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THE GRADIENT SUPERCONVERGENCE OF THE FINITE VOLUME METHOD FOR A NONLINEAR ELLIPTIC PROBLEM OF NONMONOTONE TYPE

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Dedicated to Professor Ivo Babuška on his 90th birthday

Abstract. We study the superconvergence of the finite volume method for a nonlinear elliptic problem using linear trial functions. Under the condition of C-uniform meshes, we first establish a superclose weak estimate for the bilinear form of the finite volume method. Then, we prove that on the mesh point set S, the gradient approximation possesses the superconvergence: $\max_{P\in S}|(\nabla u-\overline{\nabla}u_h)(P)|=O(h^2)|\ln h|^{3/2}$, where $\overline{\nabla}$ denotes the average gradient on elements containing vertex P. Furthermore, by using the interpolation post-processing technique, we also derive a global superconvergence estimate in the H^1-norm and establish an asymptotically exact a posteriori error estimator for the error $\|u-u_h\|_1$.

Keywords: finite volume method; nonlinear elliptic problem; local and global superconvergence in the $W^{1,\infty}$-norm; a posteriori error estimator

MSC 2010: 65M60, 65M15

1. Introduction

The finite volume method (FVM), also known as generalized difference method [13], [25], [26], box scheme [4], [29] or covolume method [16], [22], has been widely analyzed for various types of partial differential equations. The main benefit of this

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method is that it inherits some physical conservation laws of the original problem locally, which is very desirable in practical applications, for example, computational fluid mechanics. We refer to the monograph [26] for general presentation of the finite volume method, and to [16], [11], [15], [14], [19], [24], [30], [32], [34] and the references cited therein for details.

Superconvergence of the finite element method has long been an active research area in scientific computation, since it is of practical importance in enhancing the accuracy of numerical solutions [3], [1], [2], [8], [10], [9], [27], [31], [33], [35]. At present, many superconvergence results have also been obtained for the finite volume method. For linear elliptic problems, Ewing et al. [19] obtain the H^1 and $W^{1,\infty}$ superconvergence estimates between the FVM solution and the linear interpolation of the exact solution; Huang and Li [22] derive the H^1 and $W^{1,\infty}$ superconvergence estimates for the error between the FVM solution and the corresponding finite element (FEM) solution. Moreover, the superconvergence of the FVM solution in an average gradient norm has been also obtained. See, for example, [13], [26], [28]. For the linear elliptic and parabolic problems, Chou et al. [16] show the superconvergence estimates in the L_p-norm for the error between the FVM solution and the corresponding FEM solution and between their gradients. All the superconvergence results mentioned above are for linear elliptic problems.

In this paper, we consider the superconvergence of the FVM for the following nonlinear elliptic problem of nonmonotone type:

$$\begin{cases} -\operatorname{div}(a(x,u)\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a convex bounded domain in $\mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$, the coefficient function $a(x,u) \geqslant \beta_1 > 0$ in Ω. We do not assume the monotonicity condition for problem (1.1) (see (2.3)).

Some authors have studied the FVM for problem (1.1). Li [25] first obtained the error estimate in the H^1-norm. Chatzipantelidis et al. [12] establish the error estimates in the H^1-, L_2-, and L_∞-norm. Bi [6] obtains the H^1 and $W^{1,\infty}$ superclose estimates for the error between the FVM solution and the corresponding FEM solution. Bi and Ginting [7] also analyze the two-grid FVM and derived the error estimates in a broken H^1-norm. Moreover, Bergam et al. [5] give a residual type a posteriori error estimate for the FVM solution.

To the authors' best knowledge, there is no gradient superconvergence result available for the error between the FVM solution and the exact solution of nonlinear problem (1.1). Our main goal in this paper is to give some gradient superconvergence results for the linear finite volume approximation to problem (1.1). By treating the

FVM as a perturbation of the corresponding FEM, we first establish the superclose weak estimate for the bilinear form $a_h(\omega;\cdot,\cdot)$ of the FVM (see (2.8)),

$$|a_h(\omega; u-\Pi_h u, \Pi_h^* v_h)| \leqslant Ch^2\|u\|_{3,p}\|v_h\|_{1,q} \quad \forall\, v_h \in U_h,\ 1<p\leqslant\infty, \tag{1.2}$$

where $\omega \in W^{1,\infty}(\Omega)$, $1/p+1/q=1$, Π_h is the usual linear interpolation operator, Π_h^* is an interpolation operator from the trial function space U_h to the test function space V_h. It is well known that such a superclose weak estimate plays an important role in the superconvergence analysis of FEM [33], [35]. By using this weak estimate and the Green's function method [35], we can further derive

$$\|u_h-\Pi_h u\|_{1,\infty} \leqslant C(u)h^2|\ln h|^{3/2}. \tag{1.3}$$

Then, we consider the superconvergence at mesh points and prove the following superconvergence result:

$$\max_{P\in S}|(\nabla u-\overline{\nabla} u_h)(P)| \leqslant C(u)h^2|\ln h|^{3/2}, \tag{1.4}$$

where S is the set of all interior mesh points, and $\overline{\nabla}$ denotes the average gradient on elements containing point P. In order to obtain the global superconvergence, we introduce the interpolation post-processing technique [27] and prove that

$$\|u-Q_{2h}u_h\|_1 \leqslant C(u)h^2, \tag{1.5}$$

where Q_{2h} is the interpolation post-processing operator. Based on superconvergence estimate (1.5), an asymptotically exact a posteriori error estimate is also given for the error $\|u-u_h\|_1$.

This paper is organized as follows. In Section 2, we introduce the finite volume scheme and give some useful lemmas. In Section 3, we establish the superclose weak estimate and derive the superconvergence estimates for the error $u_h-\Pi_h u$ in the H^1- and $W^{1,\infty}$-norm. Section 4 is devoted to the discussion of mesh points and global superconvergence in the H^1-norm. Finally, in Section 5, some numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notation $W^{m,p}(D)$ to stand for the usual Sobolev spaces on subdomain $D\subset\Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and the semi-norm $|\cdot|_{m,p,D}$, and if $p=2$, we set $W^{m,p}(D)=H^m(D)$, $\|\cdot\|_{m,p,D}=\|\cdot\|_{m,D}$. The inner product and the norm in space $L_2(D)$ are denoted by $(\cdot,\cdot)_D$ and $\|\cdot\|_D$, respectively. When $D=\Omega$, we omit the index D. We will use the letter C to represent a generic positive constant, independent of the mesh size h.

2.1. Finite volume element approximation. Consider problem (1.1). As usual, we assume that $a(x,u) \in W^{1,\infty}(\Omega \times \mathbb{R})$ and there exist positive constants β_1, β_2, and β_3 such that

$$\beta_1 \leqslant a(x,u) \leqslant \beta_2, \quad x=(x_1,x_2) \in \Omega,\ u \in \mathbb{R}, \tag{2.1}$$

$$|a'_{x_1}(x,u)| + |a'_{x_2}(x,u)| + |a'_u(x,u)| \leqslant \beta_3, \quad x \in \Omega,\ u \in \mathbb{R}. \tag{2.2}$$

For properly smooth $a(x,u)$ and $f(x)$, the unique existence of weak and classical solutions for problem (1.1) is proved in [17], [21], [20]. We assume that the solution of problem (1.1) possesses the smoothness and boundedness required in our analysis.

Remark 2.1. In our analysis, the monotonicity condition

$$(a(x,u)\nabla u - a(x,v)\nabla v, \nabla u - \nabla v) \geqslant \alpha_0 \|\nabla u - \nabla v\|^2, \quad \alpha_0 \geqslant 0, \tag{2.3}$$

is not needed for $a(x,u)$, and since the coefficient $a(x,u)$ depends on x, the classical Kirchhoff transformation (see, for instance, [23]) cannot be used in our study.

