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On the class of positive almost weak* Dunford-Pettis operators

ABDERRAHMAN RETBI

Abstract. In this paper, we introduce and study the class of almost weak* Dunford-Pettis operators. As consequences, we derive the following interesting results: the domination property of this class of operators and characterizations of the wDP* property. Next, we characterize pairs of Banach lattices for which each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

Keywords: almost weak* Dunford-Pettis operator; almost Dunford-Pettis operator; weak Dunford-Pettis* property; positive Schur property; order continuous norm

Classification: 46A40, 46B40

1. Introduction and notation

Let us recall from [2] that a norm bounded subset A of a Banach lattice E is said to be almost limited if every disjoint weak* null sequence (f_n) of E' converges uniformly on A , that is, $\lim_{n \rightarrow \infty} \sup_{x \in A} |f_n(x)| = 0$.

An operator T from a Banach lattice E into a Banach space Y is said to be almost Dunford-Pettis if $\|T(x_n)\| \rightarrow 0$ in Y for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E [6].

A Banach space X has the Dunford-Pettis* property (DP* property for short), if $x_n \xrightarrow{w} 0$ in X and $f_n \xrightarrow{w^*} 0$ in X' imply $f_n(x_n) \rightarrow 0$.

A Banach lattice E has

- the positive Schur property, if $\|f_n\| \rightarrow 0$ for every weakly null sequence $(f_n) \subset E^+$, equivalently, $\|f_n\| \rightarrow 0$ for every weakly null sequence $(f_n) \subset E^+$ consisting of pairwise disjoint terms (see page 16 of [9]);
- the weak Dunford-Pettis* property (wDP* property for short), if every relatively weakly compact set in E is almost limited, equivalently, whenever $f_n(x_n) \rightarrow 0$ for every weakly null sequence (x_n) in E and for every disjoint weak* null sequence (f_n) in E' [2].

Recall from [4] that an operator T from a Banach space X into another Banach space Y is called weak* Dunford-Pettis if $f_n(T(x_n)) \rightarrow 0$ for every weakly null sequence $(x_n) \subset X$, and every weak* null sequence $(f_n) \subset Y'$. In this paper,

we introduce and study the disjoint version of this class of operators, that we call almost weak* Dunford-Pettis operators (Definition 2.1). It is a class which contains that of weak* Dunford-Pettis (resp. almost Dunford-Pettis).

The main results are some characterizations of almost weak* Dunford-Pettis operators (Theorem 2.3). Next, we derive the following interesting consequences: the domination property of this class of operators (Corollary 2.4), a characterization of wDP* property (Corollary 2.5). After that, we prove that each positive almost weak* Dunford-Pettis operator from a Banach lattice E into a σ -Dedekind complete Banach lattice F is almost Dunford-Pettis if and only if E has the positive Schur property or the norm of F is order continuous (Theorem 2.7). As consequence, we will give some interesting results (Corollaries 2.8 and 2.9).

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E' , endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E , the sequence (x_α) converges to 0 in the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Riesz space is said to be σ -Dedekind complete if every countable subset that is bounded above has a supremum, equivalently, whenever $0 \leq x_n \uparrow \leq x$ implies the existence of $\sup(x_n)$.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E . If T is an operator from a Banach lattice E into another Banach lattice F then its dual operator T' is defined from F' into E' by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. We refer the reader to [1] for unexplained terminology of Banach lattice theory and positive operators.

2. Main results

Next we give the definition of almost weak* Dunford-Pettis operator between Banach lattices, which is a different version of the weak* Dunford-Pettis operator.

Definition 2.1. An operator T from a Banach lattice E to a Banach lattice F is almost weak* Dunford-Pettis if $f_n(T(x_n)) \rightarrow 0$ for every weakly null sequence (x_n) in E consisting of pairwise disjoint terms, and for every weak* null sequence (f_n) in F' consisting of pairwise disjoint terms.

For proof of the next theorem, we need the following lemma which is just Lemma 2.2 of Chen in [2].

Lemma 2.2. *Let E be a σ -Dedekind complete Banach lattice, and let (f_n) be a weak* convergent sequence of E' . If (g_n) is a disjoint sequence of E' satisfying $|g_n| \leq |f_n|$ for each n , then the sequences $(g_n), (|g_n|), (g_n)^+, (g_n)^-$ are all weak* convergent to zero. In particular, if (f_n) is a disjoint weak* convergent sequence in its own right, then the sequences $(f_n), (|f_n|), (f_n)^+, (f_n)^-$ are all weak* null.*

Now, for positive operators between two Banach lattices, we give a characterization of almost weak* Dunford-Pettis operators.

