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Meromorphic Function Sharing a Small Function with its Differential Polynomial

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Abstract

In this paper we improve, generalize and extend a number of recent results related to a problem of meromorphic function sharing a small function with its differential polynomial which are the continuation of a result earlier obtained by R. Brück.

Key words: Meromorphic function, derivative, small function, weighted sharing.

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1 Introduction definitions and results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $f - a$ and $g - a$ have the same set of zeros with the same multiplicities, we say that f and g share the value a CM (counting multiplicities), and if we do not consider the multiplicities then f and g are said to share the value a IM (ignoring multiplicities).

A meromorphic function a is said to be a small function of f provided that $T(r,a) = S(r,f)$, that is $T(r,a) = o(T(r,f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

We use I to denote any set of infinite linear measure of $0 < r < \infty$.

In 1979, Mues and Steinmetz [15] proved the following theorem.

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Theorem A [15] *Let f be a non-constant entire function. If f and f' share two distinct values a, b IM, then $f' \equiv f$.*

In 1996, for one CM shared values of entire function with its first derivative Brück proposed the following famous conjecture [3]:

Conjecture *Let f be a non-constant entire function such that the hyper order $\rho_2(f)$ of f is not a positive integer or infinite. If f and f' share a finite value a CM, then $\frac{f'-a}{f-a} = c$, where c is a nonzero constant.*

Brück himself proved the conjecture for $a = 0$. For $a \neq 0$ following result was obtained in [3].

Theorem B [3] *Let f be a non-constant entire function. If f and f' share the value 1 CM and if $N(r, 0; f') = S(r, f)$ then $\frac{f'-1}{f-1}$ is a nonzero constant.*

Now it would be interesting to know whether the value 1 of Theorem B can be simply replaced by a small function $a(\not\equiv 0, \infty)$. From the following example we see that it is not possible.

Example 1.1 Let $f = 1 + e^{e^z}$ and $a(z) = \dfrac{1}{1 - e^{-z}}$.
By [6, Lemma 2.6, p. 50] we know that a is a small function of f. Also it can be easily seen that f and f' share a CM and $N(r, 0; f') = 0$ but $f - a \neq c\,(f' - a)$ for every nonzero constant c. We note that $f - a = e^{-z}(f' - a)$. So in order to replace the value 1 by a small function some extra conditions are required.

For entire function of finite order Yang [16] removed the condition $N(r, 0; f') = S(r, f)$ in Theorem B and improved the same in the following way.

Theorem C [16] *Let f be a non-constant entire function of finite order and let $a(\neq 0)$ be a finite constant. If f, $f^{(k)}$ share the value a CM, then $\frac{f^{(k)}-a}{f-a}$ is a nonzero constant, where k (≥ 1) is an integer.*

Next we consider the following examples which show that in Theorem B one can not simultaneously replace "CM" by "IM" and "entire function" by "meromorphic function".

Example 1.2 $f(z) = 1 + \tan z$.
Since $f(z) - 1 = \tan z$ and $f'(z) - 1 = \tan^2 z$ share 1 IM and $N(r, 0; f') = 0$. But the conclusion of Theorem B ceases to hold.

Example 1.3 $f(z) = \dfrac{2}{1 - e^{-2z}}$.
Clearly

$$f'(z) = -\frac{4e^{-2z}}{(1 - e^{-2z})^2}.$$

Here

$$f - 1 = \frac{1 + e^{-2z}}{1 - e^{-2z}} \quad \text{and} \quad f' - 1 = -\frac{(1 + e^{-2z})^2}{(1 - e^{-2z})^2}.$$

Here $N(r, 0; f') = 0$ but the conclusion of Theorem B does not hold.

Zhang [18] extended Theorem B to meromorphic function and also studied the value sharing of a meromorphic function with its k-th derivative counterpart.

In the mean time a new notion of scalings between CM and IM known as weighted sharing, appeared in the uniqueness literature. Below we are giving the definition.