Remark 2.2. In the case of domain Ω with a smooth boundary $\partial\Omega$, f being a α-Hölder continuous function on $\overline{\Omega}$ $(0 < \alpha \leqslant 1)$, and $a(x,u) \in C^2(\overline{\Omega} \times \mathbb{R})$, it is shown in [17], [18] that the solution of problem (1.1) has the regularity $u \in C^{2+\alpha}(\overline{\Omega})$, and the solution is unique.

Let $T_h = \bigcup\{K\}$ be a quasi-uniform triangulation of domain Ω with mesh size $h = \max h_K$, where h_K is the diameter of the triangle K. The union of the triangles of T_h determines a polygonal domain $\Omega_h \subset \Omega$ whose boundary vertices lie on $\partial\Omega$. If Ω itself is a polygonal domain, we may have $\Omega_h = \Omega$. If Ω is a domain with smooth boundary $\partial\Omega$, we assume that triangulation T_h is such that for a positive constant C, it holds that

$$\operatorname{dist}(x, \partial\Omega) \leqslant Ch^2 \quad \forall\, x \in \partial\Omega_h.$$

Concerning triangulation T_h, we construct the barycenter dual partition T_h^* by connecting the barycenter to the midpoints of edges of each $K \in T_h$ by straight lines. Thus, for each nodal point P in T_h, there exists a polygonal K_p^* surrounding P, and $K_p^* \in T_h^*$ is called the dual element or the control volume at point P, see Figure 1.

For triangulations T_h and T_h^*, we define the following trial function space U_h and test function space V_h, respectively,

$$\begin{aligned} U_h &= \{u_h \in C^0(\overline{\Omega})\colon\ u_h|_K \in P_1(K)\ \forall\, K \in T_h, u_h|_{\Omega\setminus\Omega_h} = 0\},\\ V_h &= \{v_h \in L_2(\Omega)\colon\ v_h|_{K_p^*} = \text{constant}\ \forall\, P \in N_h,\ v_h|_{\Omega\setminus\Omega_h} = 0\}, \end{aligned}$$

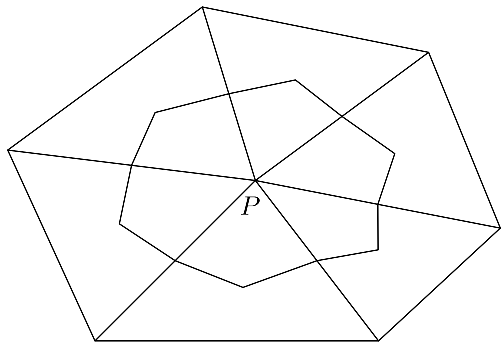

Figure 1. Dual element K_p^* around point P.

where $P_1(K)$ is the set of all linear polynomials on K and N_h is the set of all nodal points of T_h. Let $\Pi_h^*\colon U_h \to V_h$ be the interpolation operator defined by

$$\Pi_h^* u_h = \sum_{P\in N_h} u_h(P)\chi_p \quad \forall\, u_h \in U_h, \tag{2.4}$$

where χ_p is the characteristic function of the dual element K_p^*.

A standard weak form for problem (1.1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u; u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega), \tag{2.5}$$

where

$$a(\omega; u, v) = \int_\Omega a(x,\omega)\nabla u \cdot \nabla v \,\mathrm{d}x, \quad (f, v) = \int_\Omega f v \,\mathrm{d}x. \tag{2.6}$$

This weak form is usually adopted for finite element approximations. However, for the FVM, we need a new weak form. Let u be the solution of problem (1.1) and $v \in V_h$. Then, by using Green's formula, we have

$$-\int_{\partial K_p^*} \mathbf{n} \cdot (a(x,u)\nabla u) v \,\mathrm{d}s = \int_{K_p^*} f v \,\mathrm{d}x \quad \forall\, K_p^* \in T_h^*, \tag{2.7}$$

where $\mathbf{n}$ is the outward unit normal vector on the concerned boundary. Motivated by the weak form (2.7), we introduce the form

$$a_h(\omega; u, v) = -\sum_{K_p^* \in T_h^*} \int_{\partial K_p^*} \mathbf{n} \cdot (a(x,\omega)\nabla u) v \,\mathrm{d}s, \quad \omega, u \in H^1(\Omega),\ v \in V_h, \tag{2.8}$$

and define the finite volume approximation to problem (1.1) by finding $u_h \in U_h$ such that

$$a_h(u_h; u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{2.9}$$

Note that for fixed ω, $a_h(\omega; u, v)$ is linear in u and v. Since Π_h^* is a one-to-one mapping from U_h onto red V_h, the equivalent form of problem (2.9) is to find $u_h \in U_h$ such that

$$a_h(u_h; u_h, \Pi_h^* v_h) = (f, \Pi_h^* v_h) \quad \forall\, v_h \in U_h, \tag{2.10}$$

which is the finite volume scheme to be used in our analysis. From (2.7) we know that scheme (2.10) is consistent, and the following error equation holds:

$$a_h(u; u, \Pi_h^* v_h) - a_h(u_h; u_h, \Pi_h^* v_h) = 0 \quad \forall\, v_h \in U_h. \tag{2.11}$$

2.2. Some lemmas. Let $\Pi_h u$ be the usual linear interpolation approximation of a continuous function u. In our analysis, the following approximation property, trace inequality, and the inverse inequality will be used frequently [33], [35]:

$$\|u - \Pi_h u\|_{m,p,K} \leqslant C h_K^{2-m} \|u\|_{2,p,K}, \quad 0 \leqslant m \leqslant 2, \quad u \in W^{2,p}(K),\ 1 < p \leqslant \infty, \tag{2.12}$$

$$\|u\|_{0,p,\partial K} \leqslant C h_K^{-1/p} (\|u\|_{0,p,K} + h_K \|\nabla u\|_{0,p,K}), \quad u \in W^{1,p}(K),\ 1 \leqslant p \leqslant \infty, \tag{2.13}$$

$$\|u_h\|_{l,p,K} \leqslant C h_K^{m-l+2/p-2/q} \|u_h\|_{m,q,K}, \quad u_h \in P_1(K),\ m \leqslant l,\quad q \leqslant p,\ 1 \leqslant p,\ q \leqslant \infty, \tag{2.14}$$

$$\|u_h\|_{m,p,\partial K} \leqslant C h_K^{-1/p} \|u_h\|_{m,p,K}, \quad u_h \in P_1(K),\quad m = 0, 1,\ 1 \leqslant p \leqslant \infty. \tag{2.15}$$

Remark 2.3. By decomposing the dual element K_p^* into several triangles, it is easy to see that the trace inequality (2.13) and the inverse inequality (2.14) also hold on the dual element $K_p^* \in T_h^*$.

Furthermore, for the operator Π_h^*, we have the following lemma.