Theorem 2.3. *Let E and F be two Banach lattices such that F is σ -Dedekind complete. For every positive operator T from E into F , the following assertions are equivalent.*

- (1) T is almost weak* Dunford-Pettis operator.
- (2) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every disjoint weak* null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \rightarrow 0$.
- (3) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset F'$ it follows that $f_n(T(x_n)) \rightarrow 0$.
- (4) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \rightarrow 0$.
- (5) For every weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \rightarrow 0$.

PROOF: (1) \Rightarrow (2) Obvious.

(2) \Rightarrow (3) Assume by way of contradiction that there exists a disjoint weakly null sequence $(x_n) \subset E^+$, and a weak* null sequence $(f_n) \subset F'$ such that $f_n(T(x_n))$ does not converge to 0. The inequality $|f_n(T(x_n))| \leq |f_n|(T(x_n))$ implies $|f_n|(T(x_n))$ does not converge to 0. Then there exist some $\epsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we shall denote by $|f_n|(T(x_n))$ again) satisfying $|f_n|(T(x_n)) > \epsilon$ for all n .

On the other hand, since $x_n \rightarrow 0$ weakly in E , then $T(x_n) \rightarrow 0$ weakly in F . Now an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$|g_n|(T(z_n)) > \epsilon$$

and

$$(4^n \sum_{i=1}^n |g_i|)(T(z_{n+1})) < \frac{1}{n}$$

for all $n \geq 1$. Put $h = \sum_{i=1}^\infty 2^{-n} |g_n|$ and $h_n = (|g_{n+1}| - 4^n \sum_{i=1}^n |g_i| - 2^{-n}h)^+$. By Lemma 4.35 of [1] the sequence (h_n) is disjoint. Since $0 \leq h_n \leq |g_{n+1}|$ for all $n \geq 1$ and (g_n) is weak* null in F' , then from Lemma 2.2 (h_n) is weak* null in F' . From the inequality

$$\begin{aligned} h_n(T(z_{n+1})) &\geq \left(|g_{n+1}| - 4^n \sum_{i=1}^n |g_i| - 2^{-n}h \right) (T(z_{n+1})) \\ &\geq \epsilon - \frac{1}{n} - 2^{-n}h(T(z_{n+1})) \end{aligned}$$

we see that $h_n(T(z_{n+1})) \geq \frac{\epsilon}{2}$ must hold for all n sufficiently large (because $2^{-n}h(T(z_{n+1})) \rightarrow 0$), which contradicts with our hypothesis (2).

(3) \Rightarrow (4) Obvious.

(4) \Rightarrow (5) Assume by way of contradiction that there exists a weakly null sequence $(x_n) \subset E^+$ and a weak* null sequence $(f_n) \subset (F')^+$ such that $f_n(T(x_n))$ does not converge to 0. Then there exists some $\epsilon > 0$ and a subsequence of $f_n(T(x_n))$ (which we shall denote by $f_n(T(x_n))$ again) satisfying $f_n(T(x_n)) \geq \epsilon$ for all n .

On the other hand, since (f_n) is a weak* null sequence in (F') , then $T'(f_n) \rightarrow 0$ weak* in E' . Now an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$T'(g_n)(z_n) > \epsilon$$

and

$$T'(g_{n+1})(4^n \sum_{i=1}^n z_i) < \frac{1}{n}$$

for all $n \geq 1$. Put $z = \sum_{n=1}^\infty 2^{-n} z_n$ and $y_n = (z_{n+1} - 4^n \sum_{i=1}^n z_i - 2^{-n} z)^+$. By Lemma 4.35 of [1] the sequence (y_n) is disjoint. Since $0 \leq y_n \leq z_{n+1}$ for all $n \geq 1$ and (z_n) is weakly null in E , then from Theorem 4.34 of [1] $(y_n) \rightarrow 0$ weakly in E . From the inequality

$$\begin{aligned} T'(g_{n+1})(y_n) &\geq T'(g_{n+1})\left(z_{n+1} - 4^n \sum_{i=1}^n z_i - 2^{-n} z\right) \\ &\geq \epsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z) \end{aligned}$$

we see that $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) \geq \frac{\epsilon}{2}$ must hold for all n sufficiently large (because $2^{-n} T'(g_{n+1})(z) \rightarrow 0$), which contradicts with our hypothesis (4).