Definition 1.1 [7, 8] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\leq k)$ if and only if it is an a-point of g with multiplicity m $(\leq k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If a is a small function we define that f and g share a IM or a CM or with weight l according as $f - a$ and $g - a$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

Though we use the standard notations and definitions of the value distribution theory available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.2 [11] Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ $(\overline{N}(r, a; f \mid\geq p))$denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not less than p.

(ii) $N(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\leq p))$ denotes the counting function (reduced counting function) of those a-points of f whose multiplicities are not greater than p.

Definition 1.3 {6, cf. [17]} For $a \in \mathbb{C}\cup\{\infty\}$ and a positive integer p we denote by $N_p(r, a; f)$ the sum $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots + \overline{N}(r, a; f \mid\geq p)$. Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 1.4 [9] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid \; g \neq b)$ the counting function of those a-points of f, counted according to multiplicity, which are not the b-points of g.

Definition 1.5 {cf. [1], 2} Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, 1; f)$ the counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r, 1; f)$

the counting function of those 1-points of f and g where $p = q = 1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq 2$, each point in these counting functions is counted only once. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.6 [7, 8] Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly

$$\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f) \quad \text{and} \quad \overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g).$$

With the notion of weighted sharing of values the results of Zhang [18] was improved by Lahiri–Sarkar [11]. In 2005, Zhang [19] further extended the result of Lahiri–Sarkar to a small function and proved the following.

Theorem D [19] *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that $f-a$ and $f^{(k)}-a$ share $(0,l)$. If $l \geq 2$ and*

$$2\overline{N}(r,\infty;f) + N_2\big(r,0;f^{(k)}\big) + N_2\left(r,0;(f/a)'\right) < (\lambda + o(1))\, T\left(r,f^{(k)}\right) \quad (1.1)$$

or $l = 1$ and

$$2\overline{N}(r,\infty;f) + N_2\left(r,0;f^{(k)}\right) + 2\overline{N}\left(r,0;(f/a)'\right) < (\lambda + o(1))\, T\left(r,f^{(k)}\right) \quad (1.2)$$

or $l = 0$ and

$$4\overline{N}(r,\infty;f) + 3N_2\left(r,0;f^{(k)}\right) + 2\overline{N}\left(r,0;(f/a)'\right) < (\lambda + o(1))\, T\left(r,f^{(k)}\right) \quad (1.3)$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)}-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

In 2008, in connection with the results of Lahiri–Sarkar [11] and Zhang [19], Zhang and Lü [20] further investigated the analogous problem of Brück conjecture for the n-th power of a meromorphic function sharing a small function with its k-th derivative. Zhang and Lü [20] obtained the following theorem.

Theorem E [20] *Let f be a non-constant meromorphic function and k (≥ 1), n (≥ 1) and l (≥ 0) be integers. Also let $a \equiv a(z)$ $(\not\equiv 0,\infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $f^{(k)} - a$ share $(0,l)$. If $l = \infty$ and*

$$2\overline{N}(r,\infty;f) + N_2\big(r,0;f^{(k)}\big) + \overline{N}\big(r,0;(f^n/a)'\big) < (\lambda + o(1))\, T\big(r,f^{(k)}\big) \quad (1.4)$$

or $l = 0$ and

$$4\overline{N}(r,\infty;f) + \overline{N}\big(r,0;f^{(k)}\big) + 2N_2\big(r,0;f^{(k)}\big) + 2\overline{N}\big(r,0;(f^n/a)'\big) < (\lambda + o(1))\, T\big(r,f^{(k)}\big) \quad (1.5)$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{f^{(k)}-a}{f^n-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

At the end of [20] the following question was raised by Zhang and Lü [20].
What will happen if f^n and $[f^{(k)}]^m$ share a small function?

Liu [12] investigated the possible answer of the above question and obtained the following result.