Lemma 2.1. *For $K \in T_h$, let $\tau \subset \partial K$ be an edge of K. Then, for $v_h \in U_h$, $1 \leqslant q \leqslant \infty$, we have*

$$\int_K (v_h - \Pi_h^* v_h) = 0, \quad \int_\tau (v_h - \Pi_h^* v_h)\,\mathrm{d}s = 0, \tag{2.16}$$

$$\|v_h - \Pi_h^* v_h\|_{0,q,K} \leqslant C h_K \|\nabla v_h\|_{0,q,K}, \tag{2.17}$$

$$\|v_h - \Pi_h^* v_h\|_{0,q,\partial K} \leqslant C h_K^{1-1/q} \|\nabla v_h\|_{0,q,K}. \tag{2.18}$$

P r o o f. Noting that v_h is a linear function on K, formula (2.16) can be derived by a direct calculation. For (2.17), let P be a vertex of K and $K^* = K_p^* \cap K$ a third of K, then $\Pi_h^* v_h = v_h(P)$ on K^*. By using the inverse inequality, we have

$$\begin{aligned}\|v_h - \Pi_h^* v_h\|_{0,q,K^*} &= \|v_h - v_h(P)\|_{0,q,K^*} \\ &\leqslant h_{K^*}|K^*|^{1/q}|\nabla v_h|_{0,\infty,K^*} \leqslant Ch_{K^*}\|\nabla v_h\|_{0,q,K^*},\end{aligned}$$

which gives (2.17). Now let $\tau^* = \partial K \bigcap \partial K^*$. Then, from (2.13) and (2.17) we obtain that

$$\begin{aligned}\|v_h - \Pi_h^* v_h\|_{0,q,\tau^*} &\leqslant Ch_{K^*}^{-1/q}(\|v_h - \Pi_h^* v_h\|_{0,q,K^*} + h_{K^*}\|\nabla v_h\|_{0,q,K^*}) \\ &\leqslant Ch_{K^*}^{1-1/q}\|\nabla v_h\|_{0,q,K^*}.\end{aligned}$$

Thus, the proof is completed. □

Similarly, we prove the following approximation properties.

Lemma 2.2. *Let $K_p^* \in T_h^*$ and $v_h \in U_h$. Then we have*

$$\|v_h - \Pi_h^* v_h\|_{0,q,K_p^*} \leqslant Ch\|\nabla v_h\|_{0,q,K_p^*}, \quad 1 \leqslant q \leqslant \infty, \tag{2.19}$$

$$\|v_h - \Pi_h^* v_h\|_{0,q,\partial K_p^*} \leqslant Ch^{1-1/q}\|\nabla v_h\|_{0,q,K_p^*}, \quad 1 \leqslant q \leqslant \infty. \tag{2.20}$$

The basic approach of our analysis is to consider the FVM as a perturbation of the FEM [19], [34], so we need to give the difference between $a_h(\omega; u_h, \Pi_h^* v_h)$ and $a(\omega; u_h, v_h)$.

In what follows, we will omit the variable x in $a(x,u)$, except when its arising is necessary.

Lemma 2.3. *For any $\omega \in H^1(\Omega)$, $w \in U_h + H^2(\Omega)$, $v_h \in U_h$, we have*

$$\begin{aligned}a_h(\omega; w, \Pi_h^* v_h) - a(\omega; w, v_h) &= \sum_{K\in T_h}\int_{\partial K}\mathbf{n}\cdot(a(\omega)\nabla w)(\Pi_h^* v_h - v_h)\,\mathrm{d}s \\ &\quad - \sum_{K\in T_h}\int_K \operatorname{div}(a(\omega)\nabla w)(\Pi_h^* v_h - v_h)\,\mathrm{d}x.\end{aligned} \tag{2.21}$$

P r o o f. By Green's formula, we have

$$\int_K a(\omega)\nabla w\cdot\nabla v_h\,\mathrm{d}x = -\int_K \operatorname{div}(a(\omega)\nabla w)v_h\,\mathrm{d}x + \int_{\partial K}\mathbf{n}\cdot(a(\omega)\nabla w)v_h\,\mathrm{d}s,$$

and (see Figure 1)

$$\sum_{K\in T_h}\int_K \operatorname{div}(a(\omega)\nabla w)\Pi_h^* v_h \,\mathrm{d}x = \sum_{K\in T_h}\sum_{K_p^*\in T_h^*}\int_{K_p^*\cap K} \operatorname{div}(a(\omega)\nabla w)\Pi_h^* v_h \,\mathrm{d}x$$
$$= \sum_{K\in T_h}\int_{\partial K} \mathbf{n}\cdot(a(\omega)\nabla w)\Pi_h^* v_h \,\mathrm{d}s + \sum_{K_p^*\in T_h^*}\int_{\partial K_p^*} \mathbf{n}\cdot(a(\omega)\nabla w)\Pi_h^* v_h \,\mathrm{d}s.$$

Substituting the above two identities into the definitions of $a(\omega; w, v_h)$ and $a_h(\omega; w, \Pi_h^* v_h)$, the proof is completed. □

The following lemma shows that the finite volume form $a_h(\omega; u, \Pi_h^* v)$ is an h-perturbation of the finite element form $a(\omega; u, v)$.

Lemma 2.4 ([12]). *There exists a positive constant C such that for ω_h, u_h, $v_h \in U_h$,*

$$|a(\omega_h; u_h, v_h) - a_h(\omega_h; u_h, \Pi_h^* v_h)| \leqslant Ch\|\nabla\omega_h\cdot\nabla u_h\|_{0,p}\|v_h\|_{0,q},$$

where $1 \leqslant p,\ q \leqslant \infty$, $1/p + 1/q = 1$.

Lemma 2.5. *Let $\omega_h \in U_h$ and $\|\nabla\omega_h\|_{0,p} \leqslant M$, $p > 2$. Then for h small enough we have that*

$$a_h(\omega_h; u_h, \Pi_h^* u_h) \geqslant C\|\nabla u_h\|^2 \quad \forall\, u_h \in U_h. \tag{2.22}$$

Proof. It follows from Lemma 2.4 and the inverse inequality that

$$|a(\omega_h; u_h, u_h) - a_h(\omega_h; u_h, \Pi_h^* u_h)| \leqslant Ch\|\nabla\omega_h\cdot\nabla u_h\|\|\nabla u_h\|$$
$$\leqslant Ch|\nabla\omega|_{0,\infty}\|\nabla u_h\|^2 \leqslant Ch^{1-2/p}\|\nabla\omega_h\|_{0,p}\|\nabla u_h\|^2 \leqslant Ch^{1-2/p}M\|\nabla u_h\|^2.$$

Next, from the condition (2.1) we can see that

$$a(\omega_h; u_h, u_h) \geqslant \beta_1\|\nabla u_h\|^2 \quad \forall\, \omega_h, u_h \in U_h.$$

Thus, we obtain

$$a_h(\omega_h; u_h, \Pi_h^* u_h) = a(\omega_h; u_h, u_h) + a_h(\omega_h; u_h, \Pi_h^* u_h) - a(\omega_h; u_h, u_h)$$
$$\geqslant (\beta_1 - Ch^{1-2/p}M)\|\nabla u_h\|^2.$$

This gives the conclusion of Lemma 2.5. □

A direct calculation yields

$$\operatorname{div}(a(x,\omega)\nabla w) = (a'_{x_1}, a'_{x_2})\cdot\nabla w + a'_\omega \nabla\omega\cdot\nabla w + a\operatorname{div}\nabla w. \tag{2.23}$$

Particularly, for $u_h \in U_h$, since ∇u_h is a constant on K, we have

$$\operatorname{div}(a(x,\omega)\nabla u_h) = \nabla a(x,\omega)\cdot\nabla u_h,\ u_h \in U_h. \tag{2.24}$$

An analysis of the FVM for solving (1.1) with $a(x,u) = a(u)$ has been given in [12]. We note that the techniques in [12], together with the application of identity (2.24), can be used to establish the existence and uniqueness, and the H^1- and L_2-norm error estimates of the FVM solution to problem (1.1) with coefficient $a(x,u)$. We collect these results in the following two lemmas without proof.