(5) \Rightarrow (1) Let (x_n) be a weak null sequence in E consisting of pairwise disjoint terms, and let (f_n) be a weak* null sequence in F' consisting of pairwise disjoint terms, it follows from Remark(1) of [6] that $(|x_n|)$ is weakly null in E , and from lemma 2.2 that $(|f_n|)$ is weak* null in F' . So by our hypothesis (5), $|f_n|(T|x_n|) \rightarrow 0$. Now, from the inequality $|f_n(T(x_n))| \leq |f_n|(T(|x_n|))$ for each n , we deduce that $f_n(T(x_n)) \rightarrow 0$, and this completes the proof. \square

The domination property for almost weak* Dunford-Pettis operators can be derived from Theorem 2.3.

Corollary 2.4. *Let E and F be two Banach lattices such that F is σ -Dedekind complete. If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is an almost weak* Dunford-Pettis, then S is also almost weak* Dunford-Pettis.*

PROOF: Let (x_n) be a weakly null sequence in E^+ and (f_n) be a weak* null sequence in $(F')^+$. According to (5) of Theorem 2.3, it suffices to show that $f_n(S(x_n)) \rightarrow 0$. Since T is almost weak* Dunford-Pettis, then Theorem 2.3 implies that $f_n(T(x_n)) \rightarrow 0$. Now, by the inequality $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for each n , we conclude that $f_n(S(x_n)) \rightarrow 0$. \square

As consequence of Theorem 2.3 and Theorem 3.2 of Chen [2], other characterizations of Banach lattices with the wDP* property are given in the following Corollary.

Corollary 2.5. *Let E be a σ -Dedekind complete Banach lattice. Then, the following assertions are equivalent.*

- (1) E has the wDP* property.
- (2) The solid hull of every relatively weakly compact set in E is almost limited.
- (3) The identity operator $Id_E : E \rightarrow E$ is almost weak* Dunford-Pettis.
- (4) For every disjoint weakly null sequence $(x_n) \subset E$, and every disjoint weak* null sequence $(f_n) \subset E'$ it follows that $f_n(x_n) \rightarrow 0$.
- (5) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every disjoint weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \rightarrow 0$.
- (6) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset E'$ it follows that $f_n(x_n) \rightarrow 0$.
- (7) For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \rightarrow 0$.
- (8) For every weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \rightarrow 0$.

PROOF: (3) \Leftrightarrow (4) Obvious.

(3) \Leftrightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Leftrightarrow (8) follows from Theorem 2.3.

(1) \Leftrightarrow (2) \Leftrightarrow (4) follows from Theorem 3.2 of [2]. □

The proof of the next theorem is based on the following proposition.

Proposition 2.6. *Let E, F and G be three Banach lattices such that G has the DP* property. Then, each operator $T : E \rightarrow F$ that admits a factorization through the Banach lattice G is almost weak* Dunford-Pettis.*

PROOF: Let $P : E \rightarrow G$ and $Q : G \rightarrow F$ be two operators such that $T = Q \circ P$. Let (x_n) be a disjoint weakly null sequence in E and let (f_n) be a disjoint weak* null sequence in F' . It is clear that $P(x_n) \xrightarrow{w} 0$ in G and $Q'(f_n) \xrightarrow{w^*} 0$ in G' . As G has the DP* property, then

$$f_n(Tx_n) = f_n(Q \circ P(x_n)) = (Q'f_n)(P(x_n)) \rightarrow 0.$$

This proves that T is almost weak* Dunford-Pettis. □

Note that every almost Dunford-Pettis operator is almost weak* Dunford-Pettis, but the converse is not true in general. In fact, $Id_{\ell^\infty} : \ell^\infty \rightarrow \ell^\infty$ is almost weak* Dunford-Pettis operator because ℓ^∞ has the wDP* property, but it fails to be almost Dunford-Pettis because ℓ^∞ does not have the positive Schur property.

Now, we characterize Banach lattices such that each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

Theorem 2.7. *Let E and F be two Banach lattices such that F is σ -Dedekind complete. Then the following assertions are equivalent.*

- (1) Each positive almost weak* Dunford-Pettis operator $T : E \rightarrow F$ is almost Dunford-Pettis.
- (2) One of the following assertions is valid:
 - (a) E has the positive Schur property,
 - (b) the norm of F is order continuous.