Theorem F [12] *Let f be a non-constant meromorphic function and k (≥ 1), n (≥ 1), m (≥ 2) and l (≥ 0) be integers. Also let $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$. If $l = \infty$ and*

$$\frac{2}{m}\overline{N}(r,\infty;f)+\frac{2}{m}\overline{N}\big(r,0;f^{(k)}\big)+\frac{1}{m}\overline{N}\big(r,0;(f^n/a)'\big) < (\lambda+o(1))\,T\big(r,f^{(k)}\big) \quad (1.6)$$

or $l = 0$ and

$$\frac{4}{m}\overline{N}(r,\infty;f)+\frac{5}{m}\overline{N}\big(r,0;f^{(k)}\big)+\frac{2}{m}\overline{N}\big(r,0;(f^n/a)'\big) < (\lambda+o(1))\,T\big(r,f^{(k)}\big) \quad (1.7)$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{(f^{(k)})^m - a}{f^n - a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

So we see that the Brück result and the research thereafter has a long history. Several special forms on the Brück conjecture such as Nevanlinna deficiency, small functions, power functions etc. were meticulously investigated by many authors.

Next we recall the following definition.

Definition 1.7 Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non negative integers.

- The expression $M_j[f] = (f)^{n_{0j}}(f')^{n_{1j}}\ldots(f^{(k)})^{n_{kj}}$ is called a *differential monomial* generated by f of degree $d(M_j) = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.
- The sum $P[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called a *differential polynomial* generated by f of degree $\overline{d}(P) = \max\{d(M_j)\colon 1 \leq j \leq t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \leq j \leq t\}$, where $T(r, b_j) = S(r, f)$ for $j = 1, 2, \ldots, t$.
- The numbers $\underline{d}(P) = \min\{d(M_j)\colon 1 \leq j \leq t\}$ and k the highest order of the derivative of f in $P[f]$ are called respectively the lower degree and order of $P[f]$.
- $P[f]$ is said to be *homogeneous* if $\overline{d}(P) = \underline{d}(P)$.
- $P[f]$ is called a *linear differential polynomial* generated by f if $\overline{d}(P) = 1$. Otherwise $P[f]$ is called *non-linear differential polynomial.* We denote by $Q = \max\{\Gamma_{M_j} - d(M_j)\colon 1 \leq j \leq t\} = \max\{n_{1j}+2n_{2j}+\ldots+kn_{kj} : 1 \leq j \leq t\}$.

Since $(f^{(k)})^m$ is nothing but a differential monomial generated by f, it will be interesting to know whether Theorems D–F can be extended up to differential polynomial generated by f. In this direction recently Li and Yang [13] improved Theorem D in the following manner.

Theorem G [13] *Let f be a non-constant meromorphic function $P[f]$ be a differential polynomial generated by f. Also let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a small meromorphic function. Suppose that $f - a$ and $P[f] - a$ share $(0, l)$ and*

$$(t-1)\,\overline{d}(P) \leq \sum_{j=1}^{t} d(M_j).$$

If $l \geq 2$ and

$$2\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + N_2(r,0;(f/a)') < (\lambda + o(1))\,T(r,P[f]) \quad (1.8)$$

or $l = 1$ and

$$2\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + 2\overline{N}(r,0;(f/a)') < (\lambda + o(1))\,T(r,P[f]) \quad (1.9)$$

or $l = 0$ and

$$4\overline{N}(r,\infty;f) + 3N_2(r,0;P[f]) + 2\overline{N}(r,0;(f/a)') < (\lambda + o(1))\,T\,(r,P[f]) \quad (1.10)$$

for $r \in I$, where $0 < \lambda < 1$, then $\frac{P[f]-a}{f-a} = c$ for some constant $c \in \mathbb{C}/\{0\}$.

Since for a differential monomial $t = 1$, it follows that Theorem G always holds without any supposition on the degree of the monomial. But according to the statement of Theorem G for general differential polynomial, in order to obtain the conclusion, the supposition $(t-1)\overline{d}(P) \leq \sum_{j=1}^{t} d(M_j)$ is required. So it would be quite natural to investigate whether in Theorem G, the condition over the degree can be removed and at the same time (1.8), (1.9) and (1.10) can further be relaxed so that it will improve Theorems E and F to a large extent.