Lemma 2.6 ([12]). *Let $M > 0$ be such that $\|f\| \leqslant M\alpha^{-1}$. Then, for h small enough, problem* (2.10) *has a solution u_h in the ball*

$$B_M = \{v_h \in U_h\colon\ \|\nabla v_h\|_{0,p} \leqslant M,\ 2 < p \leqslant 2+\delta\},$$

where α and δ appear in the inf-sup condition [12]*: there exist constants $\alpha = \alpha(a(u),\Omega)$ and $\delta = \delta(a(u),\Omega)$ such that for $u_h \in U_h$, $\omega_h \in B_M$,*

$$\|\nabla u_h\|_{0,p} \leqslant \alpha \sup_{0\neq v_h\in U_h} \frac{a_h(\omega_h; u_h, \Pi_h^* v_h)}{\|\nabla v_h\|_{0,q}},\quad 2 < p \leqslant 2+\delta,\ 1/p + 1/q = 1.$$

Furthermore, if a'_ω is Lipschitz continuous, then for sufficiently small data f and h, the solution u_h is unique.

Lemma 2.7 ([12]). *Let u and u_h be the solutions of problems* (1.1) *and* (2.10)*, respectively. Then, if $\gamma = \alpha\beta_3 M < 1$, we have for h small enough that*

$$\|u-u_h\|_1 \leqslant Ch,\ u \in H^2(\Omega),\ f \in L_2(\Omega), \tag{2.25}$$

$$\|u-u_h\| \leqslant Ch^2,\ u \in W^{2,p}(\Omega),\ f \in H^1(\Omega),\ p > 2, \tag{2.26}$$

where $C = C(u,f)$ represents a positive constant which only depends on u and f.

The H^1- and L_2-norm error estimates of the FVM solution in Lemma 2.7 will be used in our superconvergence analysis. An application of the H^1-error estimate is to bound u_h in the $W^{1,\infty}$-norm.

Lemma 2.8. *Let $u \in W^{2,p}(\Omega)$, $p > 2$ and $u_h \in U_h$ be the solutions of problems* (1.1) *and* (2.10), *respectively. Then, for h small enough, we have*

$$|u_h|_{1,\infty} \leqslant C(u).$$

P r o o f. According to the inverse inequality, we have

$$|u_h|_{1,\infty} \leqslant |u_h - \Pi_h u|_{1,\infty} + |\Pi_h u|_{1,\infty} \leqslant Ch^{-1}\|u_h - \Pi_h u\|_1 + C|u|_{1,\infty}. \tag{2.27}$$

Note that $|u|_{1,\infty} \leqslant C|u|_{2,p},\ p > 2$. In addition, from Lemma 2.7 we know

$$\|u_h - \Pi_h u\|_1 \leqslant \|u_h - u\|_1 + \|u - \Pi_h u\|_1 \leqslant C(u)h + Ch\|u\|_2.$$

Hence, substituting this estimate into (2.27), the proof is completed. $\square$

3. Superclose estimate of the interpolation function

In this section, we will give some superclose estimates for the interpolation function, which are very useful in our superconvergence analysis.

3.1. The interpolation weak estimate. We first establish the interpolation weak estimate. To this end, we need to strengthen the triangulation condition.

Definition 3.1. Let T_h be a quasi-uniform triangulation. Then T_h is called C-uniform if any two adjacent triangle elements of T_h form an approximate parallelogram in the sense that (see Figure 2)

$$|\overrightarrow{P_1P_2} + \overrightarrow{P_3P_4}| + |\overrightarrow{P_2P_3} + \overrightarrow{P_4P_1}| = O(h^2). \tag{3.1}$$

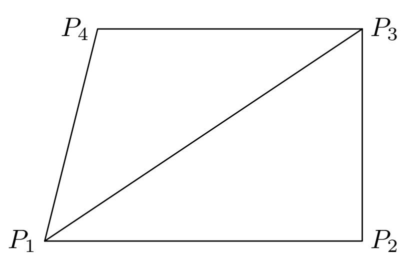

Figure 2. An approximate parallelogram.

It is well known that the interpolation weak estimate plays an important role in the superconvergence analysis of FEM [33], [35]. For FEM defined on C-uniform triangle meshes, the following interpolation weak estimate has been established (see, for example, [33], [35]): For any given $\omega \in W^{1,\infty}(\Omega)$,

$$|a(\omega; u-\Pi_h u, v)| \leqslant C(\omega)h^2\|u\|_{3,p}\|v\|_{1,q} \quad \forall\, v \in U_h,\ 1 < p \leqslant \infty,\ 1/p+1/q = 1, \tag{3.2}$$

where $C(\omega)$ is a positive constant which only depends on $\|\omega\|_{1,\infty}$.

Now we give a similar weak estimate for the FVM.

Theorem 3.1. *Let triangulation T_h be C-uniform, and $u \in W^{3,p}(\Omega)$, $\omega \in W^{1,\infty}(\Omega)$. Then, we have*

$$\begin{aligned} |a_h(\omega; u - \Pi_h u, \Pi_h^* v)| &\leqslant C(\omega)h^2\|u\|_{3,p}\|v\|_{1,q} \\ &\qquad \forall\, v \in U_h,\ 1 < p \leqslant \infty,\ 1/p + 1/q = 1, \end{aligned} \tag{3.3}$$

where $C(\omega)$ is a positive constant which depends on $\|\omega\|_{1,\infty}$.

Proof. Let a_M be the value of $a(x, \omega(x))$ at the midpoint of edge $\tau \subset \partial K$. Obviously, we have

$$|a(x,\omega) - a_M| \leqslant h_K|a(x,\omega)|_{1,\infty} \leqslant C(\omega)h_K, \quad x \in \tau.$$

From Lemma 2.3, we obtain for $v \in U_h$ that

$$\begin{aligned} &a_h(\omega; u - \Pi_h u, \Pi_h^* v) - a(\omega; u - \Pi_h u, v) \\ &\quad= \sum_{K\in T_h} \int_{\partial K} \mathbf{n}\cdot(a(\omega) - a_M)\nabla(u - \Pi_h u)(\Pi_h^* v - v)\,\mathrm{d}s \\ &\qquad+ \sum_{K\in T_h} \int_{\partial K} \mathbf{n}\cdot(a_M\nabla(u - \Pi_h u))(\Pi_h^* v - v)\,\mathrm{d}s \\ &\qquad+ \sum_{K\in T_h} (-\mathrm{div}(a(\omega)\nabla(u - \Pi_h u)), \Pi_h^* v - v)_K \\ &\quad= E_1 + E_2 + E_3. \end{aligned} \tag{3.4}$$

We need to estimate $E_1 \sim E_3$. Using (2.12)–(2.13) and (2.18), we first obtain

$$\begin{aligned} |E_1| &\leqslant \sum_{K\in T_h} h_K|a(x,\omega)|_{1,\infty}\|\nabla(u - \Pi_h u)\|_{0,p,\partial K}\|v - \Pi_h^* v\|_{0,q,\partial K} \\ &\leqslant C(\omega)h^2\|u\|_{2,p}\|v\|_{1,q}. \end{aligned}$$

Next, we write (noting that $(\Pi_h^* v - v)|_{\partial\Omega_h} = 0$)

$$E_2 = \sum_{K\in T_h} \sum_{\tau\subset\partial K\setminus\partial\Omega_h} \int_\tau \mathbf{n}\cdot(a_M \nabla(u - \Pi_h u))(\Pi_h^* v - v)\,\mathrm{d}s.$$

