \mu_\alpha.$$

After substituting (4.8) into (3.4), we obtain the relation (4.1) is valid. □

Analogously, in the case of G-almost geodesic mappings of the type $\pi_2(e)$ of the second kind we obtain

Definition 4.2. A tensor F^h_i which satisfies the conditions (3.5) is an *e-structure* which determines a G-almost geodesic mapping of the type $\underset{2}{\pi}_2(e)$.

Theorem 4.2. *An e-structure F^h_i determines a G-almost geodesic mapping of the type $\underset{2}{\pi}_2(e)$, $e = \pm 1$, if and only if it satisfies the conditions*

$$(4.9)\qquad F^h_{(i\underset{2}{|}j)} - \xi^h_{\alpha(i}F^\alpha_{j)} = \mu_{(i}F^h_{j)} - \mu_\alpha F^h_{(i}\delta^h_{j)},$$

$$(4.10)\qquad F^h_\alpha F^\alpha_i = e\delta^h_i.$$

Theorem 4.3. *An e-structure F^h_i which determines a G-almost geodesic mapping of the type $\underset{1}{\pi}_2(e)$, $e=\pm 1$, satisfies the following conditions:*

$$F^h_{i\underset{1}{|}(jk)} + \underset{1}{\xi}^h_{ijk} = \mu_{(j\underset{1}{|}k)}F^h_i + \mu_{[i\underset{1}{|}k]}F^h_j + \mu_{[i\underset{1}{|}j]}F^h_k - \mu_{\alpha\underset{1}{|}(j}F^\alpha_{k)}\delta^h_i + \mu_{\alpha\underset{1}{|}[i}F^\alpha_{k]}\delta^h_j + \mu_{\alpha\underset{1}{|}[i}F^\alpha_{j]}\delta^h_k + \underset{1}{\overset{1}{\theta}}{}^h_{ikj}, \tag{4.11}$$

where $[i,j]$ is an anti-symmetrization without division,

$$\underset{1}{\overset{1}{\theta}}{}^h_{ikj} = \underset{1}{\overset{2}{\theta}}{}^h_{ijk} + \underset{1}{\overset{2}{\theta}}{}^h_{ikj} - \underset{1}{\overset{2}{\theta}}{}^h_{jki} - \underset{1}{R}{}^h_{\alpha ij}F^\alpha_k + \underset{1}{R}{}^\alpha_{kij}F^h_\alpha - \underset{1}{R}{}^h_{\alpha ik}F^\alpha_j + \underset{1}{R}{}^\alpha_{jik}F^h_\alpha + L^\alpha_{[ij]}F^h_{k\underset{1}{|}\alpha} + L^\alpha_{[ik]}F^h_{j\underset{1}{|}\alpha},$$

$$\underset{1}{\overset{2}{\theta}}{}^h_{ijk} = \mu_i F^h_{j\underset{1}{|}k} + \mu_i F^h_{i\underset{1}{|}k} - \mu_\alpha F^\alpha_{i\underset{1}{|}k}\delta^h_j - \mu_\alpha F^\alpha_{j\underset{1}{|}k}\delta^h_i - \xi^h_{\alpha i}F^\alpha_{j\underset{1}{|}k} - \xi^h_{\alpha j}F^\alpha_{i\underset{1}{|}k},$$

$$\underset{1}{\xi}^h_{ijk} = \xi^h_{\alpha[i\underset{1}{|}k]}F^h_j + \xi^h_{\alpha[i\underset{1}{|}j]}F^h_k + \xi^h_{\alpha(j\underset{1}{|}k)]}F^h_i,$$

and

$$\underset{1}{R}{}^h_{ijk} = L^h_{ij,k} - L^h_{ik,j} + L^\alpha_{ij}L^h_{\alpha k} - L^\alpha_{ik}L^h_{\alpha j}$$

is a curvature tensor of the first kind (*see* [15]).