In this paper we shall tackle this situation by improving, unifying, generalizing and extending all the Theorems D–G. Following theorem is the main result of the paper.

Theorem 1.1 *Let f be a non-constant meromorphic function, and n (≥ 1), l (≥ 0) be integers. Let $a \equiv a(z)$ ($\not\equiv 0, \infty$) be a small meromorphic function. Suppose further that $P[f]$ be a differential polynomial generated by f such that $P[f]$ contains at least one derivative. Suppose that $f^n - a$ and $P[f] - a$ share $(0, l)$. If $l = \infty$ and*

$$2\overline{N}(r,\infty;f) + N_2\,(r,0;P[f]) + \overline{N}\,(r,0;(f^n/a)') < (\lambda + o(1))\,T\,(r,P[f]) \quad (1.11)$$

or $l \geq 2$ and

$$2\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + N_2(r,0;(f^n/a)') < (\lambda + o(1))\,T(r,P[f]) \quad (1.12)$$

or $l = 1$ and

$$2\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + \overline{N}(r,0;(f^n/a)') + \overline{N}(r,0;(f^n/a)' \mid (f^n/a) \neq 0) < (\lambda + o(1))\,T\,(r,P[f]) \quad (1.13)$$

or $l = 0$ *and*

$$4\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + 2\overline{N}(r,0;P[f]) + \overline{N}(r,0;(f^n/a)') + \overline{N}(r,0;(f^n/a)' \mid (f^n/a) \neq 0) < (\lambda + o(1))\, T(r,P[f]) \tag{1.14}$$

for $r \in I$, *where* $0 < \lambda < 1$, *then* $\frac{P[f]-a}{f^n-a} = c$ *for some constant* $c \in \mathbb{C}/\{0\}$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F, G be two non-constant meromorphic functions. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{2.1}$$

Lemma 2.1 [19] *Let* f *be a non-constant meromorphic function and* k *be a positive integer. Then*

$$N_p(r,0;f^{(k)}) \leq N_{p+k}(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.2 [10] *If* $N(r,0;f^{(k)} \mid f \neq 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \neq 0) \leq k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\geq k) + S(r,f).$$

Lemma 2.3 [14] *Let* f *be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f *with constant coefficients* $\{a_k\}$ *and* $\{b_j\}$ *where* $a_n \neq 0$ *and* $b_m \neq 0$. *Then*

$$T(r,R(f)) = dT(r,f) + S(r,f),$$

where $d = \max\{n,m\}$.

Lemma 2.4 [4] *Let* f *be a meromorphic function and* $P[f]$ *be a differential polynomial. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.5 *Let* f *be a meromorphic function and* $P[f]$ *be a differential polynomial. Then we have*

$$N\left(r,\infty;\frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\Gamma_P - \overline{d}(P))\,\overline{N}(r,\infty;f) + (\overline{d}(P) - \underline{d}(P))\,N(r,0;f \mid\geq k+1) + Q\overline{N}(r,0;f \mid\geq k+1) + \overline{d}(P)N(r,0;f \mid\leq k) + S(r,f).$$

Proof Let z_0 be a pole of f of order r, such that $b_j(z_0) \neq 0, \infty$; $1 \leq j \leq t$. Then it would be a pole of $P[f]$ of order at most $r\overline{d}(P) + \Gamma_P - \overline{d}(P)$. Since z_0 is a pole of $f^{\overline{d}(P)}$ of order $r\overline{d}(P)$, it follows that z_0 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most $\Gamma_P - \overline{d}(P)$. Next suppose z_1 is a zero of f of order s $(> k)$, such that $b_j(z_1) \neq 0, \infty; 1 \leq j \leq t$. Clearly it would be a zero of $M_j(f)$ of order

$$s \cdot n_{0j} + (s-1)n_{1j} + \ldots + (s-k)n_{kj} = s \cdot d(M_j) - (\Gamma_{M_j} - d(M_j)).$$