Let τ be an interior edge, that is, a common side of two adjacent elements K and K'. Since $\mathbf{n}|_{\tau\cap\partial K} = -\mathbf{n}|_{\tau\cap\partial K'}$ and $a_M\nabla u(\Pi_h^* v - v)$ is continuous across edge τ (except for the midpoint), it follows from (2.16) and $a_M\nabla\Pi_h u$ being constant that

$$E_2 = \sum_{K\in T_h} \sum_{\tau\subset\partial K\setminus\partial\Omega_h} \int_\tau -\mathbf{n}\cdot(a_M \nabla\Pi_h u)(\Pi_h^* v - v)\,\mathrm{d}s = 0.$$

Finally, to estimate E_3, let w^c be the piecewise constant approximation of function w on T_h, which satisfies

$$\|w - w^c\|_{0,p,K} \leqslant C h_K \|w\|_{1,p,K}, \quad 1 \leqslant p \leqslant \infty. \tag{3.5}$$

Moreover, from (2.23) and (2.16) we find that

$$\begin{aligned} E_3 &= \sum_{K\in T_h} (-\nabla a(\omega)\cdot\nabla(u - \Pi_h u) - a(\omega)\operatorname{div}\nabla(u - \Pi_h u), \Pi_h^* v - v)_K \\ &= -\sum_{K\in T_h} (\nabla a(\omega)\cdot\nabla(u - \Pi_h u), \Pi_h^* v - v)_K \\ &\quad - \sum_{K\in T_h} (a(\omega)\operatorname{div}\nabla u - (a(\omega)\operatorname{div}\nabla u)^c, \Pi_h^* v - v)_K \\ &\leqslant C|a(\omega)|_{1,\infty} \sum_{K\in T_h} h_K(\|u\|_{2,p,K} + \|u\|_{3,p,K})\|v - \Pi_h^* v\|_{0,q,K} \\ &\leqslant C(\omega)h^2\|u\|_{3,p}\|v\|_{1,q}. \end{aligned}$$

Substituting estimates $E_1 \sim E_3$ into (3.4), we complete the proof by using (3.2). $\square$

3.2. Superclose estimates for $\Pi_h u - u_h$. By means of the interpolation weak estimate in Theorem 3.1, we can obtain some important superclose results for $\Pi_h u - u_h$ in the H^1- and $W^{1,\infty}$-norm.

Theorem 3.2. *Let triangulation T_h be C-uniform, and u and u_h be the solutions of problems* (1.1) *and* (2.10), *respectively, and $u \in H^3(\Omega)$. Then we have*

$$\|\Pi_h u - u_h\|_1 \leqslant C(u)h^2\|u\|_3. \tag{3.6}$$

Proof. Let $\theta = u_h - \Pi_h u$. Then, from Lemma 2.5 and the error equation (2.11) we can obtain that

$$\begin{aligned}
C\|u_h - \Pi_h u\|_1^2 &\leqslant a_h(u_h; u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) \\
&= a_h(u_h; u_h - u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) \\
&= a_h(u_h; u_h, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) \\
&= a_h(u; u, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta) + a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) = F_1 + F_2.
\end{aligned} \tag{3.7}$$

To estimate $F_1 = a_h(u; u, \Pi_h^*\theta) - a_h(u_h; u, \Pi_h^*\theta)$, we need the following results:

$$\begin{aligned}
|a(u) - a(u_h)| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} a(tu + (1-t)u_h)\,\mathrm{d}t\right| \\
&= \left|\int_0^1 a_u'(tu + (1-t)u_h)\,\mathrm{d}t(u - u_h)\right| \leqslant |a_u'|_\infty |u - u_h|.
\end{aligned} \tag{3.8}$$

In addition, $(a(u) - a(u_h))\nabla u$ and θ are continuous across ∂K_p^*, which implies that

$$\sum_{K_p^* \in T_h^*} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u \theta \,\mathrm{d}s = 0 \quad \forall\, \theta \in U_h. \tag{3.9}$$

Thus, it follows from (3.8)–(3.9), the trace inequality, (2.20), and Lemma 2.7 that

$$\begin{aligned}
F_1 &= -\sum_{K_p^* \in T_h^*} \int_{\partial K_p^*} n \cdot (a(u) - a(u_h))\nabla u \Pi_h^*\theta \,\mathrm{d}s \\
&= -\sum_{K_p^* \in T_h^*} \int_{\partial K_p^*} n \cdot (a(u) - a(u_h))\nabla u (\Pi_h^*\theta - \theta) \,\mathrm{d}s \\
&\leqslant |a_u'|_\infty |\nabla u|_\infty \sum_{K_p^* \in T_h^*} \|u - u_h\|_{0,\partial K_p^*} \|\Pi_h\theta - \theta\|_{0,\partial K_p^*} \\
&\leqslant C \sum_{K_p^* \in T_h^*} (\|u - u_h\|_{0,K_p^*} + h\|\nabla(u - u_h)\|_{0,K_p^*})\|\theta\|_{1,K_p^*} \\
&\leqslant C(\|u - u_h\| + h\|\nabla(u - u_h)\|)\|\theta\|_1 \leqslant C(u)h^2\|\theta\|_1.
\end{aligned}$$

For F_2, from $\|u_h\|_{1,\infty} \leqslant C(u)$ and the weak estimate (3.3) we know that

$$F_2 = a_h(u_h; u - \Pi_h u, \Pi_h^*\theta) \leqslant C(u)h^2\|u\|_3\|\theta\|_1.$$

Substituting estimates F_1 and F_2 into (3.7), the proof is completed. □

A direct result of Theorem 3.2 are the following optimal error estimates in the L_p- and $W^{1,p}$-norm.

Theorem 3.3. *Let triangulation T_h be C-uniform, and u and u_h be the solutions of problems* (1.1) *and* (2.10), *respectively. Then, we have*

$$\|u-u_h\|_{0,p} \leqslant C_p(u)h^2, \quad u \in H^3(\Omega),\ 1<p<\infty, \tag{3.10}$$

$$\|u-u_h\|_{0,\infty} \leqslant C(u)|\ln h|^{1/2}h^2, \quad u \in H^3(\Omega)\cap W^{2,\infty}(\Omega), \tag{3.11}$$

$$\|u-u_h\|_{1,p} \leqslant C(u)h, \quad u \in H^3(\Omega)\cap W^{2,p}(\Omega),\ 1<p\leqslant\infty. \tag{3.12}$$

Proof. Using the embedding theory and (3.6), we obtain

$$\begin{aligned}\|u-u_h\|_{0,p} &\leqslant \|u-\Pi_h u\|_{0,p} + \|\Pi_h u - u_h\|_{0,p}\\ &\leqslant \|u-\Pi_h u\|_{0,p} + C_p\|\Pi_h u - u_h\|_1\\ &\leqslant C_p h^2(\|u\|_{2,p}+\|u\|_3), \quad 1<p<\infty.\end{aligned}$$

Furthermore, from the discrete embedding inequality in finite element space [35], we have

$$\|v_h\|_{0,\infty} \leqslant C|\ln h|^{1/2}\|v_h\|_1 \quad \forall\, v_h \in U_h,$$

which, together with (3.6), yields that

$$\begin{aligned}\|u-u_h\|_{0,\infty} &\leqslant \|u-\Pi_h u\|_{0,\infty} + \|\Pi_h u - u_h\|_{0,\infty}\\ &\leqslant \|u-\Pi_h u\|_{0,\infty} + C|\ln h|^{1/2}\|\Pi_h u - u_h\|_1\\ &\leqslant C(u)h^2(\|u\|_{2,\infty}+|\ln h|^{1/2}\|u\|_3).\end{aligned}$$

In addition, the inverse inequality implies that

$$\begin{aligned}\|u-u_h\|_{1,p} &\leqslant \|u-\Pi_h u\|_{1,p} + \|\Pi_h u - u_h\|_{1,\infty}\\ &\leqslant \|u-\Pi_h u\|_{1,p} + Ch^{-1}\|\Pi_h u - u_h\|_1\\ &\leqslant C(u)(h\|u\|_{2,p}+h\|u\|_3).\end{aligned}$$

The proof is completed. □

Remark 3.1. Bi [6] gives the following L_∞-norm error estimate:

$$\|u-u_h\|_{0,\infty} \leqslant C(u)|\ln h|h^2.$$

Obviously, our result (3.11) is better than the above one.