P r o o f. Taking the covariant derivative of the first kind of (4.1) in the direction of x^k, we get

$$F^h_{i\underset{1}{|}jk} + F^h_{j\underset{1}{|}ik} + \xi^h_{\alpha i\underset{1}{|}k}F^\alpha_j + \xi^h_{\alpha j\underset{1}{|}k}F^\alpha_i = \mu_{i\underset{1}{|}k}F^h_j + \mu_{j\underset{1}{|}k}F^h_i - \mu_{\alpha\underset{1}{|}k}F^\alpha_i\delta^h_j - \mu_{\alpha\underset{1}{|}k}F^\alpha_j\delta^h_i + \underset{1}{\overset{2}{\theta}}{}^h_{ijk}. \tag{4.12}$$

Alternating this equation with respect to i and k and using the first Ricci identity, we get

$$\begin{aligned} F^h_{i\underset{1}{|}jk} - F^h_{k\underset{1}{|}ji} + \xi^h_{\alpha i\underset{1}{|}k}F^\alpha_j - \xi^h_{\alpha k\underset{1}{|}i}F^\alpha_j + \xi^h_{\alpha j\underset{1}{|}k}F^\alpha_i - \xi^h_{\alpha j\underset{1}{|}i}F^\alpha_k \\ = \mu_{i\underset{1}{|}k}F^h_j - \mu_{k\underset{1}{|}i}F^h_j + \mu_{j\underset{1}{|}k}F^h_i - \mu_{j\underset{1}{|}i}F^h_k \\ - \mu_{\alpha\underset{1}{|}k}F^\alpha_i\delta^h_j + \mu_{\alpha\underset{1}{|}i}F^\alpha_k\delta^h_j - \mu_{\alpha\underset{1}{|}k}F^\alpha_j\delta^h_i + \mu_{\alpha\underset{1}{|}i}F^\alpha_j\delta^h_k + \underset{1}{\overset{3}{\theta}}{}^h_{ijk}, \end{aligned} \tag{4.13}$$

where

$$\underset{1}{\overset{3}{\theta}}{}^h_{ijk} = \underset{1}{\overset{2}{\theta}}{}^h_{ijk} - \underset{1}{\overset{2}{\theta}}{}^h_{kji} - \underset{1}{R}{}^h_{\alpha ik}F^\alpha_j + \underset{1}{R}{}^\alpha_{jik}F^h_\alpha + L^\alpha_{[ik]}F^h_{j\underset{1}{|}\alpha}.$$

Let us interchange indices j and k in (4.13). Then we have

$$(4.14)\qquad \begin{aligned} &F^h_{\underset{1}{i|kj}} - F^h_{\underset{1}{j|ki}} + \xi^h_{\underset{1}{\alpha i|j}} F^\alpha_k - \xi^h_{\underset{1}{\alpha j|i}} F^\alpha_k + \xi^h_{\underset{1}{\alpha k|j}} F^\alpha_i - \xi^h_{\underset{1}{\alpha k|i}} F^\alpha_j \\ &= \mu_{\underset{1}{i|j}} F^h_k - \mu_{\underset{1}{j|i}} F^h_k + \mu_{\underset{1}{k|j}} F^h_i - \mu_{\underset{1}{k|i}} F^h_j \\ &\quad - \mu_{\underset{1}{\alpha|j}} F^\alpha_i \delta^h_k + \mu_{\underset{1}{\alpha|i}} F^\alpha_j \delta^h_k - \mu_{\underset{1}{\alpha|j}} F^\alpha_k \delta^h_i + \mu_{\underset{1}{\alpha|i}} F^\alpha_k \delta^h_j + \overset{3}{\underset{1}{\theta}}{}^h_{ikj}. \end{aligned}$$

Adding the equations (4.12) and (4.14) together with some other calculations proves the equation (4.11) holds. □

We are going to proceed with the study of conditions on the e-structure that generates G-almost geodesic mappings of the type $\underset{1}{\pi}_2(e)$, $e = \pm 1$.

Definition 4.3. A G-almost geodesic mapping $f\colon GA_N \to G\bar{A}_N$ of the type $\underset{\theta}{\pi}_2(e)$ $(\theta = 1, 2)$, which satisfies the condition $F^\alpha_\alpha = 0$ is a G-almost geodesic mapping of the type $\underset{\theta}{\pi}_2(e, F)$ $(\theta = 1, 2)$.