Hence z_1 be a pole of $\frac{M_j[f]}{f^{\overline{d}(P)}}$ of order

$$s \cdot \overline{d}(P) - s \cdot d(M_j) + (\Gamma_{M_j} - d(M_j)) = s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)).$$

So z_1 would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most

$$\max\{s(\overline{d}(P) - d(M_j)) + (\Gamma_{M_j} - d(M_j)) \colon 1 \leq j \leq t)\} = s(\overline{d}(P) - \underline{d}(P)) + Q.$$

If z_1 is a zero of f of order $s \leq k$, such that $b_j(z_1) \neq 0, \infty \colon 1 \leq j \leq t$ then it would be a pole of $\frac{P[f]}{f^{\overline{d}(P)}}$ of order at most $s\overline{d}(P)$. Since the poles of $\frac{P[f]}{f^{\overline{d}(P)}}$ comes from the poles or zeros of f and poles or zeros of $b_j(z)$'s only, it follows that

$$N\left(r, \infty; \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\Gamma_P - \overline{d}(P))\, \overline{N}(r, \infty; f) + (\overline{d}(P) - \underline{d}(P))\, N(r, 0; f \mid\geq k+1)$$
$$+ Q\, \overline{N}(r, 0; f \mid\geq k+1) + \overline{d}(P) N(r, 0; f \mid\leq k) + S(r, f).$$

□

Lemma 2.6 [5] *Let $P[f]$ be a differential polynomial. Then*

$$T(r, P[f]) \leq \Gamma_P T(r, f) + S(r, f).$$

Lemma 2.7 *Let f be a non-constant meromorphic function and $P[f]$ be a differential polynomial. Then $S(r, P[f])$ can be replaced by $S(r, f)$.*

Proof From Lemma 2.6 it is clear that $T(r, P[f]) = O(T(r, f))$ and so the lemma follows. □

3 Proof of Theorem 1.1

Let $F = \frac{f^n}{a}$ and $G = \frac{P[f]}{a}$. Then

$$F - 1 = \frac{f^n - a}{a}\, G - 1 = \frac{P[f] - a}{a}.$$

Since $f^n - a$ and $P[f] - a$ share $(0, l)$ it follows that F, G share $(1, l)$ except the zeros and poles of $a(z)$.

Now we consider the following cases.

Case 1. Suppose $H \not\equiv 0$.

Subcase 1.1. Let $l \geq 1$.

From (2.1) we get

$$N(r,\infty;H) \leq \overline{N}(r,\infty;F) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + \overline{N}(r,0;a) + \overline{N}(r,\infty;a) + \sum_{j=1}^{t} N(r,0;b_j) + \sum_{j=1}^{t} N(r,\infty;b_j), \tag{3.1}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$. Then by a simple calculation we see that z_0 is a zero of H and hence

$$N_E^{1)}(r,1;F) = N(r,1;F \mid= 1) \leq N(r,0;H) \leq N(r,\infty;H) + S(r,F). \tag{3.2}$$

By the second fundamental theorem, Lemma 2.7, (3.1) and noting that $\overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) + S(r,f)$, we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,1;G) - N_0(r,0;G') + S(r,G) \\ &\leq 2\overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,0;G \mid\geq 2) + \overline{N}(r,0;F \mid\geq 2) \\ &\quad + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') + S(r,f). \end{aligned} \tag{3.3}$$

While $l = \infty$, $\overline{N}_*(r,1;F,G) = 0$. So

$$\overline{N}(r,0;F \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') \leq \overline{N}(r,0;F'). \tag{3.4}$$

So

$$T(r,G) \leq 2\,\overline{N}(r,\infty;F) + N_2(r,0;G) + \overline{N}(r,0;F') + S(r,f)$$

that is

$$T(r,P[f]) \leq 2\,\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + \overline{N}(r,0;(f^n/a)') + S(r,f),$$

which contradicts (1.11).