Below we consider the superconvergence estimate in space $W^{1,\infty}(\Omega)$. To this end, we need to introduce the regularized Green function [33], [35].

For any given $z \in \Omega$, let $\delta_h^z \in U_h$ be the regularized δ-function which satisfies

$$(\delta_h^z, v_h) = v_h(z), \ z \in \Omega \quad \forall\, v_h \in U_h.$$

For any appointed direction L, define the direction derivative

$$\partial_z v(z) = \lim_{\Delta z \to 0, \Delta z \| L} \frac{v(z + \Delta z) - v(z)}{|\Delta z|},$$

where the notation $\Delta z \parallel L$ means that the increment Δz is parallel to the direction L. Then, for given $\omega \in W^{1,\infty}(\Omega)$, there exists a regularized Green function of derivative type $\partial_z G^z(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$a(\omega; v, \partial_z G^z) = (\partial_z \delta_h^z, v) \quad \forall\, v \in H_0^1(\Omega).$$

Let $\partial_z G_h^z \in U_h$ be the finite element approximation of $\partial_z G^z$ such that

$$a(\omega; v_h, \partial_z G^z - \partial_z G_h^z) = 0 \quad \forall\, v_h \in U_h.$$

Clearly, we have

$$a(\omega; v_h, \partial_z G_h^z) = a(\omega; v_h, \partial_z G^z) = (\partial_z \delta_h^z, v_h) = \partial_z v_h(z) \quad \forall\, v_h \in U_h. \tag{3.13}$$

The following boundedness estimates have been given in [33], [35]:

$$\|\partial_z G_h^z\|_1 \leqslant Ch^{-1} |\ln h|^{1/2}; \ \|\partial_z G_h^z\|_{1,1} \leqslant C |\ln h|, \tag{3.14}$$

where C is a positive constant independent of $z \in \Omega$.

Theorem 3.4. *Let triangulation T_h be C-uniform, and u and u_h be the solutions of problems* (1.1) *and* (2.10), *respectively, and $u \in W^{3,\infty}(\Omega)$. Then, we have*

$$\|\Pi_h u - u_h\|_{1,\infty} \leqslant Ch^2 |\ln h|^{3/2} \|u\|_{3,\infty}. \tag{3.15}$$

P r o o f. Let $g_h^z = \partial_z G_h^z$. From (3.13) and Lemma 2.3 we have

$$\begin{aligned} \partial_z (u_h - \Pi_h u)(z) &= a(u_h; u_h - \Pi_h u, g_h^z) \\ &= a(u_h; u_h - \Pi_h u, g_h^z) - a_h(u_h; u_h - \Pi_h u, \Pi_h^* g_h^z) + a_h(u_h; u_h - \Pi_h u, \Pi_h^* g_h^z) \\ &= \sum_{K \in T_h} \int_{\partial K} \mathbf{n} \cdot (a(u_h) \nabla (u_h - \Pi_h u))(g_h^z - \Pi_h^* g_h^z) \, \mathrm{d}s \\ &\quad + \sum_{K \in T_h} (-\mathrm{div}(a(u_h) \nabla (u_h - \Pi_h u)), g_h^z - \Pi_h^* g_h^z)_K \\ &\quad + a_h(u_h; u_h - \Pi_h u, \Pi_h^* g_h^z) = S_1 + S_2 + S_3. \end{aligned} \tag{3.16}$$

Thus, it follows from (2.16), (2.13), (2.18), Theorem 3.2, and (3.14) that

$$\begin{aligned} S_1 &= \sum_{K\in T_h} \int_{\partial K} \mathbf{n} \cdot ((a(u_h) - a^c(u_h))\nabla(u_h - \Pi_h u))(g_h^z - \Pi_h^* g_h^z)\,\mathrm{d}s \\ &\leqslant C \sum_{K\in T_h} h_K |a(u_h)|_{1,\infty} \|\nabla(u_h - \Pi_h u)\|_{0,\partial K} \|g_h^z - \Pi_h^* g_h^z\|_{0,\partial K} \\ &\leqslant C h^{1/2} |\nabla u_h|_\infty \|\nabla(u_h - \Pi_h u)\| h^{1/2} \|g_h^z\|_1 \leqslant C h^2 |\ln h|^{1/2} \|u\|_3. \end{aligned}$$

Moreover, from (2.24), (2.17), Theorem 3.2, and (3.14) we know

$$\begin{aligned} S_2 &\leqslant C \sum_{K\in T_h} |a(u_h)|_{1,\infty} \|u_h - \Pi_h u\|_{1,K} \|g_h^z - \Pi_h^* g_h^z\|_{0,K} \\ &\leqslant C(u) h^2 \|u\|_3 h \|g_h^z\|_1 \leqslant C(u) h^2 |\ln h|^{1/2} \|u\|_3. \end{aligned}$$

Furthermore, according to the error equation (2.11), we may write

$$\begin{aligned} S_3 &= a_h(u_h; u_h - \Pi_h u, \Pi_h^* g_h^z) \\ &= a_h(u_h; u_h - u, \Pi_h^* g_h^z) + a_h(u_h; u - \Pi_h u, \Pi_h^* g_h^z) \\ &= a_h(u_h; u_h, \Pi_h^* g_h^z) - a_h(u_h; u, \Pi_h^* g_h^z) + a_h(u_h; u - \Pi_h u, \Pi_h^* g_h^z) \\ &= a_h(u; u, \Pi_h^* g_h^z) - a_h(u_h; u, \Pi_h^* g_h^z) + a_h(u_h; u - \Pi_h u, \Pi_h^* g_h^z) \\ &= S_{31} + S_{32}. \end{aligned} \tag{3.17}$$

From (3.8), (3.9), (2.20), and Theorem 3.3 we find that

$$\begin{aligned} S_{31} &= a_h(u; u, \Pi_h^* g_h^z) - a_h(u_h; u, \Pi_h^* g_h^z) \\ &= -\sum_{K_p^*\in T_h^*} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u \Pi_h^* g_h^z\,\mathrm{d}s \\ &= -\sum_{K_p^*\in T_h^*} \int_{\partial K_p^*} \mathbf{n} \cdot (a(u) - a(u_h))\nabla u (\Pi_h^* g_h^z - g_h^z)\,\mathrm{d}s \\ &\leqslant |a_u'|_\infty |\nabla u|_\infty \|u - u_h\|_{0,\infty} \sum_{K_p^*\in T_h^*} \|\Pi_h^* g_h^z - g_h^z\|_{0,1,\partial K_p^*} \\ &\leqslant C(u) \|u - u_h\|_{0,\infty} \sum_{K_p^*\in T_h^*} \|g_h^z\|_{1,1,K_p^*} \\ &\leqslant C(u) h^2 |\ln h|^{1/2} \|g_h^z\|_{1,1} \leqslant C(u) h^2 |\ln h|^{3/2}. \end{aligned}$$

For S_{32}, the weak estimate (3.3) implies that

$$S_{32} = a_h(u_h; u - \Pi_h u, \Pi_h^* \theta) \leqslant C(u_h) h^2 \|u\|_{3,\infty} \|g_h^z\|_{1,1} \leqslant C(u) h^2 |\ln h|.$$

Hence,

$$S_3 = S_{31} + S_{32} \leqslant C(u)h^2|\ln h|^{3/2}.$$

Substituting estimates $S_1 \sim S_3$ into (3.16), we complete the proof. □

4. Mesh points and global superconvergence in $W^{1,\infty}$-norm

In this section, we first give the gradient superconvergence on mesh points. Then, by using the interpolation post-processing technique, we further derive a global superconvergence result, and establish an asymptotically exact a posteriori error estimator in the $W^{1,\infty}$-norm.