Perform a contraction by indices h and i in the algebraic condition (4.2). Then we have the equation

$$F^\alpha_\beta F^\beta_\alpha = eN.$$

Let us take its second covariant derivative of the first kind in the directions x^i and x^k:

$$(4.15)\qquad F^\alpha_\beta F^\beta_{\underset{1}{\alpha|jk}} + F^\alpha_{\underset{1}{\beta|j}} F^\beta_{\underset{1}{\alpha|k}} = 0.$$

After the composing the equation (4.11) with F^i_k and using the result (4.15), we get

$$(4.16)\qquad \begin{aligned} -2F^\alpha_{\underset{1}{\beta|j}} F^\beta_{\underset{1}{\alpha|k}} + F^\alpha_\beta \xi^\beta_{\alpha jk} &= \mu_{\underset{1}{(j|k)}} eN - F^\beta_\beta \mu_{\underset{1}{\alpha|(j}} F^\alpha_{k)} + \mu_{\underset{1}{(\alpha|\beta)}} F^\alpha_k F^\beta_j \\ &\quad - e\mu_{\underset{1}{(j|k)}} + F^\alpha_\beta \overset{1}{\underset{1}{\theta}}{}^\beta_{\alpha kj}. \end{aligned}$$

Using the condition $F^\alpha_\alpha = 0$, from (4.16) we have

$$(4.17)\qquad e(N-1)\mu_{\underset{1}{(j|k)}} + \mu_{\underset{1}{(\alpha|\beta)}} F^\alpha_j F^\beta_k = \overset{4}{\underset{1}{\theta}}{}_{jk},$$

where we denoted $\overset{4}{\underset{1}{\theta}}{}_{jk} = F^\alpha_\beta \overset{1}{\underset{1}{\theta}}{}^\beta_{\alpha kj} + 2F^\alpha_{\underset{1}{\beta|j}} F^\beta_{\underset{1}{\alpha|k}} - F^\alpha_\beta \xi^\beta_{\alpha jk}$. Composing (4.17) with $F^j_{j'} F^k_{k'}$ we obtain

$$(4.18)\qquad e(N-1)\mu_{\underset{1}{(\alpha|\beta)}} F^\alpha_j F^\beta_k + \mu_{\underset{1}{(j|k)}} = \overset{4}{\underset{1}{\theta}}{}_{\alpha\beta} F^\alpha_j F^\beta_k.$$

Now, from (4.17) and (4.18) we obtain

$$\mu_{(i\underset{1}{|}j)} = \underset{1}{\overset{5}{\theta}}_{ij}, \tag{4.19}$$

where $\underset{1}{\overset{5}{\theta}}_{ij} = N^{-1}(2-N)^{-1}[\underset{1}{\overset{4}{\theta}}_{\alpha\beta}F_i^\alpha F_j^\beta - e(N-1)\underset{1}{\overset{4}{\theta}}_{ij}]$. Let us take the covariant derivative of the first kind of the (4.19) in the direction of x^k:

$$\mu_{i\underset{1}{|}jk} + \mu_{j\underset{1}{|}ik} = \underset{1}{\overset{5}{\theta}}_{ij\underset{1}{|}k} \tag{4.20}$$

and alternate this equation with respect to the indices i and k. Then we have

$$\mu_{i\underset{1}{|}jk} - \mu_{k\underset{1}{|}ji} - \underset{1}{R}^\alpha_{ijk}\mu_\alpha - L^\alpha_{[jk]}\mu_{i\underset{1}{|}\alpha} = \underset{1}{\overset{5}{\theta}}_{ij\underset{1}{|}k} - \underset{1}{\overset{5}{\theta}}_{kj\underset{1}{|}i}.$$

Switching indices j and k, we obtain

$$\mu_{i\underset{1}{|}kj} - \mu_{j\underset{1}{|}ki} - \underset{1}{R}^\alpha_{ikj}\mu_\alpha - L^\alpha_{[kj]}\mu_{i\underset{1}{|}\alpha} = \underset{1}{\overset{5}{\theta}}_{ik\underset{1}{|}j} - \underset{1}{\overset{5}{\theta}}_{jk\underset{1}{|}i}.$$