While $l \geq 2$, (3.4) changes to

$$\begin{aligned} &\overline{N}(r,0;F \mid\geq 2) + \overline{N}_*(r,1;F,G) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') \\ &\leq \overline{N}(r,0;F \mid\geq 2) + \overline{N}(r,1;F \mid\geq l+1) + \overline{N}(r,1;F \mid\geq 2) + \overline{N}_0(r,0;F') \\ &\leq N_2(r,0;F'). \end{aligned} \tag{3.5}$$

Hence

$$T(r,G) \leq 2\,\overline{N}(r,\infty;F) + N_2(r,0;G) + N_2(r,0;F') + S(r,f).$$

i.e.,

$$T(r,P[f]) \leq 2\overline{N}(r,\infty;f) + N_2(r,0;P[f]) + N_2(r,0;(f^n/a)') + S(r,f),$$

which contradicts (1.12).

While $l=1$ (3.4) changes to

$$\overline{N}(r,0;F\mid\geq 2)+2\,\overline{N}(r,1;F\mid\geq 2)+\overline{N}_0(r,0;F')\\ \leq \overline{N}(r,0;F')+\overline{N}(r,0;F'\mid F\neq 0).$$

Similarly as above we have

$$T(r,P[f])\leq 2\,\overline{N}(r,\infty;f)+N_2(r,0;P[f])+\overline{N}(r,0;(f^n/a)')\\ +\overline{N}\big(r,0;(f^n/a)'\mid (f^n/a)\neq 0\big)+S(r,f),$$

which contradicts (1.13).

Subcase 1.2. $l=0$.

In this case F and G share $(1,0)$ except the zeros and poles of $a(z)$. Also we have

$$N(r,\infty;H)\leq \overline{N}(r,\infty;F)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}(r,0;G\mid\geq 2)+\overline{N}_L(r,1;F)\\ +\overline{N}_L(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f). \tag{3.6}$$

Let z_0 be a zero of $F-1$ with multiplicity p and a zero of $G-1$ with multiplicity q. It is easy to see that

$$N_E^{1)}(r,1;F)=N_E^{1)}(r,1;G)+S(r,f),\\ \overline{N}_E^{(2}(r,1;F)=\overline{N}_E^{(2}(r,1;G)+S(r,f)$$

and

$$N_E^{1)}(r,1;F)\leq N(r,\infty;H)+S(r,f). \tag{3.7}$$

By the second fundamental theorem we get using (3.6) and (3.7) that

$$\begin{aligned}T(r,G)&\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_E^{(2}(r,1;F)\\ &\quad+\overline{N}_L(r,1;G)-\overline{N}_0(r,0;F')+S(r,f)\\ &\leq 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;F\mid\geq 2)+\overline{N}_L(r,1;F)+\overline{N}(r,0;G\mid\geq 2)\\ &\quad+\overline{N}_L(r,1;G)+\overline{N}_0(r,0;G')+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+S(r,f)\\ &\leq 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G\mid\geq 2)+2\overline{N}(r,1;G\mid\geq 2)+\overline{N}_0(r,0;G')\\ &\quad+\overline{N}(r,0;F\mid\geq 2)+2\overline{N}(r,1;F\mid\geq 2)+S(r,f)\\ &\leq 2\,\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,0;G')+\overline{N}(r,0;G'\mid G\neq 0)\\ &\quad+\overline{N}(r,0;F')+\overline{N}(r,0;F'\mid F\neq 0)+S(r,f).\end{aligned}$$

From Lemma 2.1 for $p=1$, $k=1$ and Lemma 2.2 we get

$$T(r,G)\leq 2\overline{N}(r,0;G)+N_2(r,0;G)+\overline{N}(r,0;F')\\ +\overline{N}(r,0;F'\mid F\neq 0)+4\overline{N}(r,\infty;F)+S(r,f),$$

that is,

$$T(r,P[f]) \le 4\overline{N}(r,\infty;f) + 2\overline{N}(r,0;P[f]) + N_2(r,0;P[f]) + \overline{N}(r,0;(f^n/a)')$$
$$+ \overline{N}(r,0;(f^n/a)' \mid (f^n/a) \ne 0) + S(r,f).$$

This contradicts (1.14).