4.1. Gradient superconvergence on mesh points. Let P_0 be an interior nodal point surrounded by elements $K_1, \ldots, K_m$, $K_i = \triangle P_0 P_i P_{i+1}$. Denote by N_i the midpoint of a common edge P_0P_i of two adjacent elements K_{i-1} and K_i (see Figure 3 for the $m = 6$ case). It is well known that if the triangle meshes are C-uniform, the midpoint N_i is the optimal stress point [35], that is (set $K_0 = K_m$),

$$(4.1)\quad \left|\nabla u(N_i) - \tfrac{1}{2}[\nabla \Pi_h u|_{K_{i-1}} + \nabla \Pi_h u|_{K_i}]\right| \leqslant Ch^2 \|u\|_{3,\infty,K_{i-1}\cup K_i}, \quad i = 1, \ldots, m.$$

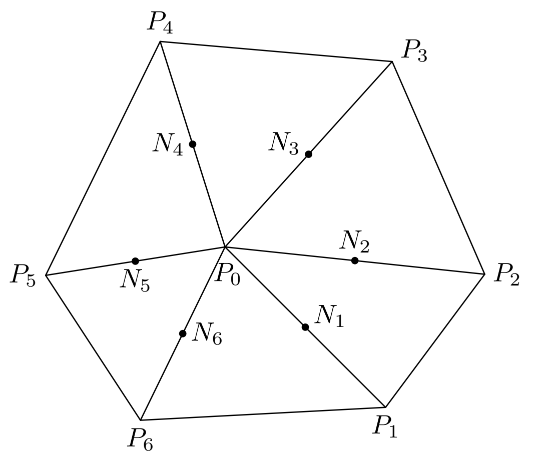

Figure 3. C-uniform triangle elements around node P_0.

In the existing literature, the set of optimal stress points on triangle meshes only has the midpoints of the interior edges. See, for example, [26], [35]. Below we prove that any interior mesh point P is also an optimal stress point if C-uniform meshes are used.

Lemma 4.1. *Assume that the triangulation T_h is C-uniform. Then*

$$\left|\frac{1}{m}\sum_{i=1}^{m}(N_i - P_0)\right| \leqslant Ch^2. \tag{4.2}$$

P r o o f. Let $P_{i_1} = P_{i_2}$ if $i_1 = i_2 \pmod{m}$, and vector $\overrightarrow{P_0P_i} = P_i - P_0$. Since $N_i = \frac{1}{2}(P_0 + P_i)$, we have

$$\frac{1}{m}\sum_{i=1}^{m}(N_i - P_0) = \frac{1}{2m}\sum_{i=1}^{m}(P_i - P_0) = \frac{1}{2m}\sum_{i=1}^{m}\overrightarrow{P_0P_i}. \tag{4.3}$$

By the vector operation rule and C-uniform condition (3.1), we can obtain (see Figure 3 for $m = 6$) that

$$\sum_{i=1}^{m}\overrightarrow{P_{i+1}P_{i+2}} = \overrightarrow{P_2P_3} + \overrightarrow{P_3P_4} + \overrightarrow{P_4P_5} + \overrightarrow{P_5P_6} + \overrightarrow{P_6P_1} + \overrightarrow{P_1P_2} = \overrightarrow{0},$$

$$\text{and}\quad |\overrightarrow{P_0P_i} + \overrightarrow{P_{i+1}P_{i+2}}| \leqslant Ch^2, \quad i = 1, 2, \ldots, m.$$

Hence,

$$\left|\sum_{i=1}^{m}\overrightarrow{P_0P_i}\right| = \left|\sum_{i=1}^{m}(\overrightarrow{P_0P_i} + \overrightarrow{P_{i+1}P_{i+2}})\right| \leqslant Ch^2. \tag{4.4}$$

Now, Lemma 4.1 follows from (4.3) and (4.4). □

Denote by $\overline{\nabla}\Pi_h u(P)$ the arithmetic mean of the gradient $\nabla\Pi_h u$ on elements containing mesh point P.

Theorem 4.1. *Assume that the triangulation T_h is C-uniform, and $u \in W^{3,\infty}(\Omega)$. Then, any interior mesh point P is also an optimal stress point, that is,*

$$|\nabla u(P) - \overline{\nabla}\Pi_h u(P)| \leqslant Ch^2\|u\|_{3,\infty,E}, \tag{4.5}$$

where E is the union of elements containing point P.

P r o o f. Let P_0 be an interior mesh point (see Figure 3). First, from (4.1) we have (setting $K_0 = K_m$) that

$$\begin{aligned}\left|\frac{1}{m}\sum_{i=1}^{m}\nabla u(N_i) - \overline{\nabla}\Pi_h u(P_0)\right| &= \left|\frac{1}{m}\sum_{i=1}^{m}\nabla u(N_i) - \frac{1}{m}\sum_{i=1}^{m}\nabla\Pi_h u|_{K_i}\right| \\ &= \frac{1}{m}\left|\sum_{i=1}^{m}\nabla u(N_i) - \frac{1}{2}\left(\sum_{i=1}^{m}\nabla\Pi_h u|_{K_{i-1}} + \sum_{i=1}^{m}\nabla\Pi_h u|_{K_i}\right)\right| \\ &\leqslant \frac{1}{m}\sum_{i=1}^{m}\left|\nabla u(N_i) - \frac{1}{2}(\nabla\Pi_h u|_{K_{i-1}} + \nabla\Pi_h u|_{K_i})\right| \leqslant Ch^2\|u\|_{3,\infty,E}.\end{aligned} \tag{4.6}$$

Next, let $\partial u = \partial_x u$ or $\partial_y u$. Then, by means of the Taylor expansion, we obtain

$$\partial u(N_i) = \partial u(P_0) + \nabla \partial u(P_0) \cdot (N_i - P_0) + R_i,$$

where the remainder term $|R_i| \leqslant Ch^2 \|u\|_{3,\infty,K_{i-1}\cup K_i}$. Hence,

$$\frac{1}{m}\sum_{i=1}^{m}(\partial u(N_i) - \partial u(P_0)) = \frac{1}{m}\nabla \partial u(P_0) \cdot \sum_{i=1}^{m}(N_i - P_0) + \frac{1}{m}\sum_{i=1}^{m} R_i.$$

So, from Lemma 4.1 we have that

$$\left|\frac{1}{m}\sum_{i=1}^{m}\nabla u(N_i) - \nabla u(P_0)\right| \leqslant Ch^2 \|u\|_{3,\infty,E},$$

which, together with (4.6), completes the proof. $\square$

Now, combining Theorems 3.4 and 4.1, we immediately obtain the following superconvergence result.