PROOF: (1) \Rightarrow (2) Assume by way of contradiction that E does not have the positive Schur property and the norm of F is not order continuous. We have to construct a positive almost weak* Dunford-Pettis operator which is not almost Dunford-Pettis. As E does not have the positive Schur property, then there exists a disjoint weakly null sequence (x_n) in E^+ which is not norm null. By choosing a subsequence we may suppose that there is $\epsilon > 0$ with $\|x_n\| > \epsilon > 0$ for all n . From the equality $\|x_n\| = \sup \{f(x_n) : f \in (E')^+, \|f\| = 1\}$, there exists a sequence $(f_n) \subset (E')^+$ such that $\|f_n\| = 1$ and $f_n(x_n) \geq \epsilon$ holds for all n . Now, consider the operator $R : E \rightarrow \ell^\infty$ defined by

$$R(x) = (f_n(x))_{n=1}^\infty$$

On the other hand, since the norm of F is not order continuous, it follows from Theorem 4.51 of [1] that ℓ^∞ is lattice embeddable in F , i.e., there exists a lattice homomorphism $S : \ell^\infty \rightarrow F$ and there exist two positive constants M and m satisfying

$$m \|(\lambda_k)_k\|_\infty \leq \|S((\lambda_k)_k)\|_F \leq M \|(\lambda_k)_k\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S \circ R$, and note by Proposition 2.6 that T is a positive almost weak* Dunford-Pettis operator because ℓ^∞ has DP* property. However, for the disjoint weakly null sequence $(x_n) \subset E^+$, we have

$$\|T(x_n)\| = \|S((f_k(x_n))_k)\| \geq m \|((f_k(x_n))_k)\|_\infty \geq m f_n(x_n) \geq m\epsilon$$

for every n . This shows that T is not almost Dunford-Pettis, and we are done.

(a) \Rightarrow (1) In this case, each operator $T : E \rightarrow F$ is almost Dunford-Pettis.

(b) \Rightarrow (1) Let $(x_n) \subset E$ be a positive disjoint weakly null sequence. We shall show that $\|T(x_n)\| \rightarrow 0$. By Corollary 2.6 of [3], it suffices to prove that $|T(x_n)| \xrightarrow{w} 0$ and $f_n(T(x_n)) \rightarrow 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Let $f \in (F')^+$ and by Theorem 1.23 of [1] there exists some $g \in [-f, f]$ with $f|T(x_n) = g(T(x_n))$. Since $x_n \xrightarrow{w} 0$ then $f|T(x_n) = g(T(x_n)) = (T'g)(x_n) \rightarrow 0$, thus $|T(x_n)| \xrightarrow{w} 0$. On the other hand, let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of F is order continuous, then by Corollary 2.4.3 of [5] $f_n \xrightarrow{w^*} 0$. Now, since T is positive almost weak* Dunford-Pettis then, $f_n(T(x_n)) \rightarrow 0$. This completes the proof. \square

Remark 1. The assumption that F is σ -Dedekind complete is essential in Theorem 2.7. In fact, if we consider $E = \ell^\infty$ and $F = c$, the Banach lattice of all convergent sequences, it is clear that $F = c$ is not σ -Dedekind complete, and it follows from the proof of Proposition 1 of [7] and Theorem 5.99 of [1] that each

operator from ℓ^∞ into c is Dunford-Pettis (and hence is almost Dunford-Pettis). But ℓ^∞ does not have the positive Schur property and the norm of c is not order continuous.

As consequences of Theorem 2.7, we have the following characterization.

Corollary 2.8. *Let E be a σ -Dedekind complete Banach lattice. Then the following assertions are equivalent.*

- (1) *Each positive almost weak* Dunford-Pettis operator $T : E \rightarrow E$ is almost Dunford-Pettis.*
- (2) *The norm of E is order continuous.*

PROOF: The result follows from Theorem 2.7 by noting that if E has the positive Schur property then the norm of E is order continuous. \square

Now, from Corollary 2.8 and Theorem 4.9 (Nakano) of [1], we obtain the following result, which is just Proposition 3.3 of [2].

Corollary 2.9. *Let E be a Banach lattice. Then E has the positive Schur property if and only if E has the wDP* property and its norm is order continuous.*

PROOF: The “only if” part is trivial.

For the “if” part, since E has wDP* property, then $Id_E : E \rightarrow E$ is almost weak* Dunford-Pettis operator. As the norm of E is order continuous, it follows from Theorem 4.9 (Nakano) of [1] that E is σ -Dedekind complete, and by Corollary 2.8 we have that $Id_E : E \rightarrow E$ is almost Dunford-Pettis. This proves that E has the positive Schur property. \square

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