Case 2. Let $H \equiv 0$.

On integration we get from (2.1)

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{3.8}$$

where C, D are constants and $C \ne 0$. From (3.8) it is clear that F and G share 1 CM. We claim that $D = 0$. If $\overline{N}(r,\infty;f) \ne S(r,f)$, then by (3.8) we get $D = 0$.

So we assume that

$$\overline{N}(r,\infty;f) = S(r,f) \tag{3.9}$$

and $D \ne 0$. Clearly

$$\overline{N}(r,\infty;G) = \overline{N}(r,\infty;f) + S(r,f).$$

From (3.8) we get

$$\frac{1}{F-1} = \frac{D\left(G - 1 + \frac{C}{D}\right)}{G-1}. \tag{3.10}$$

Clearly from (3.10) we have

$$\overline{N}\left(r, 1 - \frac{C}{D}; G\right) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G) = S(r,f). \tag{3.11}$$

If $\frac{C}{D} \ne 1$, by the second fundamental theorem, Lemma 2.7 and (3.11) we have

$$T(r,G) \le \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}\left(r, 1 - \frac{C}{D}; G\right) + S(r,G)$$
$$\le \overline{N}(r,0;G) + S(r,f) \le N_2(r,0;G) + S(r,f) \le T(r,G) + S(r,f).$$

So,

$$T(r,G) = N_2(r,0;G) + S(r,f).$$

i.e.,

$$T(r,P[f]) = N_2(r,0;P[f]) + S(r,f),$$

which contradicts (1.11).

If $\frac{C}{D} = 1$ we get from (3.8)

$$\left(F - 1 - \frac{1}{C}\right) G \equiv -\frac{1}{C}. \tag{3.12}$$

From (3.12) it follows that

$$N(r,0;f\mid\geq k+1)\leq N(r,0;P[f])\leq N(r,0;G)=S(r,f). \tag{3.13}$$

Again from (3.12) we see that

$$\frac{1}{f^{\overline{d}(P)}\left(f^n-(1+1/C)a\right)}\equiv-\frac{C}{a^2}\,\frac{P[f]}{f^{\overline{d}(P)}}$$

Hence by the first fundamental theorem, (3.9), (3.13), Lemmas 2.3, 2.4 and 2.5 we get that

$$\begin{aligned}
&(n+\overline{d}(P))T(r,f)=\\
&=T\left(r,f^{\overline{d}(P)}\left(f^n-\left(1+\frac{1}{C}\right)a\right)\right)+S(r,f)\\
&=T\left(r,\frac{1}{f^{\overline{d}(P)}(f^n-(1+\frac{1}{C})a)}\right)+S(r,f)\\
&=T\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\leq m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right)+S(r,f)\\
&\leq(\overline{d}(P)-\underline{d}(P))\left[T(r,f)-\{N(r,0;f\mid\leq k)+N(r,0;f\mid\geq k+1)\}\right]\\
&\quad+(\overline{d}(P)-\underline{d}(P))N(r,0;f\mid\geq k+1)+Q\,\overline{N}(r,0;f\mid\geq k+1)\\
&\quad+\overline{d}(P)N(r,0;f\mid\leq k)+S(r,f)\\
&\leq(\overline{d}(P)-\underline{d}(P))T(r,f)+\underline{d}(P)N(r,0;f\mid\leq k)+S(r,f).
\end{aligned}\tag{3.14}$$

From (3.14) it follows that

$$nT(r,f)\leq S(r,f),$$

which is absurd. Hence $D=0$ and so

$$\frac{G-1}{F-1}=C\quad\text{or}\quad\frac{P[f]-a}{f^n-a}=C.$$

This proves the theorem.

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