Theorem 4.2. *Assume that the triangulation T_h is C-uniform, u and u_h are the solutions of problems* (1.1) *and* (2.10), *respectively, and $u \in W^{3,\infty}(\Omega)$. Then, we have*

$$\max_{P\in S} |\nabla u(P) - \overline{\nabla} u_h(P)| \leqslant Ch^2 |\ln h|^{3/2} \|u\|_{3,\infty}, \tag{4.7}$$

where S is the set of all interior mesh points.

4.2. Global superconvergence in H^1-norm. In order to derive the global superconvergence approximation to solution u, we need to introduce the interpolation post-processing technique proposed by Lin et al. in [27].

Let T_{2h} be a coarser mesh triangulation of domain Ω with the mesh size $2h$. The triangulation T_h is a refined triangulation of T_{2h} obtained by connecting the midpoints of all edges of elements in T_{2h}. We assume that T_{2h} is C-uniform. It is easy to see that the triangulation T_h is C-uniform if T_{2h} is. Let Q_{2h} be the piecewise quadratic interpolation operator defined on T_{2h} by

$$Q_{2h}u(z_i) = u(z_i), \quad i = 1, 2, \ldots$$

where $\{z_i\}$ are the vertices and the midpoints of edges of all elements in T_{2h}. Note that each interpolation node of Q_{2h} is also the vertex of an element in T_h.

Lemma 4.2 ([27]). *The interpolation operator Q_{2h} satisfies*

$$Q_{2h}u = Q_{2h}\Pi_h u, \quad u \in H^2(\Omega),$$
$$\|u - Q_{2h}u\|_1 \leqslant Ch^2\|u\|_3,$$
$$\|Q_{2h}v_h\|_1 \leqslant C\|v_h\|_1 \quad \forall\, v_h \in U_h.$$

Let u_h be the FVM solution. Then, the interpolation post-processed solution is defined by $Q_{2h}u_h$.

Theorem 4.3. *Let the triangulation T_{2h} be C-uniform, u and u_h be the solutions of problems* (1.1) *and* (2.10)*, respectively, and $u \in H^3(\Omega)$. Then, we have the following superconvergence estimate:*

$$\|u - Q_{2h}u_h\|_1 \leqslant Ch^2\|u\|_3.$$

Proof. From Lemma 4.2 we have

$$\begin{aligned}\|u - Q_{2h}u_h\|_1 &= \|u - Q_{2h}u + Q_{2h}\Pi_h u - Q_{2h}u_h\|_1\\ &\leqslant \|u - Q_{2h}u\|_1 + \|Q_{2h}(\Pi_h u - u_h)\|_1\\ &\leqslant Ch^2\|u\|_3 + C\|\Pi_h u - u_h\|_1.\end{aligned}$$

The proof is completed by using Theorem 3.2. □

It is very important for a numerical method to have a computable a posteriori error bound, so that we can assess and enhance the accuracy of the numerical solution by an adaptive algorithm in practical applications. By virtue of the superconvergence result of Theorem 4.3, we can further derive an asymptotically exact a posteriori error estimator for the error $\|u - u_h\|_1$.

Define the error estimator $E(u_h) = \|u_h - Q_{2h}u_h\|_1$. Obviously, $E(u_h)$ is a computable quantity in terms of the FVM solution u_h.

Theorem 4.4. *Assume that the conditions of Theorem* 4.5 *hold. Then, $E(u_h)$ is an asymptotically exact a posteriori error estimator for the error $\|u - u_h\|_1$, namely*

$$\lim_{h\to 0}\frac{\|u_h - Q_{2h}u_h\|_1}{\|u - u_h\|_1} = 1. \tag{4.8}$$

Proof. Using the triangle inequality, we obtain

$$\|u - u_h\|_1 - \|u - Q_{2h}u_h\|_1 \leqslant E(u_h) \leqslant \|u - u_h\|_1 + \|u - Q_{2h}u_h\|_1,$$

or

$$1 - \frac{\|u - Q_{2h}u_h\|_1}{\|u - u_h\|_1} \leqslant \frac{E(u_h)}{\|u - u_h\|_1} \leqslant 1 + \frac{\|u - Q_{2h}u_h\|_1}{\|u - u_h\|_1}. \tag{4.9}$$

From Theorem 4.3, we know that $\|u - Q_{2h}u_h\|_1 = O(h^2)$. However, for the FVM solution, generally speaking, we only have $\|u - u_h\|_1 = O(h)$. Thus, letting $h \to 0$ in (4.9), the proof is completed. □

5. Numerical example

In this section, we will present a numerical example to illustrate the theoretical analysis.

Let us consider problem (1.1) with the data:

$$a(x, u) = 1 + \frac{x_1 x_2}{1 + u^2}, \quad u(x) = x_1^{3.5} \ln x_1 \; x_2^{3.5} \ln x_2,$$

and the source term $f = -\mathrm{div}(a(x, u)\nabla u)$. For simplicity, we take $\Omega = [0, 1]^2$.

In the numerical experiment, we first partition the domain Ω into square meshes with mesh size $h = 1/N$, and then we obtain the C-uniform triangle meshes by perturbing randomly the inner nodes of the square meshes within h^2 and dividing each derived quadrilateral into two triangles (see Figure 4). The refined meshes are obtained by successively halving the mesh size h. The finite volume equation (2.10) is reduced to a nonlinear system of algebraic equations $A(U)U = F$, where U is a vector whose entries are the values of u_h at the mesh points. We have used a fixed point type iteration to solve this system, that is, we solve the linearization system $A(U^{(k-1)})U^{(k)} = F$, where $U^{(k-1)}$ is the previous iteration vector. The fixed point iteration continues until a tolerance of $|U^{(k)} - U^{(k-1)}| < 10^{-6}$ or $|A(U^{(k-1)})U^{(k)} - F| < 10^{-6}$ is reached.

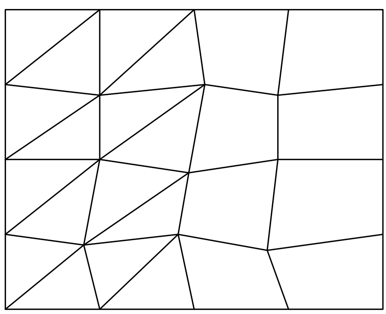

Figure 4. C-uniform triangle meshes.

Let $e_h = \max_{P\in S}|\nabla u(P) - \overline{\nabla} u_h(P)|$ (or $e_h = \|u - Q_{2h}u_h\|_1$) be the computational error with mesh size h. The numerical convergence rate is computed by using the formula $r = \ln(e_h/e_{h/2})/\ln 2$. Denote by $\sigma = \|u_h - Q_{2h}u_h\|_1/\|u - u_h\|_1$ the efficiency index of an a posteriori error estimator $\|u_h - Q_{2h}u_h\|_1$. Table I gives the numerical results with successively halved mesh size h. We see that for approximations $\overline{\nabla} u_h$ and $Q_{2h}u_h$, a convergence rate of $O(h^2)$-order is achieved as the theoretical prediction and the a posteriori error estimator is efficient, that is, $\sigma \approx 1$, as $h \to 0$.

	$\max\|\nabla u(P) - \overline{\nabla} u_h(P)\|$		$\|u - Q_{2h}u_h\|_1$		estimator
mesh h	error	rate	error	rate	σ-index
1/8	0.6635	–	0.3421	–	3.82
1/16	1.6796e-1	1.982	8.6479e-2	1.984	2.21
1/32	4.2428e-2	1.985	2.1815e-2	1.987	2.12
1/64	1.0710e-2	1.986	0.5488e-2	1.991	1.13
1/128	2.7041e-3	1.986	1.3801e-3	1.992	1.11

Table I. The convergence rate and the estimator.

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