After adding this result to (4.20), we get

$$\mu_{i\underset{1}{|}(jk)} + \mu_{j\underset{1}{|}[ki]} = \underset{1}{R}^\alpha_{ikj}\mu_\alpha + L^\alpha_{[kj]}\mu_{i\underset{1}{|}\alpha} + \underset{1}{\overset{5}{\theta}}_{ik\underset{1}{|}j} - \underset{1}{\overset{5}{\theta}}_{jk\underset{1}{|}i} + \underset{1}{\overset{5}{\theta}}_{ij\underset{1}{|}k}. \tag{4.21}$$

Finally, we obtain a system of differential equations of the Cauchy type with covariant derivatives with respect to unknown functions μ_i, μ_{ij}, F_i^h and F_{ij}^h:

$$\begin{aligned} F^h_{i\underset{1}{|}j} &= F^h_{ij}, \\ F^h_{i(j\underset{1}{|}k)} &= \underset{1}{\overset{6}{\theta}}_{ijk}, \\ \mu_{i\underset{1}{|}j} &= \mu_{ij}, \\ \mu_{i\underset{1}{|}(jk)} + \mu_{j\underset{1}{|}[ki]} &= \underset{1}{\overset{7}{\theta}}_{ijk}, \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} \underset{1}{\overset{6}{\theta}}_{ijk} = & -\underset{1}{\xi}^h_{ijk} + \mu_{(j\underset{1}{|}k)}F_i^h + \mu_{[i\underset{1}{|}k]}F_j^h + \mu_{[i\underset{1}{|}j]}F_k^h - \mu_{\alpha\underset{1}{|}(j}F^\alpha_{k)}\delta^h_i \\ & + \mu_{\alpha\underset{1}{|}[i}F^\alpha_{k]}\delta^h_j + \mu_{\alpha\underset{1}{|}[i}F^\alpha_{j]}\delta^h_k + \underset{1}{\overset{1}{\theta}}{}^h_{ikj} \end{aligned}$$

and

$$\underset{1}{\overset{7}{\theta}}_{ijk} = \underset{1}{R}{}^{\alpha}_{ikj}\mu_\alpha + L^{\alpha}_{[kj]}\mu_{i\underset{1}{|}\alpha} + \underset{1}{\overset{5}{\theta}}_{ik\underset{1}{|}j} - \underset{1}{\overset{5}{\theta}}_{jk\underset{1}{|}i} + \underset{1}{\overset{5}{\theta}}_{ij\underset{1}{|}k}.$$

On the other hand, functions μ_i, μ_{ij}, F^h_i and F^h_{ij} satisfy the algebraic formulas

$$F^h_{(i\underset{1}{|}j)} + \xi^h_{\alpha(i}F^\alpha_{j)} = \mu_{(i}F^h_{j)} - \mu_\alpha F^\alpha_{(i}\delta^h_{j)}, \tag{4.23}$$

$$F^h_\alpha F^\alpha_i = e\delta^h_i, \quad \mu_{(ij)} = \underset{1}{\overset{5}{\theta}}_{ij}.$$

The system (4.22) has at most one solution for initial conditions (4.23). Initial conditions are limited by the algebraic ones (4.23). It can be easily seen that the initial conditions have at most

$$\frac{1}{2}N(N^2-1)$$

independent parameters. In this way, the following theorems are proved.

Theorem 4.4. *The equations* (4.22) *and* (4.23) *give an algebraic differential equation system of the Cauchy type in covariant derivatives with respect to the unknown functions* μ_i, μ_{ij}, F^h_i *and* F^h_{ij}. *This system generates all e-structures* F^h_i *determining G-almost geodesic mappings of the type* $\underset{1}{\pi}_2(e,F)$, $e=\pm 1$.

Theorem 4.5. *Let* GA_n *be a non-symmetric affine connection space. A family of all e-structures* F^h_i *which determine a G-almost geodesic mapping of the type* $\underset{1}{\pi}_2(e,F)$, $e=\pm 1$, *depends on at most* $N(N^2-1)/2$ *real parameters.*

Analogously, we can consider the case of G-almost geodesic mappings of the type $\underset{2}{\pi}_2(e)$, $e=\pm 1$.

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